

AN INTEGRAL COMPARISON OF CRYSTALLINE AND DE RHAM COHOMOLOGY

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ABSTRACT. Let $\mathcal{O}_K$ be a mixed characteristic complete DVR with perfect residue field k and fraction field K . It is a celebrated result of Berthelot and Ogus that for a smooth proper formal scheme $X/\mathcal{O}_K$ there exists a comparison between the de Rham cohomology groups $H_{\mathrm{dR}}^i(X/\mathcal{O}_K)$ and the crystalline cohomology groups $H_{\mathrm{crys}}^i(X_k/W(k))$ of the special fibre, after tensoring with K . In this article, we use the stacky perspective on prismatic cohomology, due to Drinfeld and Bhatt–Lurie, to give a version of this comparison result with coefficients in a perfect complex of prismatic F -crystals on X . Our method is of an integral nature and suggests new tools to understand the relationship between torsion in de Rham and crystalline cohomology.

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1. INTRODUCTION

Let $k/\mathbb{F}_p$ be a perfect extension, set $W = W(k)$ and $K_0 = \text{Frac}(W)$, and let K/K_0 be a finite totally ramified extension of degree e and with uniformiser π . Fix X to be a smooth proper formal $\mathcal{O}_K$ -scheme. One of the major goals of p -adic Hodge theory is the comparison of various p -adic cohomology theories attached to X , and for which one of the initial motivations was the crystalline-de Rham comparison of Berthelot–Ogus.

To explain this, recall that for a smooth proper morphism $f: T \rightarrow S$ of smooth complex varieties and a sufficiently small disk $D \subseteq S^{\text{an}}$, the Gauss–Manin connection induces an isomorphism,

$$\nabla_{\text{GM}}: R^i f_* \Omega_{T^{\bullet}/S^{\text{an}}}^{\bullet}|_D \simeq \mathcal{O}_D \otimes_{\mathbb{C}} H_{\text{dR}}^i(T_{s_0}/\mathbb{C}), \quad (1)$$

for any $s_0 \in D$. Analogously, envisioning $\text{Spf}(\mathcal{O}_K)$ as a small disk around $\text{Spec}(k)$, one anticipates that the de Rham cohomology $H_{\text{dR}}^i(X/\mathcal{O}_K)$ depends only on the special fibre X_k . The crystalline cohomology groups $H_{\text{crys}}^i(X_k/W)$ were defined to realise this idea (cf. [Ber74, BO78]).

Using the formalism of crystalline cohomology, the analogue of (1) becomes the precise mathematical statement that there is a functorial isomorphism of $\mathcal{O}_K$ -modules,

$$H_{\text{dR}}^i(X/\mathcal{O}_K) \simeq H_{\text{crys}}^i(X_k/W) \otimes_W \mathcal{O}_K, \quad \text{if } e \leq p-1. \quad (2)$$

Thus, for small e , we do indeed see that $H_{\text{dR}}^i(X/\mathcal{O}_K)$ depends functorially only on X_k .

In their celebrated paper [BO83], Berthelot and Ogus observed that the analogue of (2) *always holds rationally*, regardless of e , i.e. there is a functorial isomorphism of K -vector spaces,

$$H_{\text{dR}}^i(X_K/K) = H_{\text{dR}}^i(X/\mathcal{O}_K) \otimes_{\mathcal{O}_K} K \simeq H_{\text{crys}}^i(X_k/W) \otimes_W K. \quad (3)$$

Thus, the rational de Rham cohomology $H_{\text{dR}}^i(X_K/K)$ also depend functorially only on X_k .

The Berthelot–Ogus isomorphism has become nearly indispensable in arithmetic geometry. But, this classical perspective is lacking in two significant ways:

- (I) the classical setup of the Berthelot–Ogus comparison does not allow for “(integral) coefficients”,
- (II) the method of proof does not lend itself to relating the *integral structures* $H_{\text{dR}}^i(X/\mathcal{O}_K)$ and $H_{\text{crys}}^i(X_k/W)$, e.g. relating their *torsion subgroups*.

Our goal in this article is to explain how the recent advances in integral p -adic Hodge theory, in the form of Bhatt and Scholze’s prismatic cohomology (see [BS22]) and its stacky reinterpretation by Drinfeld and Bhatt–Lurie (see [Dri24, BL22a]), helps address these insufficiencies.

For now, we state our analogue of the Berthelot–Ogus isomorphism with coefficients where we will use ϕ to denote Frobenius, in several different contexts.

Theorem A. *Let $\mathcal{E}$ be a perfect complex of prismatic crystals on X . Then, for $n \geq \lceil \log_p(\frac{e}{p-1}) \rceil$ there exists a natural isomorphism of n -twisted de Rham and crystalline cohomology groups,*

$$\iota^{(n)}: H_{\text{dR}}^{i,(n)}(\mathcal{E}) \simeq \phi^{*n} H_{\text{crys}}^i(\mathcal{E}_k^{\text{crys}}) \otimes_W \mathcal{O}_K.$$

Moreover, a Frobenius structure $\varphi: \phi^* \mathcal{E}[1/\mathfrak{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathfrak{I}_{\Delta}]$ gives rise to a natural commutative diagram of isomorphisms,

$$\begin{array}{ccc} H_{\text{dR}}^{i,(n)}(\mathcal{E}) \otimes_{\mathcal{O}_K} K & \xrightarrow[\sim]{\iota^{(n)}} & \phi^{*n} H_{\text{crys}}^i(\mathcal{E}_k^{\text{crys}}) \otimes_W K \\ \downarrow \overline{\varphi}_{\text{dR}}^{(n)} \wr & & \wr \downarrow \overline{\varphi}_{\text{crys}}^{(n)} \\ H_{\text{dR}}^i(X/\mathcal{O}_K, \mathcal{E}^{\text{dR}}) \otimes_{\mathcal{O}_K} K & \xrightarrow{\sim} & H_{\text{crys}}^i(X_k/W, \mathcal{E}_k^{\text{crys}}) \otimes_W K. \end{array}$$

Thus, the cohomology groups $H_{\text{dR}}^{i,(n)}(\mathcal{E})$ and $H_{\text{dR}}^i(X/\mathcal{O}_K, \mathcal{E}^{\text{dR}}) \otimes_{\mathcal{O}_K} K$ admit a W -descent and a K_0 -descent, respectively. Moreover, these groups only depend functorially on the pair $(X_k, \mathcal{E}_k)$.

In the rest of this introduction we shall explain Theorem A more precisely, and indicate how it provides new tools for comparing torsion in de Rham and crystalline cohomologies.

A prismatic analogue of Dwork's trick. The restriction on e appearing in (2) can be motivated via the isomorphism (1) and its setup: for $e \geq p$ the ‘radius’ of $\mathrm{Spf}(\mathcal{O}_K)$, intuitively $p^{-1/e}$, is too large to support a convergent connection.

Now, note that the ‘Frobenius’ $\phi(x) = x^p$ on the open p -adic unit disk $\mathbb{D}$ is contracting: $\phi^n(\mathbb{B}(r)) \subseteq \mathbb{B}(r^{p^n})$, where $\mathbb{B}(t) = \{|x| \leq t\} \subseteq \mathbb{D}$. So, for a vector bundle $\mathcal{F}$ on $\mathbb{D}$, we have $\phi^{*n}(\mathcal{F}|_{\mathbb{B}(r)}) = \phi^{*n}(\mathcal{F}|_{\mathbb{B}(r^{p^n})})$. Thus, twisting by Frobenius increases the radius of triviality, an observation known as *Dwork's trick*.

We would like to apply this idea to the ‘disk’ $\mathrm{Spf}(\mathcal{O}_K)$. By (2) it would suffice to twist enough number of times to reduce from radius $p^{-1/e}$ to radius $p^{-1/(p-1)}$, i.e. twist $a = \lceil \log_p(\frac{e}{p-1}) \rceil$ number of times (cf. [Kat73, Section 3]). And, indeed, by inspecting the proof of (3) reveals the use of a Dwork's-trick-like isomorphism:

$$X_{p=0}^{(a)} \simeq X_k^{(a)} \otimes_k \mathcal{O}_K/p, \quad (4)$$

where $(-)^{(a)}$ denotes the relevant Frobenius twist of the scheme. Then, (3) follows as the relative Frobenius maps $X_{p=0} \rightarrow X_{p=0}^{(a)}$ and $X_k \rightarrow X_k^{(a)}$ induce isogenies on crystalline cohomology.

Unfortunately, this idea lacks firm foundations because there is no Frobenius map on $\mathrm{Spf}(\mathcal{O}_K)$. While there have been various attempts to address this issue, most notably Ogus' introduction of the convergent site (e.g. see [Ogu84]), they fail to address (II) and (to a lesser extent) (I).

To provide full and firm foundations for an analogue of Dwork's trick, we appeal to the stacky interpretation of prismatic cohomology. From the morphism $f: X \rightarrow \mathrm{Spf}(\mathcal{O}_K)$ one produces a diagram of (formal) stacks,

$$\begin{array}{ccccc} X^{\mathrm{dR}} & \xrightarrow{\rho_{X, \mathrm{dR}}} & X^\Delta & \xleftarrow{\rho_{X, \mathrm{crys}}} & (X_k/W)^{\mathrm{crys}} \\ \downarrow f^{\mathrm{dR}} & & \downarrow f^\Delta & & \downarrow f^{\mathrm{crys}} \\ \mathrm{Spf}(\mathcal{O}_K) & \xrightarrow{\rho_{\mathrm{dR}}} & \mathcal{O}_K^\Delta & \xleftarrow{\rho_{\mathrm{crys}}} & \mathrm{Spf}(W), \end{array}$$

For a perfect complex $\mathcal{E}$ on X^Δ , i.e. a *perfect complex of prismatic crystals*, we have:

$$\rho_{\mathrm{dR}}^* f_*^\Delta \mathcal{E} \simeq f_*^{\mathrm{dR}} \mathcal{E}^{\mathrm{dR}} =: \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{E}^{\mathrm{dR}}), \quad \rho_{\mathrm{crys}}^* f_*^\Delta \mathcal{E} \simeq f_*^{\mathrm{crys}} \mathcal{E}_k^{\mathrm{crys}} =: \mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{E}_k^{\mathrm{crys}}),$$

where we are suppressing derived notation and, by definition, $\mathcal{E}^{\mathrm{dR}} = \rho_{\mathrm{dR}, X}^* \mathcal{E}$ and $\mathcal{E}_k^{\mathrm{crys}} = \rho_{\mathrm{crys}, X}^* \mathcal{E}$.¹

Unlike $\mathrm{Spf}(\mathcal{O}_K)$, the stack $\mathcal{O}_K^\Delta$ admits a *Frobenius* $F_{\mathcal{O}_K}$, and so we can form the twisted maps,

$$\rho_{\mathrm{dR}}^{(n)} := F_{\mathcal{O}_K}^n \circ \rho_{\mathrm{dR}}, \quad \rho_{\mathrm{crys}}^{(n)} := F_{\mathcal{O}_K}^n \circ \rho_{\mathrm{crys}}.$$

Then, we define the n -twisted de Rham and crystalline cohomology complexes,

$$\mathrm{R}\Gamma_{\mathrm{dR}}^{(n)}(\mathcal{E}) := (\rho_{\mathrm{dR}}^{(n)})^* f_*^\Delta \mathcal{E}, \quad \mathrm{R}\Gamma_{\mathrm{crys}}^{(n)}(\mathcal{E}) := (\rho_{\mathrm{crys}}^{(n)})^* f_*^\Delta \mathcal{E} \simeq \phi^{*n} \mathrm{R}\Gamma((X_k/W)_{\mathrm{crys}}, \mathcal{E}_k^{\mathrm{crys}}),$$

and the n -twisted de Rham and crystalline cohomology groups are their cohomology groups. Subsequently, we view the following as a prismatic incarnation of Dwork's trick.

Theorem B (see Theorem 3.12). *For any $n \geq a$, the following compositions,*

$$\mathrm{Spf}(\mathcal{O}_K) \xrightarrow{\rho_{\mathrm{dR}}^{(n)}} \mathcal{O}_K^\Delta \xleftarrow{\rho_{\mathrm{crys}}^{(n)}} \mathrm{Spf}(W) \xleftarrow{\mathrm{str.}} \mathrm{Spf}(\mathcal{O}_K),$$

(where the rightmost map is the natural one) are naturally identified in $\mathrm{Map}(\mathrm{Spf}(\mathcal{O}_K), \mathcal{O}_K^\Delta)$.

Remark 1. When $e = 1$, and thus $a = 0$, Theorem B is due to Bhatt–Lurie when $X = \mathrm{Spf}(\mathbb{Z}_p)$ (see [BL22a, Proposition 3.6.6]), and in general due to Imai–Kato–Yousis (see [IKY25, Theorem 3.19]). So, we really do view the main import of Theorem B as an analogue of Dwork's trick.

From Theorem B, the first part of Theorem A easily follows. For the second part, the main observation is that a Frobenius structure φ on $\mathcal{E}$ induces one on $Rf_*^\Delta \mathcal{E}$, and that pulling it back along $\rho_{\mathrm{dR}}^{(n)}$ and $\rho_{\mathrm{crys}}^{(n)}$, induces isogenies labelled $\overline{\varphi}_{\mathrm{dR}}^{(n)}$ and $\overline{\varphi}_{\mathrm{crys}}^{(n)}$, as in loc. cit., respectively.

¹If $\mathcal{E}$ is a vector bundle, then $\mathcal{E}^{\mathrm{dR}}$ is equivalent to a vector bundle with integrable connection on X , and $\mathcal{E}_k^{\mathrm{crys}}$ to a crystal on $(X_k/W)_{\mathrm{crys}}$. Moreover, their cohomologies recover the usual de Rham and crystalline cohomologies.

A Breuil–Kisin reinterpretation. One can make the analogy with Dwork’s trick even stronger, and in doing so connect our work to previous incarnations of Dwork’s trick appearing in integral p -adic Hodge theory, e.g. those appearing in [HK94, Section 5], [Tsu99, Section 4.4], [Fal99, Section 2], [Bre97, Section 6] and [Kis06, Section 1.2].

Namely, there exists a Frobenius-equivariant flat *Breuil–Kisin* covering $\rho_{\mathfrak{S}}: \mathrm{Spf}(\mathfrak{S}) \rightarrow \mathcal{O}_K^{\Delta}$ where $\mathfrak{S} := W[[u]]$ is equipped with the Witt vector Frobenius on W and $\phi(u) = u^p$. We think of $\mathrm{Spf}(\mathfrak{S})$ as a closed-times-open bi-disk, where $\mathrm{Spf}(W)$ has radius $p^{-1/p}$. Let E denote the minimal polynomial for π relative to W . Then, as E has degree e we indeed see that the closed embedding,

$$\mathrm{Spf}(\mathcal{O}_K) \xhookrightarrow{(E=0)} \mathrm{Spf}(\mathfrak{S}).$$

does act as a closed subdisk of radius $p^{-1/e}$. In the prismatic world, the role of an appropriately-sized closed subdisk (i.e. where we expect convergence) is played by a morphism,

$$\rho_{\tilde{S}}: \mathrm{Spf}(\tilde{S}) \rightarrow \mathcal{O}_K^{\Delta}, \quad \text{for } \tilde{S} := \mathfrak{S}[\{ \frac{u^{m\tilde{e}}}{m!} \}_{m \geq 1}]_p^{\wedge},$$

and where $\tilde{e} = \lceil \frac{e}{p-1} \rceil$ is the ratio of the convergent radius and the radius of $\mathrm{Spf}(\mathcal{O}_K)$.²

We then have the following diagram, which conveys the full strength of Dwork’s trick:

$$\begin{array}{ccccc} & & \text{str.} & & \\ & \curvearrowright & & \curvearrowleft & \\ \mathrm{Spf}(\mathcal{O}_K) & \xrightarrow{\rho_{\mathrm{dR}}^{(n)}} & \mathcal{O}_K^{\Delta} & \xleftarrow{\rho_{\mathrm{crys}}^{(n)}} & \mathrm{Spf}(W) \\ \downarrow i & \nearrow \rho_{\tilde{S}}^{(n)} & \uparrow \rho_{\mathfrak{S}}^{(n+1)} & \nwarrow \rho_{\tilde{S}}^{(n)} & \downarrow (u=0) \\ & \mathrm{Spf}(\tilde{S}) & \xrightarrow{\quad} & \mathrm{Spf}(\mathfrak{S}) & \xleftarrow{\quad} \mathrm{Spf}(\tilde{S}), \end{array} \quad (5)$$

where i corresponds to the map $\tilde{S} \twoheadrightarrow \mathcal{O}_K$ sending u to π , the lower horizontal arrows correspond to the natural inclusion $\mathfrak{S} \hookrightarrow \tilde{S}$, and s comes from the natural map $W \rightarrow \tilde{S}$, which induces a map in the absolute prismatic site $(\mathcal{O}_K)_{\Delta}$.

A more refined version of Dwork’s trick in the prismatic setting, is the following.

Theorem C (see Proposition 3.24). *For any uniformiser π and any integer $n \geq a$, the diagram (5) is naturally 2-commutative.*

To explain the implications of Theorem (C), fix a perfect complex $\mathcal{E}$ on X^{Δ} and $n \geq a$. For notational simplicity, we further write $\mathfrak{M}^{(n+1)} := (\rho_{\mathfrak{S}}^{(n+1)})^* f_*^{\Delta} \mathcal{E}$ and $\tilde{M}^{(n)} := (\rho_{\tilde{S}}^{(n)})^* f_*^{\Delta} \mathcal{E}$. The commutativity of the left square of (5) implies that we have isomorphisms,

$$(\rho_{\mathrm{dR}}^{(n)})^* f_*^{\Delta} \mathcal{E} \simeq \mathfrak{M}^{(n+1)}/E \simeq i^* \tilde{M}^{(n)}.$$

The second identification here is the analogue of the equality $\phi^{*n}(\mathcal{F}|_{\mathbb{B}(r)}) = \phi^{*n}(\mathcal{F}|_{\mathbb{B}(r^p)})$ from the classical version of Dwork’s trick. The commutativity of the right square gives us identifications,

$$(\rho_{\mathrm{crys}}^{(n)})^* f_*^{\Delta} \mathcal{E} \simeq \mathfrak{M}^{(n+1)}/u \simeq \tilde{M}^{(n)}/u.$$

On the other hand, the commutativity of the right square with the *section* s further gives us that,

$$\tilde{M}^{(n)}/u \simeq s^*(\rho_{\mathrm{crys}}^{(n)})^* f_*^{\Delta} \mathcal{E}. \quad (6)$$

The map $\rho_{\mathrm{crys}}^{(n)}$ is obtained by applying $(-)^{\Delta}$ to the map $\mathrm{Spec}(k) \xrightarrow{\pi=0} \mathrm{Spf}(\mathcal{O}_K)$ precomposed with the $(n+1)^{\mathrm{th}}$ power of the Frobenius. Thus, we view (6) precisely as the expected *constancy* after restricting to the smaller subdisk $\mathrm{Spf}(\tilde{S})$, as suggested by Dwork’s trick.

Thus, we can ultimately give a refinement of Theorem A. Namely, there is:

- (a) a canonical isomorphism of perfect complexes over $\mathcal{O}_K$:

$$\mathfrak{M}^{(n+1)}/E \simeq \mathfrak{M}^{(n+1)}/u \otimes_W \mathcal{O}_K, \quad (7)$$

²The prism $(\tilde{S}, (p))$ is a modification (better suited to the ramified situation) of the *Breuil prism*.

- (b) an identification of the source of (7) with $(\rho_{\text{dR}}^{(n)})^* f_*^\Delta \mathcal{E}$,
- (c) the ‘constancy’ of $\mathfrak{M}^{(n+1)}/u$, with constant value $(\rho_{\text{crys}}^{(n)})^* f_*^\Delta \mathcal{E}$.

Implications for crystalline and de Rham torsion. The isomorphism (3) tells us that the (free) ranks of de Rham and crystalline cohomology must always agree. A much more subtle question is to determine the relationship between their respective *torsion submodules*.

For clarity, let us write

$$H_{\text{crys}}^i(X_k/W)[p^\infty] \simeq \bigoplus_{i=1}^r W/p^{a_i}, \quad H_{\text{dR}}^i(X/\mathcal{O}_K)[p^\infty] \simeq \bigoplus_{j=1}^s \mathcal{O}_K/\pi^{b_j}.$$

The equality $r = s$ holds (e.g. see [ČK19, Remark 7.8]), but relating $\ell_{\text{crys}}^i = \sum_{i=1}^r a_i$ and $\ell_{\text{dR}}^i = \sum_{j=1}^s b_j$ seems difficult. To wit, [ČK19, Question 7.13]) asks if $\ell_{\text{dR}}^i \neq e \cdot \ell_{\text{crys}}^i$ can occur. We discuss such examples below (see Examples 4.21 and 4.32) giving an affirmative answer to this question, and we expect that equality does not hold ‘generically’ for large i and e .

But, in fact, Theorem A and the isomorphism (7) gives us tools to study the precise relationship.

Conjecture α (see Conjecture 4.19). *The inequality $\ell_{\text{crys}}^i \leq \ell_{\text{dR}}^i \leq e \cdot \ell_{\text{crys}}^i$ always holds.*

Conjecture α seems difficult even in the simplest non-trivial cases. But, in Section 4.4 we explain how Dwork’s trick, specifically the isomorphism (7), allows us to transform Conjecture α into questions about u^∞ -torsion in prismatic cohomology, which possesses finer structures given the extensive study in recent years (e.g. [LL23, GL25]).

In Appendix A, we give evidence for these conjectural properties of the u^∞ -torsion in Breuil–Kisin cohomology, by verifying that our main such conjecture (see Conjecture 4.26) holds in a small number of non-trivial cases (see Theorem A.20).

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Global notation and conventions

Throughout this article we fix the following notation.

- ◇ p is a prime,
- ◇ k is a perfect field of characteristic p ,
- ◇ F_X (resp. F_R) is the absolute Frobenius on an $\mathbb{F}_p$ -scheme X (resp. $\mathbb{F}_p$ -algebra R),
- ◇ for a ring R and R -module M we write its length as $\ell_R(M)$,
- ◇ $W := W(k)$ the ring of Witt vectors,
- ◇ $K_0 = \text{Frac}(W)$,
- ◇ K a finite totally ramified extension of K_0 ,
- ◇ $\mathcal{O}_K$ the ring of integers of K ,
- ◇ π a uniformiser of K ,
- ◇ $E_\pi = E := \min.\text{poly}_{\mathcal{O}_K/W}(\pi) \in W[u]$,
- ◇ $e = [K : K_0]$,
- ◇ $\tilde{e} = \lceil \frac{e}{p-1} \rceil$,
- ◇ $a = \lceil \log_p \left(\frac{e}{p-1} \right) \rceil$.
- ◇ $\bar{K}$ is an algebraic closure of K and $C = \widehat{\bar{K}}$,
- ◇ $A_{\text{inf}} = A_{\text{inf}}(\mathcal{O}_C)$,
- ◇ $A_{\text{crys}} = A_{\text{crys}}(\mathcal{O}_C)$,
- ◇ $\pi^b := (\pi, \pi^{1/p}, \dots)$,
- ◇ $\theta : A_{\text{inf}} \rightarrow \mathcal{O}_C$ is Fontaine’s map,
- ◇ $\theta^{(-1)} := \theta \circ \phi^{-1}$,
- ◇ $\xi = p - p^b$ and $\xi^{(1)} = \phi(\xi)$.

We also use the phrase *formal stack over $\mathbb{Z}_p$* to mean a stack on the category of p -nilpotent rings, equipped with the fpqc topology. For a stack S on all rings with the fpqc topology, we denote by $\hat{S}$ its p -adic completion, which means its restriction to the full subcategory of p -nilpotent rings. A morphism of formal stacks $f : X \rightarrow Y$ over $\mathbb{Z}_p$ is called a *flat surjection* if for every p -nilpotent ring R and object y of $Y(R)$, there exists a faithfully flat map $R \rightarrow R'$ and an object x' of $X(R')$ such that $f(x') \simeq y|_{R'}$ is in $Y(R')$.

2. PRELIMINARIES ON PRISMATIC THEORY

In this section, we give a minimal recollection on parts of the prismatic theory we need in the sequel. We do this mainly as a means to set notation and to refer the reader to more in-depth discussions (e.g. [Bha23, BL22a, BL22b, BS22, BS23, Dri24]) for details.

In addition to [global notation and conventions](#), we use the following notation in this section.

Notation 2.1. Let us fix the following notation:

- X is a quasi-syntomic p -adic formal scheme,
- $\mathbb{W}$ is the p -typical Witt vector group scheme,
- $\delta: \mathbb{W} \rightarrow \mathbb{W}$ the usual δ -structure,
- $F: \mathbb{W} \rightarrow \mathbb{W}$ is the usual Frobenius lift,
- $V: \mathbb{W} \rightarrow \mathbb{W}$ is the usual Verschiebung map,
- $[-]: \widehat{\mathbb{A}}^1 \rightarrow \mathbb{W}$ the Teichmüller lift,
- $\gamma_0: \mathbb{W} \rightarrow \mathbb{G}_a$ the zeroth component map.

Finally, we shall freely use the notion of quasi-ideals $d: I \rightarrow A$ (which we sometimes write as $[I \rightarrow A]$ for clarity) and their quotients $\text{Cone}(d)$ (where we occasionally write A/I) as in [Dri21].

2.1. Prismatisation and prismatic (F -)crystals. We begin by recalling the prismatisation of X , a formal stack over $\mathbb{Z}_p$, due to Drinfeld and Bhatt–Lurie.

Definition 2.2 (Prismatisation, [Bha23, Definition 5.1.6]). Let us recall the following:

- (1) For a p -nilpotent ring R a *Cartier–Witt divisor over R* is a map of invertible $\mathbb{W}(R)$ -modules $\alpha: I \rightarrow \mathbb{W}(R)$ such that $\gamma_0(\alpha(I))$ is nilpotent and $\delta(\alpha(I))$ generates $\mathbb{W}(R)$.
- (2) The *prismatisation* X^Δ of X is the formal stack over $\mathbb{Z}_p$ associating to any p -nilpotent ring R the groupoid of pairs $(\alpha: I \rightarrow \mathbb{W}(R), s)$ where α is a Cartier–Witt divisor over R , and $s: \text{Spec}(\mathbb{W}(R)/I) \rightarrow X$ is a morphism of (derived) schemes.³

If $X = \text{Spf}(R)$, then we write R^Δ instead of X^Δ . We write $\mathcal{O}^\Delta$ for the structure sheaf of X^Δ .

The formal stack X^Δ over $\mathbb{Z}_p$ carries a Frobenius lift denoted F_X (or F_R when $X = \text{Spf}(R)$).⁴ For a p -nilpotent ring R , the map $F_X: X^\Delta(R) \rightarrow X^\Delta(R)$ associates to the pair $(\alpha: I \rightarrow \mathbb{W}(R), s)$ the pair $(F^*(\alpha): F^*(I) \rightarrow \mathbb{W}(R), F^*(s))$, where F is the Frobenius on $\mathbb{W}$. Here $F^*(s)$ denotes the composition $\text{Spec}(\mathbb{W}(R)/F^*(I)) \xrightarrow{F} \text{Spec}(\mathbb{W}(R)/I) \xrightarrow{s} X$.

By a *prismatic crystal* on X we mean a quasi-coherent object of the derived ∞ -category,

$$\mathbf{D}(X^\Delta) := \lim_{\text{Spec}(R) \rightarrow X^\Delta} \mathbf{D}(R),$$

where $\text{Spec}(R) \rightarrow X^\Delta$ travels over morphisms to X^Δ from the spectra of p -nilpotent rings. By a *perfect prismatic crystal* we mean an object of the full (∞) -subcategory,

$$\mathbf{Perf}(X^\Delta) := \lim_{\text{Spec}(R) \rightarrow X^\Delta} \mathbf{Perf}(R).$$

We define the category $\mathbf{Vect}(X^\Delta)$ of *prismatic crystals in vector bundles* analogously.

We frequently use another more down-to-earth interpretation of prismatic crystals using objects (A, I) of the absolute prismatic site X_Δ . We let ϕ_A denote the Frobenius lift on a prism (A, I) .

Definition 2.3 (Absolute prismatic site, [BS23]). The *absolute prismatic site* X_Δ of X is opposite to the category⁵ of bounded prisms (A, I) equipped with a morphism $s: \text{Spf}(A/I) \rightarrow X$ and endowed with the flat topology.⁶ If $X = \text{Spf}(R)$, we write R_Δ instead of X_Δ . We denote by $\mathcal{O}_\Delta$ the sheaf associating A to (A, I) , and let ϕ_Δ denote the endomorphism restricting to ϕ_A on each (A, I) .

³Technically, $\mathbb{W}(R)/I = \text{cone}(\alpha)$ need not be a discrete ring and, in general, it is a 1-truncated animated ring. See [Dri24, Paragraph 1.3.5] for an elementary description of the groupoid of maps $\text{Spec}(\mathbb{W}(R)/I) \rightarrow X$.

⁴Technically, this notation is overloaded if X is an $\mathbb{F}_p$ -scheme, as it could also mean the absolute Frobenius there. But, we hope the meaning is clear to the reader from the context.

⁵As is standard, we will conflate X_Δ and its opposite category, often writing morphisms in X_Δ as if they were in X_Δ^{op} . Additionally, we will almost always omit the structure map s from the notation, just writing (A, I) .

⁶More precisely, the topology generated by those collection of morphisms $\{(A, I) \rightarrow (A_m, I_m)\}_{m \in M}$ such that either $\{\text{Spf}(A_m) \rightarrow \text{Spf}(A)\}$ is a Zariski cover or $M = \{1\}$ and $A \rightarrow A_1$ is (p, I) -adically faithfully flat.

For any object (A, I) of X_Δ , note that one can construct a Frobenius-equivariant morphism $\rho_{(A, I)}: \mathrm{Spf}(A) \rightarrow X^\Delta$ as follows: Consider a morphism $g: \mathrm{Spec}(R) \rightarrow \mathrm{Spf}(A)$, where R is a p -nilpotent ring. As A is a δ -ring, therefore, the universal property of Witt vectors gives us a morphism $A \rightarrow \mathbb{W}(R)$. Then, the Cartier–Witt divisor over R corresponding to the composition $\rho_{(A, I)} \circ g$ is the map $\alpha: I \otimes_A \mathbb{W}(R) \rightarrow \mathbb{W}(R)$. Moreover, as $\mathrm{cone}(\alpha)$ annihilates A , so we get an induced map,

$$\mathrm{Spec}(\mathrm{cone}(\alpha)) \xrightarrow{g} \mathrm{Spf}(A/I) \xrightarrow{s} X.$$

Thus, we have produced an element $\rho_{(A, I)} \circ g$ in $X^\Delta(R)$ as desired. This map is functorial in the sense that if $(A, I) \rightarrow (B, J)$ is a map in X_Δ , then the composition $\mathrm{Spf}(B) \rightarrow \mathrm{Spf}(A) \xrightarrow{\rho_{(A, I)}} X^\Delta$ is naturally identified with the map $\rho_{(B, J)}$.

Example 2.4. Suppose that R is a quasi-regular semi-perfectoid algebra (e.g. perfectoid) in the sense of [BMS19, Definition 4.20]. Then, from [BS22, Section 7.1] there exists an initial object (Δ_R, I_R) of R_Δ (e.g. if R is perfectoid, then $\Delta_R = A_{\mathrm{inf}}(R)$). Additionally, the natural map $\rho_{(\Delta_R, I_R)}: \mathrm{Spf}(\Delta_R) \rightarrow R^\Delta$ is an isomorphism (see [BL22a, Lemma 6.1]).

Proposition 2.5 ([BL22b, Theorem 6.5]). *The morphisms $\rho_{(A, I)}$ induce equivalences*

$$\mathbf{D}(X^\Delta) \simeq \mathbf{D}(X_\Delta) \simeq \lim_{(A, I) \in X_\Delta} \mathbf{D}(A),$$

which restrict to equivalences on the full subcategories of perfect complexes and vector bundles.

There is a natural map $u: X^\Delta \rightarrow \mathbf{Cart} := [\widehat{\mathbb{A}^1}/\mathbb{G}_m]$ associating to a pair $(\alpha: I \rightarrow \mathbb{W}(R), s)$ the generalised Cartier divisor $I \otimes_{\mathbb{W}(R), \gamma_0} R \rightarrow R$, using the interpretation of the target as in [BL22a, Section 3.1]. We let $\mathcal{J}^\Delta \subseteq \mathcal{O}^\Delta$ denote the pullback under u of the tautological invertible ideal sheaf on $\mathbf{Cart}$. Under the equivalence in Proposition 2.5, the pair $\mathcal{J}^\Delta \subseteq \mathcal{O}^\Delta$ and endomorphism F_X of $\mathcal{O}^\Delta$ correspond to the pair $\mathcal{J}_\Delta \subseteq \mathcal{O}_\Delta$ and endomorphism ϕ_Δ of $\mathcal{O}_\Delta$, where $\mathcal{J}_\Delta(A, I) := I$.

Definition 2.6. A *perfect prismatic F -crystal* on X^Δ (resp. X_Δ) is a perfect complex $\mathcal{E}$ on X^Δ (resp. X_Δ) and an isomorphism $\varphi_\mathcal{E}: F_X^*(\mathcal{E})[1/\mathcal{J}^\Delta] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{J}^\Delta]$ of $\mathcal{O}^\Delta[1/\mathcal{J}^\Delta]$ -modules (resp. an isomorphism $\varphi_\mathcal{E}: \phi_\Delta^*(\mathcal{E})[1/\mathcal{J}_\Delta] \rightarrow \mathcal{E}[1/\mathcal{J}_\Delta]$ of $\mathcal{O}_\Delta[1/\mathcal{J}_\Delta]$ -modules),⁷ called a *Frobenius structure*. Denote the category of perfect prismatic F -crystals on X^Δ (resp. X_Δ) by $\mathbf{Perf}^\varphi(X^\Delta)$ (resp. $\mathbf{Perf}^\varphi(X_\Delta)$).

From Proposition 2.5, it easily follows that there are natural identifications,

$$\mathbf{Perf}^\varphi(X^\Delta) \simeq \mathbf{Perf}^\varphi(X_\Delta) \simeq \lim_{(A, I) \in X_\Delta} \mathbf{Perf}^\varphi(A, I), \quad (2.1)$$

where $\mathbf{Perf}^\varphi(A, I)$ denotes the category of perfect complexes of A -modules M equipped with an isomorphism of A -modules $\varphi_M: (\phi_A^* M)[1/I] \xrightarrow{\sim} M[1/I]$. Due to these equivalences, we shall often implicitly identify the categories in (2.1).

2.2. Absolute pushforwards and relative prismatic cohomology. Let $f: X \rightarrow Y$ be a morphism of quasi-syntomic p -adic formal schemes, and note that we have the following:

- (1) A morphism of formal stacks over $\mathbb{Z}_p$,

$$f^\Delta: X^\Delta \rightarrow Y^\Delta, \quad (\alpha: I \rightarrow \mathbb{W}(R), s) \mapsto (\alpha: I \rightarrow \mathbb{W}(R), f \circ s).$$

- (2) A cocontinuous morphism of ringed sites,

$$f_\Delta: X_\Delta \rightarrow Y_\Delta, \quad ((A, I), s) \mapsto ((A, I), f \circ s).$$

⁷Observe that as $\mathcal{J}^\Delta \subseteq \mathcal{O}^\Delta$ is a Cartier divisor, therefore, for any morphism $\mathrm{Spec}(R) \rightarrow X^\Delta$, the open subscheme $\mathrm{Spec}(R) - V(\mathcal{J}^\Delta(R)) \subseteq \mathrm{Spec}(R)$ is affine (see [SP, Tag 07ZT]) and we denote by $\mathcal{O}^\Delta[1/\mathcal{J}^\Delta](R)$ its global sections. The sheaf $\mathcal{O}_\Delta[1/\mathcal{J}_\Delta]$ admits a similar definition.

The preceding morphisms thus naturally induce morphisms of ∞ -categories,

$$Rf_*^\Delta: \mathbf{D}(X^\Delta) \rightarrow \mathbf{D}(Y^\Delta), \quad \text{and} \quad R(f_\Delta)_*: \mathbf{D}(X_\Delta, \mathcal{O}_\Delta) \rightarrow \mathbf{D}(Y_\Delta, \mathcal{O}_\Delta),$$

These two pushforwards match under the equivalence in Proposition 2.5, and so we shall often implicitly identify them.

Let (A, I) be an object of Y_Δ and $\mathcal{E}$ an object of $\mathbf{D}(X^\Delta)$. Then, we define

$$R\Gamma(X/(A, I), \mathcal{E}) := L\rho_{(A, I)}^* Rf_*^\Delta \mathcal{E}.$$

When there is no chance for confusion, we shall shorten this notation to $R\Gamma_A(\mathcal{E})$ and denote its cohomology groups by $H_A^i(\mathcal{E})$. These cohomology groups may be computed site-theoretically as the cohomology of (the pullback of) $\mathcal{E}$ on the relative prismatic site $(X/(A, I))_\Delta$ as in [BS22].

This interpretation in terms of relative prismatic cohomology allows one to produce from a Frobenius structure $\varphi_\mathcal{E}$ on $\mathcal{E}$ a morphism $F_Y^*(Rf_*^\Delta \mathcal{E})[1/\mathfrak{p}^\Delta] \rightarrow (Rf_*^\Delta \mathcal{E})[1/\mathfrak{p}^\Delta]$ of $\mathcal{O}^\Delta[1/\mathfrak{p}^\Delta]$ -modules (e.g. see [GR24, Construction 7.6]). We abuse notation and denote this map by $Rf_*^\Delta \varphi_\mathcal{E}$.

Theorem 2.7 ([GR24, Corollary 5.16 and Theorem 8.1]). *Let $f: X \rightarrow Y$ be a smooth proper morphism of smooth formal $\mathcal{O}_K$ -schemes. Then, $Rf_*^\Delta \mathcal{E}$ is a perfect prismatic crystal on Y^Δ for any perfect prismatic crystal $\mathcal{E}$ on X . Moreover, $\mathcal{E}$ is equipped with a Frobenius structure $\varphi_\mathcal{E}$, then $Rf_*^\Delta \varphi_\mathcal{E}$ is a Frobenius structure on $Rf_*^\Delta \mathcal{E}$.*

2.3. The prismatisation of $\mathcal{O}_K$. We now record some points of $\mathcal{O}_K^\Delta$ that play a central role in our proof of the twisted crystalline-de Rham comparison (see Theorem B).

2.3.1. The (n -twisted) Breuil–Kisin prism. Let $\mathfrak{S} := W[[u]]$ and recall that $E = E_\pi$ in $\mathfrak{S}$ is the Eisenstein polynomial of a uniformiser π of $\mathcal{O}_K$. We define the *Breuil–Kisin prism* to be the pair $(\mathfrak{S}, E)$, where $\mathfrak{S}$ is equipped with the Frobenius lift

$$\phi_\mathfrak{S}: \mathfrak{S} \rightarrow \mathfrak{S}, \quad \sum_{k \in \mathbb{N}} a_k u^k \mapsto \sum_{k \in \mathbb{N}} \phi_W(a_k) u^{pk}.$$

Observe that the map sending u to π induces an isomorphism $\mathfrak{S}/E \xrightarrow{\sim} \mathcal{O}_K$, and we denote the *inverse* of this map as $\text{nat}: \mathcal{O}_K \xrightarrow{\sim} \mathfrak{S}/E$. Then, we may view the pair $(\mathfrak{S}, E)$ as an object of $(\mathcal{O}_K)_\Delta$ with the map $\text{Spf}(\mathfrak{S}/E) \rightarrow \text{Spf}(\mathcal{O}_K)$ induced by nat . Additionally, we shorten the notation $\rho_{(\mathfrak{S}, E)}$ to $\rho_\mathfrak{S}$ or ρ_π .

Proposition 2.8. *The map $\rho_\mathfrak{S}: \text{Spf}(\mathfrak{S}) \rightarrow \mathcal{O}_K^\Delta$ is a flat surjection.*

Proof. Note that $\phi_\mathfrak{S}: \text{Spf}(\mathfrak{S}) \rightarrow \text{Spf}(\mathfrak{S})$ is a flat surjection. So, it is enough to show that $\rho_\mathfrak{S} \circ \phi_\mathfrak{S}$ is a flat surjection. But, from the commutativity of triangle (2) in diagram (2.8) of Proposition 2.13, it is then sufficient to show that the composition $\text{Spf}(A_{\text{inf}}) \rightarrow \mathcal{O}_K^\Delta$ (see Section 2.3.5 for this map), is a flat surjection. By using the identification in Example 2.4, we are further reduced to showing that the map $\mathcal{O}_C^\Delta \rightarrow \mathcal{O}_K^\Delta$ is a flat surjection. The latter follows from [BL22b, Lemma 6.3], as $\mathcal{O}_K \rightarrow \mathcal{O}_C$ is quasi-syntomic (e.g. see [IKY24, Lemma 1.15]). $\square$

For any $n \geq 0$, set $E^{(n)} := E_\pi^{(n)} := \phi_\mathfrak{S}^n(E_\pi)$. Then, with $\mathfrak{S}$ still equipped with the Frobenius lift $\phi_\mathfrak{S}$, the pair $(\mathfrak{S}, E^{(n)})$ is a prism called the (n -twisted) *Breuil–Kisin prism*, which coincides with the usual Breuil–Kisin prism when $n = 0$. The n -fold self-composition of the Frobenius $\phi_\mathfrak{S}^n$ induces a map,

$$\overline{\phi}_\mathfrak{S}^n: \mathfrak{S}/E \rightarrow \mathfrak{S}/E^{(n)},$$

and composing this with nat endows $(\mathfrak{S}_R, (E^{(n)}))$ with the structure of an object of $(\mathcal{O}_K)_\Delta$. Additionally, we shorten $\rho_{(\mathfrak{S}, E^{(n)})}$ to $\rho_\mathfrak{S}^{(n)}$ or $\rho_\pi^{(n)}$, noting the change in index.

It is clear that $\phi_\mathfrak{S}^n: \mathfrak{S} \rightarrow \mathfrak{S}$ is a flat surjection which induces a morphism $(\mathfrak{S}, (E)) \rightarrow (\mathfrak{S}, E^{(n)})$ in $(\mathcal{O}_K)_\Delta$. Using this and the fact that $\rho_\mathfrak{S}$ is Frobenius-equivariant, we have the following identification in $\text{Map}(\text{Spf}(\mathfrak{S}), \mathcal{O}_K^\Delta)$:

$$\rho_\mathfrak{S}^{(n)} \simeq F_{\mathcal{O}_K}^n \circ \rho_\mathfrak{S}. \quad (2.2)$$

2.3.2. The product of Breuil–Kisin prisms. Breuil–Kisin prisms depend on the choice of $E = E_\pi$, and thus on the choice of the uniformiser π . To show the independence of our later constructions of this choice, in this section, we study the product of Breuil–Kisin prisms for different choices of the uniformiser.

Let us fix another uniformiser π' of $\mathcal{O}_K$. Let $\mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}$ be shorthand for the $(p, E_\pi, E_{\pi'})$ -adic completion of $\mathfrak{S} \otimes_{\mathbb{Z}_p} \mathfrak{S}$, and set

$$J := \ker(\mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}) \rightarrow \mathcal{O}_K,$$

where the surjective map denotes the composition of the multiplication map with nat . We define the *mixed Breuil–Kisin prism* to be

$$\mathfrak{S}_{\pi, \pi'} = (\mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}) \{ \frac{J}{E_\pi} \}_\delta^\wedge,$$

where the right-hand side denotes the (p, E_π) -adic completion of the δ -algebra obtained by freely adjoining to $(\mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S})$ the elements $\frac{j}{E_\pi}$, for j in J .⁸ Let us set $I_{\pi, \pi'} \subseteq \mathfrak{S}_{\pi, \pi'}$ to be the ideal generated by E_π , or equivalently, by $E_{\pi'}$.

Let us observe that there are natural maps of prisms $\mathfrak{S} \xrightarrow{p_1} \mathfrak{S}_{\pi, \pi'} \xleftarrow{p_2} \mathfrak{S}$ coming from projection and, by our setup, the following two compositions coincide:

$$\mathcal{O}_K \xrightarrow{\text{nat}} \mathfrak{S}_\pi / E_\pi \xrightarrow{p_1} \mathfrak{S}_{\pi, \pi'} / I_{\pi, \pi'} \xleftarrow{p_2} \mathfrak{S} / E_{\pi'} \xleftarrow{\text{nat}} \mathcal{O}_K.$$

Thus, $(\mathfrak{S}_{\pi, \pi'}, I_{\pi, \pi'})$ has an unambiguous structure as an element of $(\mathcal{O}_K)_\Delta$, and we use $\rho_{\pi, \pi'}$ to denote the corresponding map $\text{Spf}(\mathfrak{S}_{\pi, \pi'}) \rightarrow \mathcal{O}_K^\Delta$.

Proposition 2.9. *The object $(\mathfrak{S}_{\pi, \pi'}, I_{\pi, \pi'})$ is the product of $(\mathfrak{S}, E_\pi)$ and $(\mathfrak{S}, E_{\pi'})$ in $(\mathcal{O}_K)_\Delta$. Moreover, the following diagram*

$$\begin{array}{ccc} \mathfrak{S}_{\pi, \pi'} & \xrightarrow{p_1} & \mathfrak{S} \\ p_2 \downarrow & \searrow \rho_{\pi, \pi'} & \downarrow \rho_\pi \\ \mathfrak{S} & \xrightarrow{\rho_{\pi'}} & \mathcal{O}_K^\Delta, \end{array}$$

is 2-commutative, and its outer square is 2-cartesian.

Proof. The first claim follows from the same argument given in [DLMS24, Example 3.4]. For the second claim, consider morphisms $f, g: \mathfrak{S} \rightrightarrows R$ for a p -nilpotent ring R , such that the compositions $\rho_\pi \circ f$ and $\rho_{\pi'} \circ g$ are identified in $\text{Map}(\text{Spec}(R), \mathcal{O}_K^\Delta)$. This gives us an isomorphism of generalised Cartier–Witt divisors $E_\pi \mathbb{W}(R) \rightarrow \mathbb{W}(R)$ and $E_{\pi'} \mathbb{W}(R) \rightarrow \mathbb{W}(R)$, such that the induced maps $\mathcal{O}_K \rightarrow \mathfrak{S}/E_\pi \rightarrow \mathbb{W}(R)/E_\pi$ and $\mathcal{O}_K \rightarrow \mathfrak{S}/E_{\pi'} \rightarrow \mathbb{W}(R)/E_{\pi'}$ are identified. From the two maps $\mathfrak{S} \rightarrow \mathbb{W}(R)$ induced by f and g , we obtain a unique morphism $\mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S} \rightarrow \mathbb{W}(R)$. Our identification of Cartier–Witt divisors guarantees that this map uniquely upgrades to a morphism $\mathfrak{S}_{\pi, \pi'} \rightarrow \mathbb{W}(R)$. This map gives rise to a morphism $h: \mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S} \rightarrow R$ and it is easy to see that $p_1 \circ h = f$ and $p_2 \circ h = g$, as desired. $\square$

2.3.3. The modified Breuil prism. We continue with the notation from Section 2.3.1. Define

$$\begin{aligned} \tilde{S} &= \mathfrak{S} \{ \frac{\phi(u^{\tilde{e}})}{p} \}_\delta^\wedge = \mathfrak{S} [\{ \frac{u^{m\tilde{e}}}{m!} \}_{m \geq 1}]_p^\wedge \\ &= \{ \sum_{m \in \mathbb{N}} a_m \frac{u^m}{[m/\tilde{e}]!} \in K_0[[u]] : a_m \text{ converges } p\text{-adically to } 0 \}, \end{aligned}$$

endowed with the natural Frobenius lift

$$\phi_{\tilde{S}}: \tilde{S} \rightarrow \tilde{S}, \quad \sum_{m \in \mathbb{N}} a_m \frac{u^m}{[m/\tilde{e}]!} \mapsto \sum_{m \in \mathbb{N}} \phi_W(a_m) \frac{u^{pm}}{[m/\tilde{e}]!}.$$

The pair $(\tilde{S}, p)$ is a prism which we call the *modified Breuil prism*.

⁸The ostensible asymmetry in the definition of $\mathfrak{S}_{\pi, \pi'}$ is remedied by observing that it is naturally equal to $(\mathfrak{S} \widehat{\otimes}_{\mathbb{Z}_p} \mathfrak{S}) \{ \frac{J}{E_{\pi'}} \}_\delta^\wedge$ as $E_\pi/E_{\pi'}$ is a unit in $\mathfrak{S}_{\pi, \pi'}$ (using [BS22, Lemma 2.24] and the fact that $E_{\pi'} = 0 \pmod{E_\pi}$).

Remark 2.10. There is another more well-known prism closely related to $(\tilde{S}, p)$: the *Breuil prism* (S, p) (hence the name ‘modified Breuil prism’ for $(\tilde{S}, p)$). The ring S is defined in exactly the same manner as the ring $\tilde{S}$ but with each instance of $\tilde{e}$ replaced by e . There is a natural injective homomorphism $S_R \rightarrow \tilde{S}_R$ which is an isomorphism if and only if $e = 1$ or $p = 2$.

Observe that

$$E^{(1)} = E^p = u^{pe} = 0 \bmod p\tilde{S},$$

where the first equality follows because $\phi_{\mathfrak{S}}$ is a Frobenius lift, the second because E is an Eisenstein polynomial of degree e , and the final equality follows because

$$u^{pe} = pu^{p(e-\tilde{e})}(p-1)! \frac{u^{p\tilde{e}}}{p!} \in p\tilde{S}_R.$$

Thus, if $\mathfrak{S}/E^{(1)} \rightarrow \tilde{S}/p$ is the induced map, then we obtain a map $r: \mathcal{O}_K \rightarrow \tilde{S}/p$, which is defined to be the following composition:

$$\mathcal{O}_K \xrightarrow{\text{nat}} \mathfrak{S}/E \xrightarrow{\bar{\phi}_{\mathfrak{S}}} \mathfrak{S}/E^{(1)} \rightarrow \tilde{S}/p. \quad (2.3)$$

For $n \geq 0$, we may view $(\tilde{S}, p)$ as an object of $(\mathcal{O}_K)_{\Delta}$ with the $\mathcal{O}_K$ -structure map $F_{\tilde{S}/p}^n \circ r$. Denote the corresponding map $\text{Spf}(\tilde{S}) \rightarrow \mathcal{O}_K^{\Delta}$ by $\rho_{\tilde{S}}^{(n)}$ or $\tilde{\rho}_{\pi}^{(n)}$, dropping the superscript when $n = 0$. Then, by the same logic as for Equation (2.2), we have the following identification in $\text{Map}(\text{Spf}(\tilde{S}), \mathcal{O}_K^{\Delta})$:

$$\rho_{\tilde{S}}^{(n)} \simeq F_{\mathcal{O}_K}^n \circ \rho_{\tilde{S}}. \quad (2.4)$$

Finally, the following proposition follows by noting that $\frac{\pi^{m\tilde{e}}}{m!}$ belongs to $\mathcal{O}_K$ for all $m \geq 1$ (e.g. see [BO83, Lemma 3.9]).

Proposition 2.11. *There exists a unique arrow i making the following diagram commute:*

$$\begin{array}{ccc} W & \longrightarrow & \tilde{S} \longleftarrow \mathfrak{S} \\ & \searrow & \downarrow \scriptstyle (E=0) \\ & & \mathcal{O}_K \end{array}$$

i

where the unlabelled arrows are the natural maps. Moreover, i is a PD-thickening.

2.3.4. The prism (W, p) . The pair (W, p) , where W is equipped with the natural Witt vector Frobenius ϕ_W , is a prism. Via the natural map $q: \mathcal{O}_K \rightarrow k$, we may view (W, p) as an element of $(\mathcal{O}_K)_{\Delta}$, and we denote the corresponding map $\text{Spf}(W) \rightarrow \mathcal{O}_K^{\Delta}$ by ρ_W .

Observe that we may also view (W, p) naturally as an object of k_{Δ} , and thus we also obtain a map $\text{Spf}(W) \rightarrow k^{\Delta}$ and denote it as $\bar{\rho}_W$. The following proposition is clear by Example 2.4.

Proposition 2.12. *The map $\rho_W: \text{Spf}(W) \rightarrow \mathcal{O}_K^{\Delta}$ naturally factorises as*

$$\text{Spf}(W) \xrightarrow{\bar{\rho}_W} k^{\Delta} \rightarrow \mathcal{O}_K^{\Delta},$$

and the first map is an isomorphism.

Now, for any $n \geq 0$, we may also consider (W, p) as an object of $(\mathcal{O}_K)_{\Delta}$ using the $\mathcal{O}_K$ -structure map given by the composition

$$\mathcal{O}_K \xrightarrow{q} k \xrightarrow{F_k^n} k,$$

and denote the corresponding map $\text{Spf}(W) \rightarrow \mathcal{O}_K^{\Delta}$ by $\rho_W^{(n)}$. Then, similarly to Equation (2.2), we have the following identifications in $\text{Map}(\text{Spf}(W), \mathcal{O}_K^{\Delta})$ and $\text{Map}(\text{Spf}(W), k^{\Delta})$, respectively:

$$\rho_W^{(n)} \simeq F_{\mathcal{O}_K}^n \circ \rho_W, \quad \bar{\rho}_W^{(n)} \simeq F_k^n \circ \bar{\rho}_W. \quad (2.5)$$

2.3.5. The prism $(A_{\text{inf}}, \xi^{(1)})$. Endowed with the usual Frobenius lift ϕ , the pair $(A_{\text{inf}}, \xi^{(1)})$ is a prism. We may then consider the following composition:

$$\mathcal{O}_K \rightarrow \mathcal{O}_C \xrightarrow{\sim} A_{\text{inf}}/\xi^{(1)},$$

where the last isomorphism is induced by the map $\theta^{(-1)} = \theta \circ \phi^{-1}$. This endows $(A_{\text{inf}}, \xi^{(1)})$ with the structure of an object of $(\mathcal{O}_K)_{\Delta}$. We denote the corresponding map $\text{Spf}(A_{\text{inf}}) \rightarrow \mathcal{O}_K^{\Delta}$ by ρ_{inf} .

Moreover, for any $n \geq 0$, we may consider the pair $(A_{\text{inf}}, \xi^{(n+1)})$, where as per usual $\xi^{(n+1)} = \phi^n(\xi^{(1)}) = \phi^{n+1}(\xi)$. This is also a prism with the same Frobenius lift ϕ . We may endow it with the structure of an object of $(\mathcal{O}_K)_{\Delta}$ via the $\mathcal{O}_K$ -structure map

$$\mathcal{O}_K \rightarrow \mathcal{O}_C \xrightarrow{\sim} A_{\text{inf}}/\xi^{(1)} \xrightarrow{\phi^n} A_{\text{inf}}/\xi^{(n+1)}.$$

We denote the corresponding map $\text{Spf}(A_{\text{inf}}) \rightarrow \mathcal{O}_K^{\Delta}$ by $\rho_{\text{inf}}^{(n)}$. Then, by the same logic as for Equation (2.2), one has the following identification in $\text{Map}(\text{Spf}(A_{\text{inf}}), \mathcal{O}_K^{\Delta})$:

$$\rho_{\text{inf}}^{(n)} \simeq F_{\mathcal{O}_K}^n \circ \rho_{\text{inf}}. \quad (2.6)$$

2.3.6. The prism (A_{crys}, p) . Endowed with the usual Frobenius lift ϕ , the pair (A_{crys}, p) is a prism. We may then consider the composition,

$$\mathcal{O}_K \rightarrow \mathcal{O}_C \xrightarrow{\sim} A_{\text{inf}}/\xi^{(1)} \rightarrow A_{\text{crys}}/p,$$

where the last map makes sense as $\xi^{(1)} \in pA_{\text{crys}}$,⁹ and we endow (A_{crys}, p) with the structure of an object of $(\mathcal{O}_K)_{\Delta}$. We denote the corresponding map $\text{Spf}(A_{\text{crys}}) \rightarrow \mathcal{O}_K^{\Delta}$ by ρ_{crys} .

Moreover, for any $n \geq 0$, we may consider the pair (A_{crys}, p) as an object of $(\mathcal{O}_K)_{\Delta}$ with the $\mathcal{O}_K$ -structure map

$$\mathcal{O}_K \rightarrow \mathcal{O}_C \xrightarrow{\sim} A_{\text{inf}}/\xi^{(1)} \rightarrow A_{\text{crys}}/p \xrightarrow{\phi^n} A_{\text{crys}}/p.$$

We denote the corresponding map $\text{Spf}(A_{\text{crys}}) \rightarrow \mathcal{O}_K^{\Delta}$ by $\rho_{\text{crys}}^{(n)}$. Then, by the same logic as for (2.2), one has the following identification in $\text{Map}(\text{Spf}(A_{\text{crys}}), \mathcal{O}_K^{\Delta})$:

$$\rho_{\text{crys}}^{(n)} \simeq F_{\mathcal{O}_K}^n \circ \rho_{\text{crys}}. \quad (2.7)$$

2.3.7. Relationship between various prisms. In this section, our goal is to show the precise relationship between various prisms discussed above. We do this in the form of the following proposition, whose precise meaning and proof will be explained immediately afterwards.

Proposition 2.13. *Let $n \geq 0$ and consider the following diagram:*

$$\begin{array}{ccc} \text{Spf}(A_{\text{inf}}) & \xleftarrow{\quad} & \text{Spf}(A_{\text{crys}}) \\ \downarrow \alpha_{\text{inf}} & \searrow \rho_{\text{inf}}^{(n)} & \downarrow \alpha_{\text{crys}} \\ & \text{(2)} \quad \mathcal{O}_K^{\Delta} & \text{(3)} \quad \text{Spf}(W) \\ & \nearrow \rho_{\mathfrak{S}}^{(n+1)} & \nearrow \rho_W^{(n+1)} \\ \text{Spf}(\mathfrak{S}) & \xleftarrow{\quad} & \text{Spf}(\tilde{S}) \end{array} \quad (2.8)$$

(1) $\mathcal{O}_K^{\Delta} \xleftarrow{\rho_{\mathfrak{S}}^{(n)}} \text{Spf}(\tilde{S})$

Triangles (2), (3), (4), (5) and (6) commute for all $n \geq 0$, and triangle (1) commutes if and only if $n \geq a$.

In Proposition 2.13 the unlabelled morphisms are the obvious ones, and the remaining morphisms are explained as follows:

- The map $\alpha_{\text{inf}}: \mathfrak{S} \rightarrow A_{\text{inf}}$ is a homomorphism of W -algebras, where we set $\alpha_{\text{inf}}(u) = [\pi^b]$,

⁹Indeed, $\xi^p = p! \frac{\xi^p}{p!}$ belongs to pA_{crys} , and so $\xi^{(1)} = \xi^p = 0 \pmod{pA_{\text{crys}}}$.

- the map $\alpha_{\text{crys}}: \tilde{S} \rightarrow A_{\text{crys}}$ is the unique extension of α_{inf} as the divided powers $\frac{[\pi^b]^{m\tilde{e}}}{m!}$ belong to A_{crys} for all $m \geq 0$ (e.g. this follows from [BO83, Lemma 3.9]),¹⁰
- $s: \tilde{S} \rightarrow W$ is the identity on W and sends u to 0.

Proof of Proposition 2.13. The proof of this proposition is largely straightforward, so we only comment on the more non-obvious parts of the claim below. In particular, by definitions it is clear that triangles (3), (4) and (5) are commutative.

Let us first show that the triangle (1) in diagram (2.8) commutes if and only if $n \geq a$. This amounts to the claim that the following morphisms,

$$(\mathfrak{S}, E^{(n+1)}, \bar{\phi}_{\mathfrak{S}}^{n+1} \circ \text{nat}) \longrightarrow (\tilde{S}/p, F_{\tilde{S}/p}^n \circ r) \longleftarrow (W, p, F_k^{n+1} \circ q),$$

where the third entry of each term indicates the $\mathcal{O}_K$ -structure map, are morphisms in $(\mathcal{O}_K)_{\Delta}$ if and only if $n \geq a$. It is easy to see that the maps above are morphisms of prisms, and it remains to show that the $\mathcal{O}_K$ -structure agree. Diagrammatically, it is equivalent to the claim that the following square:

$$\begin{array}{ccccc} \mathfrak{S}/E^{(1)} & \longrightarrow & \tilde{S}/p & \xrightarrow{F_{\tilde{S}/p}^n} & \tilde{S}/p & \longleftarrow & k \\ \bar{\phi}_{\mathfrak{S}} \uparrow & & & & & & \uparrow F_k^{n+1} \\ \mathfrak{S}/E & \xleftarrow{\sim} & \mathcal{O}_K & \longrightarrow & \mathcal{O}_K/\pi & \xrightarrow{\sim} & k, \end{array}$$

where all the non-labelled arrows are the natural ones, commutes if and only if $n \geq a$. Writing $\mathcal{O}_K = W[\pi]$, it suffices to verify that starting from $\mathcal{O}_K$ and travelling around the diagram in both directions produces the same result for any x in W and π . It is easy to verify that the composition of the arrows in both directions sends x to $x^{p^{n+1}}$ in $\tilde{S}/p$. Moreover, the composition of the arrows along the right-hand side of the diagram sends π to 0, but the composition of the arrows along the left hand side sends π to $u^{p^{n+1}}$ in $\tilde{S}/p$. Thus, the diagram commutes if and only if p divides $u^{p^{n+1}}$ in $\tilde{S}$. This clearly happens if and only if $p^{n+1} \geq p\tilde{e}$, which is equivalent to having that $p^n \geq \frac{e}{p-1}$, or equivalently, $n \geq \lceil \log_p(\frac{e}{p-1}) \rceil = a$, as claimed.

Next, to show that the triangle (2) in diagram (2.8) is commutative, by definitions it is enough to establish that α_{inf} is a map in $(\mathcal{O}_K)_{\Delta}$. To get this claim, we must first show that α_{inf} maps the ideal $(\phi_{\mathfrak{S}}(E))$ into the ideal (ξ) . This amounts to showing that $\theta(\alpha_{\text{inf}}(E)) = 0$, but this precisely translates to having $E(\pi) = 0$. To show that the $\mathcal{O}_K$ -structure maps are compatible, it suffices to observe that

$$\mathcal{O}_K \xrightarrow{\sim} \mathfrak{S}/E \xrightarrow{\bar{\phi}_{\mathfrak{S}}} \mathfrak{S}/E^{(1)} \xrightarrow{\alpha_{\text{inf}}} A_{\text{inf}}/\xi^{(1)},$$

sends w in W to $\phi(w)$ in $A_{\text{inf}}/\xi^{(1)}$ and sends π to $[\pi^b]^p$. But, this is precisely the same as the map $\mathcal{O}_K \rightarrow A_{\text{inf}}/\xi^{(1)}$ described in Section 2.3.5.

Finally, to establish that the triangle (6) in diagram (2.8) commutes, it is enough to show that s is a morphism in $(\mathcal{O}_K)_{\Delta}$. To this end, observe that the map $\mathfrak{S} \rightarrow W$ sending u to 0 does send $(E^{(n)})$ to (p) for any $n \geq 1$. Indeed, note that $\phi^n(E(u)) = E(u^{p^n})$, and so it is sent to $E(0)$ in W . But, as E is an Eisenstein polynomial, therefore, we must have $E(0) = pv$, for some unit v in W . This establishes the claim and allows us to conclude. $\square$

3. THE TWISTED CRYSTALLINE-DE RHAM COMPARISON

In this section we provide a general comparison between the so-called *twisted crystalline realisation* and *twisted de Rham realisation* of a perfect complex of prismatic F -crystals over $\mathcal{O}_K$.

We continue to use notation from [global notation and conventions](#) and Notation 2.1.

¹⁰Indeed, as ξ has divided powers in A_{crys} , it suffices to show that the image of $[\pi^b]^{\tilde{e}}$ under the map $\theta: A_{\text{inf}} \rightarrow \mathcal{O}_C$ has divided powers. But, $\theta([\pi^b]^{\tilde{e}}) = \pi^{\tilde{e}}$ which has divided powers in $\mathcal{O}_K$, and thus $\mathcal{O}_C$, by [BO83, Lemma 3.9].

3.1. A stack-theoretic approach. We begin by explaining a conceptual way of proving the twisted crystalline-de Rham comparison that is more in the spirit of [BO78].

3.1.1. The twisted crystalline realisation functor. We begin by studying several ways to explicitly understand the stack k^Δ . The interested reader may consult [IKY25, Section 1.1.3] for a more general discussion in an unramified and non-twisted setting.

Definition 3.1. The *reduced n -twisted crystalline point* of $\mathcal{O}_K$ is the morphism of formal stacks over $\mathbb{Z}_p$,

$$\bar{\rho}_{\text{crys}}^{(n)}: \text{Spf}(W) \longrightarrow k^\Delta,$$

for $n \geq 0$, which for a p -nilpotent ring R associates to a map $\text{Spec}(R) \rightarrow \text{Spf}(W)$ the Cartier–Witt divisor $\mathbb{W}(R) \xrightarrow{p} \mathbb{W}(R)$ with structure map,

$$k \xrightarrow{\sim} W/p \xrightarrow{\phi_W} W/p \longrightarrow \mathbb{W}(R)/p \xrightarrow{F^n} \mathbb{W}(R)/p, \quad (3.1)$$

where $W/p \rightarrow \mathbb{W}(B)/p$ is the map induced by the unique δ -ring map $W \rightarrow \mathbb{W}(B)$ lifting $W \rightarrow B$. The *n -twisted crystalline point* of $\mathcal{O}_K$ is the map $\rho_{\text{crys}}^{(n)}: \text{Spf}(W) \rightarrow \mathcal{O}_K^\Delta$ obtained as the following composition:

$$\text{Spf}(W) \xrightarrow{\bar{\rho}_{\text{crys}}^{(n)}} k^\Delta \rightarrow \mathcal{O}_K^\Delta,$$

where the latter map is the obvious one.

Let us shorten $\bar{\rho}_{\text{crys}}^{(0)}$ to $\bar{\rho}_{\text{crys}}$ and $\rho_{\text{crys}}^{(0)}$ to ρ_{crys} . Then, by the definition of the Frobenius maps $F_k: k^\Delta \rightarrow k^\Delta$ and $F_{\mathcal{O}_K}: \mathcal{O}_K^\Delta \rightarrow \mathcal{O}_K^\Delta$, we have that for any $n \geq 0$.

$$\rho_{\text{crys}}^{(n)} = F_{\mathcal{O}_K}^n \circ \rho_{\text{crys}}, \quad \bar{\rho}_{\text{crys}}^{(n)} = F_k^n \circ \bar{\rho}_{\text{crys}} \quad (3.2)$$

Remark 3.2. The appearance of the ϕ_W in (3.1) may seem artificial. To help demystify it, observe that in general for a quasi-syntomic k -scheme Z there is an isomorphism between the formal stack over W given by $(Z/W)^{\text{crys}}$ (e.g. as defined in [Bha23, Remark 2.5.12]) and the pullback stack $\phi_W^*(Z^\Delta)$ (e.g. see [Bha23, Corollary 2.6.8 & Construction 3.1.1] or [IKY25, Lemma 1.6]). From this perspective, we may view $\bar{\rho}_{\text{crys}}$ as the composition

$$\text{Spf}(W) \xrightarrow{\sim} (k/W)^{\text{crys}} \xrightarrow{\sim} \phi_W^*(k^\Delta) \xrightarrow{F_k} k^\Delta.$$

This observation is important as the first isomorphism $\text{Spf}(W) \rightarrow (k/W)^{\text{crys}}$ is the one well-suited to crystalline theory (e.g. it corresponds to the equivalence between (F) -crystals on $(k/W)^{\text{crys}}$ and (ϕ_W) -modules over W), and is also the reason why a Frobenius twist appears in the crystalline comparison theorem (e.g. as in [BS22, Theorem 1.8]).

We define our twisted crystalline realisation functor as follows.

Definition 3.3. The *n -twisted crystalline realisation* is the functor,

$$\mathbb{D}_{\text{crys}}^{(n)}: \mathbf{Perf}((\mathcal{O}_K)_\Delta) \longrightarrow \mathbf{Perf}(W),$$

obtained as the pullback $\rho_{\text{crys}}^{(n)}$.

Remark 3.4. For any object $\mathcal{V}$ of $\mathbf{Perf}((\mathcal{O}_K)_\Delta)$, by Remark 3.2 one may also interpret $\mathbb{D}_{\text{crys}}^{(n)}(\mathcal{V})$ as $(\phi^n \mathcal{V}^{\text{crys}})(W \twoheadrightarrow k)$, where $\mathcal{V}^{\text{crys}}$ is the perfect complex of crystals on $(k/W)^{\text{crys}}$ associated to $\mathcal{V}_k$ via the equivalence in [GR24, Theorem 6.4], ϕ is the Frobenius morphism of topoi $(k/W)^{\text{crys}} \rightarrow (k/W)^{\text{crys}}$, and $W \twoheadrightarrow k \simeq W/p$ is the natural PD-thickening.

Proposition 3.5. For any $n \geq 0$, there is a natural identification of $\bar{\rho}_{\text{crys}}^{(n)}$ and $\rho_{\text{crys}}^{(n)}$ with $\bar{\rho}_W^{(n+1)}$ and $\rho_W^{(n+1)}$, respectively.

Proof. By Equations (2.5) and (3.2), our definition of ρ_{crys} , and Proposition 2.12 we are reduced to showing that $\rho_{\text{crys}} = \rho_W^{(1)}$. But, by definition, to a p -nilpotent ring R and a map $\text{Spec}(R) \rightarrow \text{Spf}(W)$, the map $\rho_W^{(1)}$ associates the Cartier–Witt divisor $d: \mathbb{W}(R) \otimes_W (p) \rightarrow \mathbb{W}(R)$, where $W \rightarrow \mathbb{W}(B)$ is the map of δ -rings obtained from $W \rightarrow B$, and with the structure map

$$k \xrightarrow{F_k} k \longrightarrow \text{cone}(d) \longrightarrow \mathbb{W}(B)/p,$$

which clearly agrees with the object of $\mathcal{O}_K^\Delta(R)$ associated to $\text{Spec}(R) \rightarrow \text{Spf}(W)$ by ρ_{crys} . $\square$

Combining this observation with Proposition 2.13 and the equivalence in [BL22a, Proposition 3.3.5], we obtain the following identifications.

Corollary 3.6. *For any $n \geq 0$, uniformiser π of $\mathcal{O}_K$, and any object $\mathcal{V}$ of $\mathbf{Perf}((\mathcal{O}_K)_\Delta)$, there is a natural identification in $\mathbf{Perf}(W)$:*

$$\mathbb{D}_{\text{crys}}^{(n)}(\mathcal{V}) \simeq \phi_W^{*(n+1)} \mathcal{V}(W, p) \xrightarrow[\psi_{\text{crys}, \pi}^{(n)}]{\sim} (\phi_{\mathfrak{S}}^{*(n+1)}(\mathcal{V}(\mathfrak{S}, E_\pi)))/u. \quad (3.3)$$

Remark 3.7. The first isomorphism in (3.3) is truly canonical, but the second (while natural in $\mathcal{V}$) implicitly depends on the choice of a uniformiser π to define E_π , and therefore the prism $(\mathfrak{S}, E_\pi)$ in $(\mathcal{O}_K)_\Delta$. This explains our reasoning for labelling the second map but not the first.

3.1.2. The twisted de Rham realisation functor. We now discuss a de Rham analogue of the material from Section 3.1.1. The interested reader may consult [IKY25, Section 1.2] for a more general discussion in the non-twisted setting.

Definition 3.8. The n -twisted de Rham point of $\mathcal{O}_K$ is the morphism of formal stacks over $\mathbb{Z}_p$,

$$\rho_{\text{dR}}^{(n)}: \text{Spf}(\mathcal{O}_K) \longrightarrow \mathcal{O}_K^\Delta,$$

for $n \geq 0$, which for a p -nilpotent ring R associates to a map $\text{Spec}(R) \rightarrow \text{Spf}(\mathcal{O}_K)$ the Cartier–Witt divisor $\mathbb{W}(R) \xrightarrow{p} \mathbb{W}(R)$ with the structure map

$$\mathcal{O}_K \longrightarrow R = \mathbb{W}(R)/V\mathbb{W}(R) \xrightarrow{F} \mathbb{W}(R)/p \xrightarrow{F^n} \mathbb{W}(R)/p.$$

Let us shorten $\rho_{\text{dR}}^{(0)}$ to ρ_{dR} . Then, by the definition of the Frobenius map $F_{\mathcal{O}_K}: \mathcal{O}_K^\Delta \rightarrow \mathcal{O}_K^\Delta$ we have that $\rho_{\text{dR}}^{(n)} = F_{\mathcal{O}_K}^n \circ \rho_{\text{dR}}$, for any $n \geq 0$.

Definition 3.9. The n -twisted de Rham realisation is the functor,

$$\mathbb{D}_{\text{dR}}^{(n)}: \mathbf{Perf}((\mathcal{O}_K)_\Delta) \longrightarrow \mathbf{Perf}(\mathcal{O}_K),$$

given by pullback along $\rho_{\text{dR}}^{(n)}: \text{Spf}(\mathcal{O}_K) \rightarrow \mathcal{O}_K^\Delta$.

We again wish to more concretely understand the twisted de Rham realisation using the Breuil–Kisin prism, as in Corollary 3.6.

Proposition 3.10 ([IKY25, Proposition 1.15]). *There is a canonical identification in $\text{Map}(\text{Spf}(\mathcal{O}_K), \mathcal{O}_K^\Delta)$:*

$$\rho_{\text{dR}} \simeq \rho_{\mathfrak{S}}^{(1)}.$$

Corollary 3.11. *For any $n \geq 0$, uniformiser π of $\mathcal{O}_K$, and any object $\mathcal{V}$ of $\mathbf{Perf}((\mathcal{O}_K)_\Delta)$, the following is a natural identification in $\mathbf{Perf}(\mathcal{O}_K)$:*

$$\psi_{\text{dR}, \pi}^{(n)}: \mathbb{D}_{\text{dR}}^{(n)}(\mathcal{V}) \xrightarrow{\sim} \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_\pi)/E_\pi. \quad (3.4)$$

3.1.3. The twisted crystalline-de Rham comparison (stacky form). We are now ready to state and prove the stacky version of the twisted crystalline-de Rham comparison.

Theorem 3.12 (Twisted crystalline-de Rham comparison). *Suppose that $n \geq a = \lceil \log_p \left(\frac{e}{p-1} \right) \rceil$.*

(1) *There is an identification between $\rho_{\text{dR}}^{(n)}$ and the composition*

$$\text{Spf}(\mathcal{O}_K) \longrightarrow \text{Spf}(W) \xrightarrow{\rho_{\text{crys}}^{(n)}} \mathcal{O}_K^\Delta,$$

as objects of $\text{Map}(\text{Spf}(\mathcal{O}_K), \mathcal{O}_K^\Delta)$.

(2) For any object $\mathcal{V}$ of $\mathbf{Perf}((\mathcal{O}_K)_\Delta)$, there is a canonical isomorphism in $\mathbf{Perf}(\mathcal{O}_K)$:

$$\iota^{(n)}: \mathbb{D}_{\text{crys}}^{(n)}(\mathcal{V}) \otimes_W \mathcal{O}_K \xrightarrow{\sim} \mathbb{D}_{\text{dR}}^{(n)}(\mathcal{V}).$$

Proof. Assertion (2) is an immediate consequence of assertion (1). To prove assertion (1), let R be a p -nilpotent ring and α a morphism $\text{Spec}(R) \rightarrow \text{Spf}(\mathcal{O}_K)$. The underlying Cartier–Witt divisors of both $\rho_{\text{dR}}^{(n)}(\alpha)$ and $\rho_{\text{crys}}^{(n)}(\alpha)$ is $\mathbb{W}(R) \xrightarrow{p} \mathbb{W}(R)$, and their $\mathcal{O}_K$ -structures are given respectively as follows (see Definition 3.1 and Definition 3.8):

$$\mathcal{O}_K \xrightarrow{\alpha} R = \mathbb{W}(R)/V\mathbb{W}(R) \xrightarrow{F} \mathbb{W}(R)/p \xrightarrow{F^n} \mathbb{W}(R)/p, \quad (3.5)$$

$$\mathcal{O}_K \rightarrow k \xrightarrow{\sim} W/p \xrightarrow{\phi_W} W/p \xrightarrow{\mathbb{W}(\alpha)} \mathbb{W}(R)/p \xrightarrow{F^n} \mathbb{W}(R)/p. \quad (3.6)$$

To show that the composition of these respective maps agree, first observe that the following respective composition of the maps,

$$W \rightarrow \mathcal{O}_K \xrightarrow{\alpha} R = \mathbb{W}(R)/V\mathbb{W}(R) \xrightarrow{F} \mathbb{W}(R)/p, \quad (3.7)$$

$$W \rightarrow k \xrightarrow{\sim} W/p \xrightarrow{\phi_W} W/p \xrightarrow{\mathbb{W}(\alpha)} \mathbb{W}(R)/p, \quad (3.8)$$

agree by [IKY25, Theorem 1.19]. But, for $n \geq a$, the map $F_{\mathcal{O}_K/p}^n: \mathcal{O}_K/p \rightarrow \mathcal{O}_K/p$ factorises as $\mathcal{O}_K/p \rightarrow k \xrightarrow{F_k^n} k \rightarrow \mathcal{O}_K/p$, where the last map is induced by $W \rightarrow \mathcal{O}_K$. As the composition of (3.5) factorises through $\mathcal{O}_K/p$, the appearance of F^n in both (3.5) and (3.6) thus implies the agreement of their compositions from the agreement of the compositions of (3.7) and (3.8). $\square$

3.2. A non-stacky approach. We now explain a more down-to-earth approach to Theorem 3.12 using Breuil–Kisin theory. In the next section, we shall compare this to the stacky approach.

3.2.1. The twisted crystalline realisation. We begin by giving a Breuil–Kisin-theoretic definition of the twisted crystalline realisation. To distinguish it from the (a priori different) stack-theoretic twisted crystalline realisation, we temporarily use a prime in the notation.

Definition 3.13. Let π be a uniformiser of $\mathcal{O}_K$ and $n \geq 0$ an integer. The π -indexed n -twisted crystalline realisation functor is defined as follows:

$$\mathbb{D}_{\text{crys}, \pi}^{\prime, (n)}: \mathbf{Perf}((\mathcal{O}_K)_\Delta) \rightarrow \mathbf{Perf}(W), \quad \mathcal{V} \mapsto \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_\pi)/u.$$

To remove the π -dependency of the n -twisted crystalline realisation functors, we consider the following construction.

Construction 3.14. Let $\mathcal{V}$ be an object of $\mathbf{Perf}((\mathcal{O}_K)_\Delta)$ and $n \geq 0$ an integer. Then, the following commutative diagram (see Proposition 2.13):

$$\begin{array}{ccccc} \text{Spf}(\mathfrak{S}) & \xrightarrow{(u=0)} & \text{Spf}(W) & \xleftarrow{(u=0)} & \text{Spf}(\mathfrak{S}) \\ & \searrow \rho_\pi^{(n+1)} & \downarrow \rho_W^{(n+1)} & \swarrow \rho_{\pi'}^{(n+1)} & \\ & & \mathcal{O}_K^\Delta & & \end{array}$$

together with Equations (2.2) and (2.5), yield isomorphisms:

$$\phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_\pi)/u \xrightarrow{\sim} \phi_W^{*(n+1)} \mathcal{V}(W, p) \xleftarrow{\sim} \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_{\pi'})/u.$$

The preceding diagram induces an isomorphism,

$$j_{\text{crys}, \pi, \pi'}^{(n), \mathcal{V}}: \mathbb{D}_{\text{crys}, \pi}^{\prime, (n)}(\mathcal{V}) \xrightarrow{\sim} \mathbb{D}_{\text{crys}, \pi'}^{\prime, (n)}(\mathcal{V}).$$

It is easy to see that the isomorphisms $j_{\text{crys}, \pi, \pi'}^{(n), \mathcal{V}}$ are functorial in $\mathcal{V}$ and satisfy the following cocycle condition:

$$j_{\text{crys}, \pi', \pi''}^{(n), \mathcal{V}} \circ j_{\text{crys}, \pi, \pi'}^{(n), \mathcal{V}} = j_{\pi, \pi''}^{(n), \mathcal{V}}.$$

Thus, we obtain natural equivalences,

$$j_{\text{crys},\pi,\pi'}^{(n)}: \mathbb{D}_{\text{crys},\pi}'^{(n)} \xrightarrow{\sim} \mathbb{D}_{\text{crys},\pi'}'^{(n)}$$

in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(W))$ satisfying the cocycle condition.

Definition 3.15. The n -twisted crystalline realisation functor is defined as

$$\mathbb{D}_{\text{crys}}'^{(n)} := \varprojlim_{\pi} \mathbb{D}_{\text{crys},\pi}'^{(n)}: \mathbf{Perf}((\mathcal{O}_K)_\Delta) \longrightarrow \mathbf{Perf}(W).$$

We end this section by comparing $\mathbb{D}_{\text{crys}}'^{(n)}$ and $\mathbb{D}_{\text{crys}}^{(n)}$. Namely, via Corollary 3.6, for any π we have a canonical identification

$$\psi_{\text{crys},\pi}^{(n)}: \mathbb{D}_{\text{crys}}^{(n)} \xrightarrow{\sim} \mathbb{D}_{\text{crys},\pi}'^{(n)}.$$

By combining Propositions 2.13 and 3.5, we have the following commutative diagram of stacks:

$$\begin{array}{ccccc} \mathrm{Spf}(\mathfrak{S}) & \xleftarrow{(u=0)} & \mathrm{Spf}(W) & \xrightarrow{(u=0)} & \mathrm{Spf}(\mathfrak{S}) \\ & \searrow \rho_{\pi}^{(n+1)} & \downarrow \rho_{\text{crys}}^{(n)} & \swarrow \rho_{\pi'}^{(n+1)} & \\ & & \mathcal{O}_K^\Delta & & \end{array}$$

Thus, it is easy to deduce that $j_{\text{crys},\pi,\pi'}^{(n)} \circ \psi_{\text{crys},\pi}^{(n)} = \psi_{\text{crys},\pi'}^{(n)}$. From the preceding discussion we obtain that,

Proposition 3.16. *The following morphism is an equivalence in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(W))$:*

$$\psi_{\text{crys}}^{(n)} := (\psi_{\text{crys},\pi}^{(n)}): \mathbb{D}_{\text{crys}}^{(n)} \longrightarrow \varprojlim_{\pi} \mathbb{D}_{\text{crys},\pi}'^{(n)} = \mathbb{D}_{\text{crys}}'^{(n)}.$$

3.2.2. The twisted de Rham realisation. We now perform a similar analysis to define the twisted de Rham realisation using the Breuil–Kisin theory, and to compare this to the stack-theoretic definition.

Definition 3.17. Let π be a uniformiser of $\mathcal{O}_K$ and $n \geq 0$ an integer. The π -indexed n -twisted de Rham realisation functor is defined as follows:

$$\mathbb{D}_{\text{dR},\pi}'^{(n)}: \mathbf{Perf}((\mathcal{O}_K)_\Delta) \longrightarrow \mathbf{Perf}(\mathcal{O}_K), \quad \mathcal{V} \longmapsto \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_\pi) / E_\pi.$$

Observe that, by the setup as in Section 2.3.2, there is a commuting diagram of maps,

$$\begin{array}{ccccc} (\mathfrak{S}, E_\pi) & \xrightarrow{p_1} & (\mathfrak{S}_{\pi,\pi'}, I_{\pi,\pi'}) & \xleftarrow{p_2} & (\mathfrak{S}, E_{\pi'}) \\ & \searrow & \downarrow & \swarrow & \\ & & \mathcal{O}_K & & \end{array} \tag{3.9}$$

where the horizontal arrows are the two projection maps in $(\mathcal{O}_K)_\Delta$ and the rest are given by reduction modulo the prismatic ideal.

Construction 3.18. Let $\mathcal{V}$ be an object of $\mathbf{Perf}((\mathcal{O}_K)_\Delta)$ and $n \geq 0$ an integer. Applying the crystal property for the diagram in (3.9), we obtain a commutative diagram of equivalences:

$$\phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_\pi) \otimes_{\mathfrak{S}} \mathfrak{S}_{\pi,\pi'} \xrightarrow{\sim} \phi_{\mathfrak{S}_{\pi,\pi'}}^{*(n+1)} \mathcal{V}(\mathfrak{S}_{\pi,\pi'}, I_{\pi,\pi'}) \xleftarrow{\sim} \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_{\pi'}) \otimes_{\mathfrak{S}} \mathfrak{S}_{\pi,\pi'}.$$

From the preceding diagram and commutativity of (3.9), note that by tensoring along $\mathfrak{S}_{\pi,\pi'} \rightarrow \mathcal{O}_K$ we obtain an isomorphism,

$$j_{\text{dR},\pi,\pi'}^{(n),\mathcal{V}}: \mathbb{D}_{\text{dR},\pi}'^{(n)}(\mathcal{V}) \xrightarrow{\sim} \mathbb{D}_{\text{dR},\pi'}'^{(n)}(\mathcal{V}).$$

The isomorphisms $j_{\text{dR},\pi,\pi'}^{(n),\mathcal{V}}$ are functorial in $\mathcal{V}$ and satisfy the following cocycle condition:

$$j_{\text{dR},\pi',\pi''}^{(n),\mathcal{V}} \circ j_{\text{dR},\pi,\pi'}^{(n),\mathcal{V}} = j_{\text{dR},\pi,\pi''}^{(n),\mathcal{V}}.$$

Thus, we obtain natural equivalences

$$j_{\mathrm{dR}, \pi, \pi'}^{(n)} : \mathbb{D}_{\mathrm{dR}, \pi}^{\prime, (n)} \xrightarrow{\sim} \mathbb{D}_{\mathrm{dR}, \pi'}^{\prime, (n)}$$

in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(\mathcal{O}_K))$ satisfying the cocycle condition.

Definition 3.19. The n -twisted de Rham realisation functor is defined as

$$\mathbb{D}_{\mathrm{dR}}^{\prime, (n)} := \varprojlim_{\pi} \mathbb{D}_{\mathrm{dR}, \pi}^{\prime, (n)} : \mathbf{Perf}((\mathcal{O}_K)_\Delta) \longrightarrow \mathbf{Perf}(\mathcal{O}_K).$$

We end this section by comparing $\mathbb{D}_{\mathrm{dR}}^{\prime, (n)}$ and $\mathbb{D}_{\mathrm{dR}}^{(n)}$. Via Corollary 3.11, for any π , we have a canonical identification

$$\psi_{\mathrm{dR}, \pi}^{(n)} : \mathbb{D}_{\mathrm{dR}}^{(n)} \xrightarrow{\sim} \mathbb{D}_{\mathrm{dR}, \pi}^{\prime, (n)}.$$

Moreover, we have the following (2-)commutative diagram:

$$\begin{array}{ccccc} & & \mathrm{Spf}(\mathfrak{S}) & & \\ & \nearrow q_1 & \uparrow E_\pi=0 & \searrow \rho_\pi^{(1)} & \\ \mathrm{Spf}(\mathfrak{S}_{\pi, \pi'}) & \xleftarrow{(E_\pi=E_{\pi'}=0)} & \mathrm{Spf}(\mathcal{O}_K) & \xrightarrow{\rho_{\mathrm{dR}}} & \mathcal{O}_K^\Delta, \\ & \searrow q_2 & \downarrow E_{\pi'}=0 & \nearrow \rho_{\pi'}^{(1)} & \\ & & \mathrm{Spf}(\mathfrak{S}) & & \end{array}$$

and therefore, it is easy to deduce that $j_{\mathrm{dR}, \pi, \pi'}^{(n)} \circ \psi_{\mathrm{dR}, \pi}^{(n)} = \psi_{\mathrm{dR}, \pi'}^{(n)}$. From this we obtain the following.

Proposition 3.20. The following morphism is an equivalence in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(\mathcal{O}_K))$:

$$\psi_{\mathrm{dR}}^{(n)} =: (\psi_{\mathrm{dR}, \pi}^{(n)}) : \mathbb{D}_{\mathrm{dR}}^{(n)} \longrightarrow \varprojlim \mathbb{D}_{\mathrm{dR}, \pi}^{\prime, (n)} = \mathbb{D}_{\mathrm{dR}}^{\prime, (n)}.$$

3.2.3. The twisted crystalline-de Rham comparison (non-stacky form). We shall now give a non-stacky version of the twisted crystalline-de Rham comparison, and compare it to the stack-theoretic one. We first require the following construction:

Construction 3.21. Fix a uniformiser π and an integer $n \geq a$. From Equations (2.2) and (2.5), and Proposition 2.13 we obtain the following equivalences in $\mathbf{Perf}(\tilde{S})$:

$$\phi_W^{*(n+1)} \mathcal{V}(W, p) \otimes_W \tilde{S} \xrightarrow{\sim} \phi_{\tilde{S}}^{*n} \mathcal{V}(\tilde{S}, p) \xleftarrow{\sim} \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_\pi) \otimes_{\mathfrak{S}} \tilde{S}.$$

Base change of the preceding isomorphisms, along the map $\tilde{S} \rightarrow \mathcal{O}_K$ from Proposition 2.11, furnishes us with an equivalence in $\mathbf{Perf}(\mathcal{O}_K)$:

$$\phi_W^{*(n+1)} \mathcal{V}(W, p) \otimes_W \mathcal{O}_K \xrightarrow{\sim} \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_\pi) / E_\pi.$$

Finally, using the isomorphism (see Construction 3.14),

$$\phi_W^{*(n+1)} \mathcal{V}(W, p) \xrightarrow{\sim} \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_\pi) / u,$$

gives rise to an isomorphism

$$\iota_\pi^{\prime, (n), \mathcal{V}} : \mathbb{D}_{\mathrm{crys}, \pi}^{\prime, (n)}(\mathcal{V}) \otimes_W \mathcal{O}_K \xrightarrow{\sim} \mathbb{D}_{\mathrm{dR}, \pi}^{\prime, (n)}(\mathcal{V}).$$

By construction, it is clear that the isomorphisms $\iota_\pi^{\prime, (n), \mathcal{V}}$ are functorial in $\mathcal{V}$, and thus give rise to an isomorphism $\iota_\pi^{\prime, (n)} : \mathbb{D}_{\mathrm{crys}, \pi}^{\prime, (n)} \otimes_W \mathcal{O}_K \rightarrow \mathbb{D}_{\mathrm{dR}, \pi}^{\prime, (n)}$ in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(\mathcal{O}_K))$.

Proposition 3.22. *For uniformisers π and π' of $\mathcal{O}_K$, and an integer $n \geq a$, the following diagram in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(\mathcal{O}_K))$ naturally commutes:*

$$\begin{array}{ccc}
\mathbb{D}_{\text{crys}, \pi}^{(n)} \otimes_W \mathcal{O}_K & \xrightarrow{\iota_{\pi}^{(n)}} & \mathbb{D}_{\text{dR}, \pi}^{(n)} \\
\downarrow j_{\text{crys}, \pi, \pi'}^{(n)} \otimes 1 & & \downarrow j_{\text{dR}, \pi, \pi'}^{(n)} \\
\mathbb{D}_{\text{crys}, \pi'}^{(n)} \otimes_W \mathcal{O}_K & \xrightarrow{\iota_{\pi'}^{(n)}} & \mathbb{D}_{\text{dR}, \pi'}^{(n)}.
\end{array} \tag{3.10}$$

Proof. To prove the commutativity of (3.10), it suffices to check it after base change along the faithfully flat map $\mathcal{O}_K \rightarrow \mathcal{O}_C$. So, consider the following commutative diagram in $(\mathcal{O}_K)_\Delta$:

$$\begin{array}{ccccc}
& & (W, p) & & \\
& \swarrow \phi^{n+1} & \uparrow & \nwarrow \phi^{n+1} & \\
(\tilde{S}_\pi, p) & \xrightarrow{\quad} & (A_{\text{crys}}, p) & \xleftarrow{\quad} & (\tilde{S}_{\pi'}, p) \\
\uparrow \phi^n & \nearrow u=0 & \uparrow & \nwarrow u=0 & \uparrow \phi^n \\
(\mathfrak{S}, E_\pi^{(1)}) & \xrightarrow{\phi^n \circ \alpha_{\text{inf}, \pi}} & (A_{\text{inf}}, \phi^n(\xi)) & \xleftarrow{\phi^n \circ \alpha_{\text{inf}, \pi'}} & (\mathfrak{S}, \phi(E_{\pi'})),
\end{array} \tag{3.11}$$

where we have used Proposition 2.13. Also, while the E_π in $(\mathfrak{S}, E_\pi)$ indicates the exact structure as an object of $(\mathcal{O}_K)_\Delta$, the notation $(\tilde{S}, p)$ does not indicate the π -dependence of the modified Breuil prism, and so we have written it as $\tilde{S}_\pi$ (and likewise for π') to indicate this dependency.

For an object $\mathcal{V}$ of $\mathbf{Perf}((\mathcal{O}_K)_\Delta)$, the crystal property yields from (3.11) a large commutative diagram of equivalences of objects in $\mathbf{Perf}(A_{\text{crys}})$. Base changing along $A_{\text{crys}} \rightarrow \mathcal{O}_C$ yields the base change of (3.10) along $\mathcal{O}_K \rightarrow \mathcal{O}_C$ as a commutative subdiagram, allowing us to conclude. $\square$

From Proposition 3.22, it is clear that we obtain an equivalence,

$$\iota'^{(n)}: \mathbb{D}_{\text{crys}}'^{(n)} \otimes_W \mathcal{O}_K \xrightarrow{\sim} \mathbb{D}_{\text{dR}}'^{(n)}$$

in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(\mathcal{O}_K))$, for any $n \geq a$. We end this section by comparing this construction to the equivalence $\iota^{(n)}$ from Theorem 3.12.

Proposition 3.23. *For any $n \geq a$, the following diagram commutes in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(\mathcal{O}_K))$:*

$$\begin{array}{ccc}
\mathbb{D}_{\text{crys}}^{(n)} \otimes_W \mathcal{O}_K & \xrightarrow{\iota^{(n)}} & \mathbb{D}_{\text{dR}}^{(n)} \\
\downarrow \psi_{\text{crys}}^{(n)} \otimes 1 & & \downarrow \psi_{\text{dR}}^{(n)} \\
\mathbb{D}_{\text{crys}}'^{(n)} \otimes_W \mathcal{O}_K & \xrightarrow{\iota'^{(n)}} & \mathbb{D}_{\text{dR}}'^{(n)}.
\end{array}$$

Proof. To prove the claim, note that it is enough to show that for a fixed uniformiser π , the following diagram commutes in $\mathbf{Func}(\mathbf{Perf}((\mathcal{O}_K)_\Delta), \mathbf{Perf}(\mathcal{O}_K))$:

$$\begin{array}{ccc}
\mathbb{D}_{\text{crys}}^{(n)} \otimes_W \mathcal{O}_K & \xrightarrow{\iota^{(n)}} & \mathbb{D}_{\text{dR}}^{(n)} \\
\downarrow \psi_{\text{crys}}^{(n)} \otimes 1 & & \downarrow \psi_{\text{dR}}^{(n)} \\
\mathbb{D}_{\text{crys}, \pi}^{(n)} \otimes_W \mathcal{O}_K & \xrightarrow{\iota_{\pi}^{(n)}} & \mathbb{D}_{\text{dR}, \pi}^{(n)}.
\end{array}$$

As in the proof of Proposition 3.22, let us denote by $(\tilde{S}_\pi, p)$ the modified Breuil prism with a choice of uniformiser π . Then, given the construction of $\psi_{\text{crys}, \pi}^{(n)}$ and $\psi_{\text{dR}, \pi}^{(n)}$, Proposition 3.23 follows from Proposition 3.24 below. This allows us to conclude. $\square$

For notational simplicity, in the following let us write $\rho_{\text{crys}}^{(n)} \otimes 1$ to denote the composition,

$$\text{Spf}(\mathcal{O}_K) \rightarrow \text{Spf}(W) \xrightarrow{\rho_{\text{crys}}^{(n)}} \mathcal{O}_K^\Delta,$$

appearing in Theorem 3.12.

Proposition 3.24. *The identification from Theorem 3.12 naturally factors as,*

$$\rho_{\text{crys}}^{(n)} \otimes 1 \xrightarrow{\sim} \tilde{\rho}_\pi^{(n)} \circ i \xleftarrow{\sim} \rho_{\text{dR}}^{(n)},$$

where $i: \text{Spf}(\mathcal{O}_K) \rightarrow \text{Spf}(\tilde{S})$ is the map described in Proposition 2.11.

Proof. Consider the following diagram:

$$\begin{array}{ccccc} & & \text{Spf}(\mathfrak{S}) & \xrightarrow{\phi_{\mathfrak{S}}^{n+1}} & \text{Spf}(\mathfrak{S}) \\ & \nearrow^{(E_\pi=0)} & \uparrow & & \searrow^{\rho_\pi} \\ \text{Spf}(\mathcal{O}_K) & \xrightarrow{i} & \text{Spf}(\tilde{S}_\pi) & \xrightarrow{\tilde{\rho}_\pi^{(n)}} & \mathcal{O}_K^\Delta \\ & \searrow_{\text{nat.}} & \downarrow & & \nearrow_{\rho_W} \\ & & \text{Spf}(W) & \xrightarrow{\phi_W^{n+1}} & \text{Spf}(W). \end{array},$$

where the left two triangles are commutative by Proposition 2.11, and the right two trapeziums are 2-commutative with obvious identifications of the compositions.

Let $\rho_{\text{dR}}^{\prime(n)}$ (resp. $\rho_{\text{crys}}^{\prime(n)} \otimes 1$) denote the upper (resp. lower) composite maps $\text{Spf}(\mathcal{O}_K) \rightarrow \mathcal{O}_K^\Delta$. By the commutativity of the above diagram, we obtain isomorphisms,

$$\rho_{\text{crys}}^{\prime(n)} \otimes 1 \xrightarrow{\sim} \tilde{\rho}_\pi^{(n)} \circ \text{sp}_{\text{dR}} \xleftarrow{\sim} \rho_{\text{dR}}^{\prime(n)}.$$

By Corollaries 3.11 and 3.6, we have isomorphisms $\rho_{\text{dR}}^{\prime(n)} \xrightarrow{\sim} \rho_{\text{dR}}^{(n)}$ and $\rho_{\text{crys}}^{\prime(n)} \otimes 1 \xrightarrow{\sim} \rho_{\text{crys}}^{(n)} \otimes 1$, respectively. Thus, we have obtained the following diagram of isomorphisms:

$$\begin{array}{ccc} \rho_{\text{crys}}^{(n)} \otimes 1 & \xrightarrow{\sim} & \rho_{\text{dR}}^{(n)} \\ \downarrow \wr & & \downarrow \wr \\ \rho_{\text{crys}}^{\prime(n)} \otimes 1 & \xrightarrow{\sim} \tilde{\rho}_\pi^{(n)} \circ i \xleftarrow{\sim} & \rho_{\text{dR}}^{\prime(n)}. \end{array} \quad (3.12)$$

This diagram commutes as each isomorphism is defined by the identity map on $\mathbb{W}$ (and a canonical identification of its quasi-ideals), thus proving the assertion. $\square$

4. TWISTED CRYSTALLINE AND DE RHAM COHOMOLOGY

In this section, we shall apply the material of Section 3 to perfect prismatic crystals of the form $Rf_*^\Delta \mathcal{O}^\Delta$. In particular, we will deduce our extension of the Berthelot–Ogus isomorphism to the case of coefficients. To formulate this result correctly, we introduce the notion of *twisted crystalline and de Rham cohomology*, and study some of their properties.

Notation 4.1. We continue to use the notation from [global notation and conventions](#), Notation 2.1 and Section 2.3. We further assume that $f: X \rightarrow \text{Spf}(\mathcal{O}_K)$ is a smooth proper morphism.

4.1. Twisted crystalline cohomology. We begin with the easier of the two definitions, twisted crystalline cohomology, which ultimately reduces to classical notions.

Definition 4.2. For $n \geq 0$, the n -twisted crystalline cohomology functor is given by

$$\text{R}\Gamma_{\text{crys}}^{(n)}(-) = \text{R}\Gamma_{\text{crys}}^{(n)}(X_k/W, -): \mathbf{Perf}(X^\Delta) \longrightarrow \mathbf{Perf}(W), \quad \mathcal{V} \longmapsto \mathbb{D}_{\text{crys}}^{(n)}(Rf_*^\Delta \mathcal{V}).$$

We denote the associated cohomology groups by $H_{\text{crys}}^{i,(n)}(\mathcal{V}) = H_{\text{crys}}^{i,(n)}(X_k/W, \mathcal{V})$, and when $\mathcal{V} = \mathcal{O}_\Delta$ we omit $\mathcal{V}$ from the notation.

Recall that for a quasi-syntomic k -scheme Z there is an equivalence of ∞ -categories,

$$(-)^{\text{crys}}: \mathbf{Perf}(Z^\Delta) \xrightarrow{\sim} \mathbf{Perf}((Z/W)_{\text{crys}}), \quad \mathcal{V} \mapsto \mathcal{V}^{\text{crys}},$$

(e.g. see [GR24, Theorem 6.4]). We then define the functor,

$$(-)^{\text{crys}}: \mathbf{Perf}(X^\Delta) \longrightarrow \mathbf{Perf}((X_k/W)_{\text{crys}}), \quad \mathcal{V} \mapsto (\mathcal{V}|_{(X_k)_\Delta})^{\text{crys}}.$$

These functors are compatible with Frobenius morphisms, i.e., $(F_X^* \mathcal{V})^{\text{crys}}$ is naturally identified with $\phi_{\text{crys}}^* \mathcal{V}^{\text{crys}}$, where ϕ_{crys} is the Frobenius endomorphism of the crystalline topos.

Proposition 4.3. *For each $n \geq 0$, there are natural identifications*

$$\mathrm{R}\Gamma_{\text{crys}}^{(n)}(\mathcal{V}) \simeq \phi_W^{*n} \mathrm{R}\Gamma((X_k/W)_{\text{crys}}, \mathcal{V}^{\text{crys}}) \simeq \mathrm{R}\Gamma((X_k^{(n)}/W)_{\text{crys}}, (\mathcal{V}^{\text{crys}})^{(n)}).$$

If $\mathcal{V}$ is equipped with a Frobenius structure, then these isomorphisms are Frobenius-equivariant.

Proof. By the discussion in Section 2.2 and Corollary 3.6, we have natural identifications,

$$\mathrm{R}\Gamma_{\text{crys}}^{(n)}(\mathcal{V}) \simeq \phi_W^{*n} (F_X^* \mathrm{R}f_*^\Delta \mathcal{V})(W, p) \simeq \phi_W^{*(n+1)} \mathrm{R}\Gamma_\Delta(X_k/W, \mathcal{V}),$$

and the last term is naturally isomorphic to $\phi_W^{*n} \mathrm{R}\Gamma((X_k/W)_{\text{crys}}, \mathcal{V}^{\text{crys}})$ by [GR24, Theorem 6.4]. This shows the first natural isomorphism, and the second one follows from [SP, Tag 07MS, Tag 07MU, Tag 07MY]. The last claim follows by a similar argument and the same references. $\square$

In particular, we see that there is a natural identification,

$$\mathrm{R}\Gamma_{\text{crys}}^{(0)}(\mathcal{V}) \simeq \mathrm{R}\Gamma((X_k/W)_{\text{crys}}, \mathcal{V}^{\text{crys}}),$$

and we shall often implicitly make this identification. Finally, note that a Frobenius structure $\varphi_{\mathcal{V}}: F_X^* \mathcal{V}[1/\mathfrak{J}_\Delta] \xrightarrow{\sim} \mathcal{V}[1/\mathfrak{J}_\Delta]$ induces an isomorphism in $\mathbf{Perf}(K_0)$:

$$\overline{\varphi}_{\mathcal{V}}^{(n)}: \mathrm{R}\Gamma_{\text{crys}}^{(n)}(\mathcal{V})[1/p] \xrightarrow{\sim} \mathrm{R}\Gamma_{\text{crys}}^{(0)}(\mathcal{V})[1/p] = \mathrm{R}\Gamma((X_k/W)_{\text{crys}}, \mathcal{V}^{\text{crys}})[1/p], \quad (4.1)$$

interpreted either prismatically, or in terms of the Frobenius structure on crystalline cohomology (e.g. see [SP, Tag 07N5]), which agree via Proposition 4.3.

4.2. Twisted de Rham cohomology. We now come to the more subtle definition of twisted de Rham cohomology, which does not have a classical interpretation.

Definition 4.4. For $n \geq 0$, the n -twisted de Rham cohomology functor is given by

$$\mathrm{R}\Gamma_{\text{dR}}^{(n)}(-) = \mathrm{R}\Gamma_{\text{dR}}^{(n)}(X/\mathcal{O}_K, -): \mathbf{Perf}(X^\Delta) \longrightarrow \mathbf{Perf}(\mathcal{O}_K), \quad \mathcal{V} \mapsto \mathbb{D}_{\text{dR}}^{(n)}(\mathrm{R}f_*^\Delta \mathcal{V}).$$

We denote the associated cohomology groups by $H_{\text{dR}}^{i,(n)}(\mathcal{V}) = H_{\text{dR}}^{i,(n)}(X/\mathcal{O}_K, \mathcal{V})$, and when $\mathcal{V} = \mathcal{O}_\Delta$ we omit $\mathcal{V}$ from the notation.

We now observe that twisted de Rham cohomology recovers the de Rham cohomology in the non-twisted (i.e. $n = 0$) case. To state this precisely, recall that we have the relative de Rham stack $(X/\mathcal{O}_K)^{\text{dR}}$ as in [Bha23, Definition 2.5.3]. A coefficient sheaf for de Rham cohomology is a perfect complex $\mathcal{W}$ on $(X/\mathcal{O}_K)^{\text{dR}}$. Additionally, if $\mathcal{W}$ is a vector bundle, then the data of $\mathcal{W}$ is equivalent to the data of a vector bundle with integrable connection $(\mathcal{W}_0, \nabla)$ on X , and the cohomology of $\mathcal{W}$ is the usual cohomology of the de Rham complex associated to $(\mathcal{W}_0, \nabla)$ (see [Bha23, Remark 2.5.8]).

Next, recall that we have the following morphism of stacks (see [Bha23, Constructions 5.3.13 & 5.3.5 and Footnote 62]):

$$i_{\text{dR}, X/\mathcal{O}_K}: (X/\mathcal{O}_K)^{\text{dR}} \longrightarrow X^\Delta.$$

For an object $\mathcal{V}$ of $\mathbf{Perf}(X_\Delta)$, let us write $\mathcal{V}^{\text{dR}}$ for $i_{\text{dR}, X/\mathcal{O}_K}^*(\mathcal{V})$, and we claim the following:

Proposition 4.5. *There exists a natural identification in $\mathbf{Perf}(\mathcal{O}_K)$:*

$$\mathrm{R}\Gamma_{\text{dR}}^{(0)}(X/\mathcal{O}_K, \mathcal{V}) \simeq \mathrm{R}\Gamma_{\text{dR}}(\mathcal{V}^{\text{dR}}).$$

Proof. Note that we have the following commutative diagram of formal stacks:

$$\begin{array}{ccccc}
(X/\mathcal{O}_K)^{\mathrm{dR}} & \longrightarrow & X^{\mathrm{dR}} & \xrightarrow{i_{\mathrm{dR},X}} & X^{\Delta} \\
\downarrow & & \downarrow f^{\mathrm{dR}} & & \downarrow f^{\Delta} \\
\mathrm{Spf}(\mathcal{O}_K) & \longrightarrow & \mathcal{O}_K^{\mathrm{dR}} & \xrightarrow{i_{\mathrm{dR},\mathcal{O}_K}} & \mathcal{O}_K^{\Delta} \\
& & \downarrow & & \downarrow \\
& & \mathrm{Spf}(\mathbb{Z}_p) & \xrightarrow{i_{\mathrm{dR}}} & \mathbb{Z}_p^{\Delta},
\end{array} \tag{4.2}$$

where the composition of the top horizontal arrows defines $i_{\mathrm{dR},X/\mathcal{O}_K}$ (e.g. see [Bha23, Footnote 62]), and the bottom horizontal arrow i_{dR} is the restriction to $\mathbb{G}_m/\mathbb{G}_m$ of the map $i_{\mathrm{dR}}: \mathbb{A}^1/\mathbb{G}_m \rightarrow \mathbb{Z}_p^N$ from [Bha23, Construction 5.3.4], and this restriction identifies with the de Rham point of [BL22a, Example 3.2.6]. Additionally, using the definitions, it is easy to verify that in (4.2) the bottom right square and the outer square on the right are cartesian, and so, it follows that the top right square is also cartesian. Moreover, the top left square is cartesian by definition and thus it follows that the outer square on the top is also cartesian.

Next, we claim that the composition of the middle horizontal arrows in (4.2) identifies with the de Rham point ρ_{dR} of Definition 3.8. Consider the following commutative diagram of stacks:

$$\begin{array}{ccccccc}
& & \text{id} & & & & \\
& \nearrow & & \searrow & & & \\
\mathrm{Spf}(\mathcal{O}_K) = \mathrm{Spf}(\mathfrak{S}/E_{\pi}) & \xrightarrow{\rho_{(\mathfrak{S},E_{\pi})}^{\mathrm{HT}}} & \mathcal{O}_K^{\mathrm{HT}} & \longrightarrow & \mathrm{Spf}(\mathcal{O}_K) & \longrightarrow & \mathcal{O}_K^{\mathrm{dR}} \\
\downarrow & & \downarrow & & & & \downarrow i_{\mathrm{dR},\mathcal{O}_K} \\
\mathrm{Spf}(\mathfrak{S}) & \xrightarrow{\rho_{(\mathfrak{S},E_{\pi})}} & \mathcal{O}_K^{\Delta} & \xrightarrow{F_{\Delta}} & \mathcal{O}_K^{\Delta}, & &
\end{array}$$

where the composition of the left vertical arrow with the bottom horizontal arrows identifies with the de Rham point ρ_{dR} by Proposition 3.10, and it is clear that the composition of the top horizontal arrows with the right vertical arrow coincides with the composition of the middle horizontal arrows in (4.2). In the preceding diagram, the left square commutes by definition and the right square commutes by Lemma 4.6 below, hence, the claim follows.

Finally, to show the claimed identification on cohomologies, note that if we have derived base change for formal stacks, we have the following identifications:

$$R\Gamma_{\mathrm{dR}}(X/\mathcal{O}_K, \mathcal{V}^{\mathrm{dR}}) \simeq Rf_*^{\mathrm{dR}} i_{X,\mathrm{dR}}^* \simeq \rho_{\mathrm{dR}} Rf_*^{\Delta} \simeq R_{\mathrm{dR}}^{(0)}(X/\mathcal{O}_K, \mathcal{V}),$$

and thus it suffices to verify that derived base change holds. As the top outer square in (4.2) is Cartesian, by applying [Hau24, Proposition A.0.2] (using Proposition 2.8 and [BL22b, Lemma 6.3]), we see that it suffices to show that ρ_{dR} is a locally closed regular immersion. But, using the flat covering $\rho_{\mathfrak{S}}: \mathrm{Spf}(\mathfrak{S}) \rightarrow \mathcal{O}_K^{\Delta}$, we are reduced to observing that $\mathrm{Spf}(\mathcal{O}_K) \rightarrow \mathrm{Spf}(\mathfrak{S})$ is cut-out by the regular element E . This completes our proof. $\square$

The following observation was used above:

Lemma 4.6. *Let X be a bounded p -adic formal scheme. Then, the following diagram of formal stacks 2-commutes up to unique isomorphism:*

$$\begin{array}{ccccc}
X^{\mathrm{HT}} & \longrightarrow & X & \longrightarrow & X^{\mathrm{dR}} \\
\downarrow & & & & \downarrow i_{\mathrm{dR},X} \\
X^{\Delta} & \xrightarrow{F_{\Delta}} & & & X^{\Delta}.
\end{array}$$

Proof. By an argument similar to the proof of [BL22a, Proposition 3.6.6], it is easy to see that the following diagram of stacks commutes up to unique isomorphism:

$$\begin{array}{ccccc} \mathbb{G}_a^{\mathrm{HT}} & \longrightarrow & \mathbb{G}_a & \longrightarrow & \mathbb{G}_a^{\mathrm{dR}} \\ \downarrow & & & & \downarrow i_{\mathrm{dR}, \mathbb{G}_a} \\ \mathbb{G}_a^{\Delta} & \xrightarrow{F_{\Delta}} & & & \mathbb{G}_a^{\Delta}. \end{array}$$

Then, the claim follows by transmutation (cf. [Bha23, Remark 2.3.8]). $\square$

Moreover, we observe that in the case of trivial coefficients (i.e. $\mathcal{V} = \mathcal{O}_{\Delta}$) one can interpret the twisted de Rham cohomology as a more classical object:

Proposition 4.7. *There exists a functorial identification,*

$$\mathrm{R}\Gamma_{\mathrm{dR}}^{(n)}(X/\mathcal{O}_K) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_{p=0}^{(n)}/\mathcal{O}_K).$$

Proof. Choose a uniformiser π of $\mathcal{O}_K$. Then, by Corollary 3.11 we have an identification,

$$\mathrm{R}\Gamma_{\mathrm{dR}}^{(n)}(X/\mathcal{O}_K) \simeq \phi_{\mathfrak{S}}^{*(n+1)} \mathrm{R}\Gamma_{\mathfrak{S}}(X/\mathcal{O}_K)/E_{\pi}.$$

Applying the crystal property to the morphism $(\mathfrak{S}, E_{\pi}^{(1)}) \rightarrow (\tilde{S}, p)$ in $(\mathcal{O}_K)_{\Delta}$ from Proposition 2.13 and using the morphism $\tilde{S} \rightarrow \mathcal{O}_K$ from Proposition 2.11, gives us the following identification:

$$\phi_{\mathfrak{S}}^{*(n+1)} \mathrm{R}\Gamma_{\mathfrak{S}}(X/\mathcal{O}_K)/E_{\pi} \simeq \phi_{\tilde{S}}^{*n} \mathrm{R}\Gamma_{\tilde{S}}(X/\mathcal{O}_K) \otimes_{\tilde{S}} \mathcal{O}_K.$$

But, we have a further identification,

$$\phi_{\tilde{S}}^{*n} \mathrm{R}\Gamma_{\tilde{S}}(X/\mathcal{O}_K) \otimes_{\tilde{S}} \mathcal{O}_K \simeq \mathrm{R}\Gamma_{\mathrm{crys}}((X \otimes_{\mathcal{O}_K} \tilde{S}/p)^{(n)}/\tilde{S}) \otimes_{\tilde{S}} \mathcal{O}_K.$$

Moreover, the map $(\tilde{S}, p) \rightarrow (\mathcal{O}_K, p)$ is a morphism of PD-rings, so by [SP, Tag 07MS] we have an identification,

$$\mathrm{R}\Gamma_{\mathrm{crys}}((X \otimes_{\mathrm{Spf}(\mathcal{O}_K)} \mathrm{Spec}(\tilde{S}/p))^{(n)}/\tilde{S}) \otimes_{\tilde{S}} \mathcal{O}_K \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_{p=0}^{(n)}/\mathcal{O}_K).$$

Finally, putting everything together yields the following identification:

$$\mathrm{R}\Gamma_{\mathrm{dR}}^{(n)}(X/\mathcal{O}_K) \simeq \mathrm{R}\Gamma_{\mathrm{crys}}(X_{p=0}^{(n)}/\mathcal{O}_K),$$

and using an argument similar to that in the proof of Proposition 3.22 shows that the preceding identification is independent of the choice of uniformiser π . This allows us to conclude. $\square$

Remark 4.8. Despite Proposition 4.7 and in contrast to twisted crystalline cohomology, it is not possible to define twisted de Rham cohomology for general coefficients in a non-prismatic way. It may be tempting to say, in analogy with Proposition 4.7, that $\mathrm{R}\Gamma_{\mathrm{dR}}^{(n)}(\mathcal{V})$ should be isomorphic to $\mathrm{R}\Gamma_{\mathrm{crys}}(X_{p=0}^{(n)}/\mathcal{O}_K, ((\mathcal{V}_{p=0})^{\mathrm{crys}})^{(n)})$, where $(\mathcal{V}_{p=0})^{\mathrm{crys}}$ has the obvious meaning. However, the natural map $X_{p=0}^{(n)} \rightarrow X_{p=0}$ does not upgrade to a morphism of topoi,

$$(X_{p=0}^{(n)}/\mathcal{O}_K)_{\mathrm{crys}} \cdots \times \cdots \rightarrow (X_{p=0}/\mathcal{O}_K)_{\mathrm{crys}},$$

and so it does not seem possible to define $((\mathcal{V}_{p=0})^{\mathrm{crys}})^{(n)}$ in a purely crystalline way.

4.3. Berthelot–Ogus comparison isomorphism with coefficients. We now explain how to obtain an analogue of the Berthelot–Ogus comparison isomorphism (see [BO83, Corollary 2.5]) with coefficients in perfect prismatic F -crystals.

To begin, we observe the following cohomological consequence of Theorem 3.12.

Theorem 4.9. *Let $\mathcal{V}$ be a perfect prismatic crystal on X . Then, there exists a natural integral generalised Berthelot–Ogus isomorphism,*

$$\mathrm{R}\Gamma_{\mathrm{crys}}^{(n)}(\mathcal{V}) \otimes_W \mathcal{O}_K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}^{(n)}(\mathcal{V}).$$

To obtain a generalisation of the classical Berthelot–Ogus isomorphism with coefficients, we need to *rationally untwist* this isomorphism. The procedure to do this begins with the following construction.

Construction 4.10. Let $(\mathcal{V}, \varphi_{\mathcal{V}})$ be an object of $\mathbf{Perf}^{\varphi}((\mathcal{O}_K)_{\Delta})$ and $i \geq 0$. Evaluating $\mathcal{V}$ on the prism $(\mathfrak{S}, E_{\pi}^{(i)})$ and applying the Frobenius morphism $\varphi_{\mathcal{V}}$ gives us the following isomorphism:

$$\phi_{\mathfrak{S}}^* \mathcal{V}(\mathfrak{S}, E_{\pi}^{(i)})[1/E_{\pi}^{(i)}] \xrightarrow{\sim} \mathcal{V}(\mathfrak{S}, E_{\pi}^{(i)})[1/E_{\pi}^{(i)}]. \quad (4.3)$$

Using the morphism $\phi_{\mathfrak{S}}^i: (\mathfrak{S}, E_{\pi}) \rightarrow (\mathfrak{S}, E_{\pi}^{(i)})$ in $(\mathcal{O}_K)_{\Delta}$, we obtain that $\phi_{\mathfrak{S}}^{*i} \mathcal{V}(\mathfrak{S}, E_{\pi}) \simeq \mathcal{V}(\mathfrak{S}, E_{\pi}^{(i)})$. Therefore, we may interpret (4.3) as an isomorphism,

$$\phi_{\mathfrak{S}}^{*(i+1)} \mathcal{V}(\mathfrak{S}, E_{\pi})[1/E_{\pi}^{(i)}] \xrightarrow{\sim} \phi_{\mathfrak{S}}^{*i} \mathcal{V}(\mathfrak{S}, E_{\pi})[1/E_{\pi}^{(i)}].$$

Thus, iterating this procedure for any $n \geq 0$, yields the following isomorphism:

$$\varphi_{\mathcal{V}, \pi}^{(n)}: \phi_{\mathfrak{S}}^{*(n+1)} \mathcal{V}(\mathfrak{S}, E_{\pi})[1/E_{\pi}^{(n)} \dots E_{\pi}^{(1)}] \xrightarrow{\sim} \phi_{\mathfrak{S}}^* \mathcal{V}(\mathfrak{S}, E_{\pi})[1/E_{\pi}^{(n)} \dots E_{\pi}^{(1)}].$$

Our goal is to apply the morphism $\varphi_{\mathcal{V}, \pi}^{(n)}$ to rationally untwist the twists appearing in the twisted crystalline and de Rham realisations. For this, we make the following easy observation.

Lemma 4.11. *For every $i \geq 1$:*

$$E_{\pi}^{(i)} = p \cdot (\text{unit}) \bmod E_{\pi}, \quad E_{\pi}^{(i)} = p \cdot (\text{unit}) \bmod u.$$

Now, consider the following isomorphisms:

$$\varphi_{\text{crys}, \mathcal{V}, \pi}^{(n)}: \mathbb{D}_{\text{crys}, \pi}^{(n)}(\mathcal{V})[1/p] \xrightarrow{\sim} \mathbb{D}_{\text{crys}, \pi}(\mathcal{V})[1/p], \quad (\text{resp. } \varphi_{\text{dR}, \mathcal{V}, \pi}^{(n)}: \mathbb{D}_{\text{dR}}^{(n)}(\mathcal{V})[1/p] \xrightarrow{\sim} \mathbb{D}_{\text{dR}}(\mathcal{V})[1/p]),$$

in $\mathbf{Perf}(K_0)$ (resp. $\mathbf{Perf}(K)$) induced by the reduction of $\varphi_{\mathcal{V}, \pi}^{(n)}$ modulo u (resp. E_{π}). The preceding isomorphisms are functorial in $\mathcal{V}$, and so we obtain “rational untwisting” maps.

Proposition 4.12. *For any $n \geq 0$, there exists a natural equivalence in $\mathbf{Func}(\mathbf{Perf}^{\varphi}((\mathcal{O}_K)_{\Delta}), \mathbf{Perf}(K_0))$ (resp. $\mathbf{Func}(\mathbf{Perf}^{\varphi}((\mathcal{O}_K)_{\Delta}), \mathbf{Perf}(K))$),*

$$\varphi_{\text{crys}, \pi}^{(n)}: \mathbb{D}_{\text{crys}, \pi}^{(n)}[1/p] \xrightarrow{\sim} \mathbb{D}_{\text{crys}}[1/p], \quad (\text{resp. } \varphi_{\text{dR}, \pi}^{(n)}: \mathbb{D}_{\text{dR}, \pi}^{(n)}[1/p] \xrightarrow{\sim} \mathbb{D}_{\text{dR}}(\mathcal{V})[1/p]).$$

The following is proven similarly to Proposition 3.22.

Proposition 4.13. *For uniformisers π and π' of $\mathcal{O}_K$ and any integer $n \geq 0$, we have the following identification:*

$$j_{\text{crys}, \pi, \pi'}^{(0)}[1/p] \circ \varphi_{\text{crys}, \pi}^{(n)} = \varphi_{\text{crys}, \pi'}^{(n)} \circ j_{\text{crys}, \pi, \pi'}^{(n)}[1/p] \\ (\text{resp. } j_{\text{dR}, \pi, \pi'}^{(0)}[1/p] \circ \varphi_{\text{dR}, \pi}^{(n)} = \varphi_{\text{dR}, \pi'}^{(n)} \circ j_{\text{dR}, \pi, \pi'}^{(n)}[1/p]).$$

Using Propositions 4.12 and 4.13, the next definitions follows naturally.

Definition 4.14. For any $n \geq 0$, we define the following equivalence in $\mathbf{Func}(\mathbf{Perf}^{\varphi}((\mathcal{O}_K)_{\Delta}), \mathbf{Perf}(K_0))$ (resp. $\mathbf{Func}(\mathbf{Perf}^{\varphi}((\mathcal{O}_K)_{\Delta}), \mathbf{Perf}(K))$):

$$\varphi_{\text{crys}}^{(n)} := (\varphi_{\text{crys}, \pi}^{(n)}): \mathbb{D}_{\text{crys}}^{(n)}[1/p] \xrightarrow{\sim} \mathbb{D}_{\text{crys}}[1/p], \quad (\text{resp. } \varphi_{\text{dR}}^{(n)} := (\varphi_{\text{dR}, \pi}^{(n)}): \mathbb{D}_{\text{dR}}^{(n)}[1/p] \xrightarrow{\sim} \mathbb{D}_{\text{dR}}(\mathcal{V})[1/p]),$$

Thus, we obtain the following Berthelot–Ogus-type isomorphism, where the tensor products are computed entrywise (i.e. after applying the relevant functor to a perfect prismatic F -crystal).

Theorem 4.15. *Let $(\mathcal{V}, \varphi_{\mathcal{V}})$ be an object of $\mathbf{Perf}^{\varphi}((\mathcal{O}_K)_{\Delta})$. Then, for any $n \geq a$, we have a generalised Berthelot–Ogus isomorphism,*

$$\iota_{\text{BO}}^{(n)}: \mathbb{D}_{\text{crys}} \otimes_W K \xrightarrow{\sim} \mathbb{D}_{\text{dR}} \otimes_{\mathcal{O}_K} K,$$

such that the following diagram commutes,

$$\begin{array}{ccc} \mathbb{D}_{\text{crys}}^{(n)} \otimes_W K & \xrightarrow{\iota^{(n)}[1/p]} & \mathbb{D}_{\text{dR}}^{(n)} \otimes_{\mathcal{O}_K} K \\ \varphi_{\text{crys}}^{(n)} \downarrow & & \downarrow \varphi_{\text{dR}}^{(n)} \\ \mathbb{D}_{\text{crys}} \otimes_W K & \xrightarrow{\iota_{\text{BO}}^{(n)}} & \mathbb{D}_{\text{dR}} \otimes_{\mathcal{O}_K} K. \end{array}$$

As an application of Theorem 4.15, and using Corollaries 3.6 and 3.11, we recover the following analogue of the classical Berthelot–Ogus isomorphism for cohomology with coefficients.

Corollary 4.16. *Let $(\mathcal{V}, \varphi_{\mathcal{V}})$ be an object of $\mathbf{Perf}^{\varphi}((X)_{\Delta})$. Then, there exists a canonical isomorphism,*

$$\iota_{\mathrm{BO}}^{(n)}: \mathrm{R}\Gamma_{\mathrm{crys}}(\mathcal{V}^{\mathrm{crys}}) \otimes_W K \xrightarrow{\sim} \mathrm{R}\Gamma_{\mathrm{dR}}(\mathcal{V}^{\mathrm{dR}}) \otimes_{\mathcal{O}_K} K.$$

Remark 4.17. The isomorphism $\iota_{\mathrm{BO}}^{(a)}$ from Corollary 4.16 agrees with the isomorphism from [BO83] when $\mathcal{V} = \mathcal{O}^{\Delta}$. Indeed, this follows ultimately from two observations. The first is that the isomorphisms $\varphi_{\mathrm{crys}}^{(n)}$ and $\varphi_{\mathrm{dR}}^{(n)}$ agree with Frobenius maps on crystalline cohomology coming from the identifications in Propositions 4.3 and 4.7, respectively. The second is the observation that the argument for equivalency of structure maps in the proof of Theorem 3.12 precisely correspond to the Dwork-like isomorphism as in Equation (4) from the introduction.

4.4. A conjectural framework to study torsion in crystalline and de Rham cohomology.

In this final subsection we state a conjecture concerning the relationship between torsion in crystalline and de Rham cohomology, and explain how Theorem 3.12 gives a framework for its study. In Appendix A, we shall verify one of these conjectures in a small number of small cases.

The main conjecture. To state our conjecture, it is helpful to first set the following notation.

Notation 4.18. For any $n \geq 0$, we set

$$\ell_{\mathrm{crys}}^{i,(n)}(X/\mathcal{O}_K) := \ell_W(H_{\mathrm{crys}}^{i,(n)}(X_k/W)[p^{\infty}]), \quad \ell_{\mathrm{dR}}^{i,(n)}(X/\mathcal{O}_K) := \ell_{\mathcal{O}_K}(H_{\mathrm{dR}}^{i,(n)}(X/\mathcal{O}_K)[p^{\infty}]).$$

As $X/\mathcal{O}_K$ is fixed, we shall always omit it from the notation (i.e. just writing $\ell_{\mathrm{crys}}^{i,(n)}$ and $\ell_{\mathrm{dR}}^{i,(n)}$). Moreover, we shorten $\ell_{\mathrm{crys}}^{i,(0)}$ and $\ell_{\mathrm{dR}}^{i,(0)}$ to ℓ_{crys}^i and ℓ_{dR}^i , respectively.

Conjecture 4.19 (Main Conjecture). *For all $i \geq 0$, we have the following inequalities:*

$$\ell_{\mathrm{crys}}^i \leq \ell_{\mathrm{dR}}^i \leq e \cdot \ell_{\mathrm{crys}}^i.$$

For clarity, let us write

$$H_{\mathrm{crys}}^i(X_k/W)[p^{\infty}] \simeq \bigoplus_{i=1}^r W/p^{a_i}, \quad H_{\mathrm{dR}}^i(X/\mathcal{O}_K)[p^{\infty}] \simeq \bigoplus_{j=1}^s \mathcal{O}_K/\pi^{b_j}.$$

Then, $r = s$ by [ČK19, Remark 7.8], and thus Conjecture 4.26 can be understood as giving a rough relationship between the size of a_i and the size of b_j , as $\ell_{\mathrm{crys}}^i = \sum_{i=1}^r a_i$ and $\ell_{\mathrm{dR}}^i = \sum_{j=1}^s b_j$.

To help understand this conjecture, we now examine several non-trivial examples. In particular, we discuss an example where the second inequality in Conjecture 4.19 is strict for which it is unclear whether this has been previously recorded in the literature (cf. [ČK19, Question 7.13]).

Example 4.20. Suppose X admits a descent to $\mathcal{O}_{K'}$ for some subextension K' of K/K_0 of ramification index at most $p-1$. Then, $\ell_{\mathrm{crys}}^i = \ell_{\mathrm{dR}}^i$. Indeed, by base change we are reduced to the case $K = K'$. We have $\lceil \log_p(\frac{e}{p-1}) \rceil = 0$, and so the claim follows from Theorem 4.9 (or, more simply, from [BO78, Corollary 7.4]).

Example 4.21 (Li–Petrov). There exists a finite extension K/K_0 with $e = p^4 - p^2$ and an elliptic scheme $E/\mathcal{O}_K$ such that E_k is supersingular, and such that $E[p^2](K)$ contains a point x of order p^2 . For this E , let H denote the integral closure in E of the subgroup generated by x . Set $X = BH$ to be the classifying stack for H , a smooth proper formal $\mathcal{O}_K$ -stack. Then, Li and Petrov compute that,

$$\begin{aligned} H_{\mathrm{crys}}^2(X_k/W) &= k^2, & H_{\mathrm{crys}}^3(X/\mathcal{O}_K) &= k, \\ H_{\mathrm{dR}}^2(X/\mathcal{O}_K) &= \mathcal{O}_K/\pi^{2e-p^3+p^2} \oplus \mathcal{O}_K/\pi^{p^3-p^2}, & H_{\mathrm{dR}}^3(X/\mathcal{O}_K) &= \mathcal{O}_K/\pi^{p^3-p^2}. \end{aligned}$$

From this, we observe that,

$$e \cdot \ell_{\mathrm{crys}}^2 = 2e, \quad \ell_{\mathrm{dR}}^2 = (2e - p^3 + p^2) + (p^3 - p^2) = 2e,$$

so that $\ell_{\mathrm{crys}}^2 < \ell_{\mathrm{dR}}^2 = e \cdot \ell_{\mathrm{crys}}^2$, although it is clear that,

$$H_{\mathrm{crys}}^2(X/\mathcal{O}_K) \otimes_W \mathcal{O}_K \not\simeq H_{\mathrm{dR}}^2(X/\mathcal{O}_K).$$

Furthermore, we observe that,

$$e \cdot \ell_{\text{crys}}^3 = e = p^4 - p^2, \quad \ell_{\text{dR}}^3 = p^3 - p^2,$$

so that $1 = \ell_{\text{crys}}^3 < \ell_{\text{dR}}^3 < e \cdot \ell_{\text{crys}}^3$. One may further replace X by an actual formal scheme by a standard approximation technique (cf. [LL23, Construction 6.12 and Proposition 6.13]).

Example 4.22. Suppose that $H_{\text{dR}}^i(X/\mathcal{O}_K)[p^\infty]$ is p -torsion. In this case, we have that $1 \leq b_j \leq e$ for all j , and $a_i \geq 1$ for all i . Thus, we see that,

$$\ell_{\text{dR}}^i = \sum_{j=1}^s b_j \leq e \cdot s = e \cdot r \leq e \cdot \sum_{i=1}^r a_i = e \cdot \ell_{\text{crys}}^i.$$

Additionally, if $H_{\text{crys}}^i(X_k/W)[p^\infty]$ is also p -torsion, then $a_i = 1$ for all i , and we see that,

$$\ell_{\text{crys}}^i = \sum_{i=1}^r a_i = r = s \leq \sum_{j=1}^s b_j = \ell_{\text{dR}}^i.$$

Remark 4.23. One can use Example 4.22 to illustrate the subtlety of Conjecture 4.19. Suppose that $H_{\text{crys}}^i(X_k/W)[p^\infty]$ and $H_{\text{dR}}^i(X/\mathcal{O}_K)[p^\infty]$ are both p^2 -torsion, so $a_i \in \{1, 2\}$ for all i . Write r_1 for the number of i with $a_i = 1$ and define r_2 analogously. Thus, $\ell_{\text{crys}}^i = r_1 + 2r_2$. Our assumptions imply only that $b_j \leq 2e$ for all j . Then, the naive approach taken in Conjecture 4.19 only implies that $\ell_{\text{dR}}^i \leq 2er$, whereas $e \cdot \ell_{\text{crys}}^i = er_1 + 2er_2$, and thus this is not helpful if $r_1 \neq 0$.

Questions about torsion in Breuil–Kisin cohomology. We now explain how Theorem 4.9 allows one to reduce Conjecture 4.19 to questions about torsion in Breuil–Kisin cohomology.

We begin by making the following simple but crucial observation, coming from Proposition 4.3 and the fact that $\phi_W: W \xrightarrow{\sim} W$ is an isomorphism of rings.

Proposition 4.24. *The equality $\ell_{\text{crys}}^{i,(n)} = \ell_{\text{crys}}^i$ holds for all $n \geq 0$.*

Now, from Theorem 3.12 it follows that we have the equality,

$$\ell_{\text{dR}}^{i,(n)} = e \cdot \ell_{\text{crys}}^{i,(n)}, \quad \text{for all } n \geq 0. \quad (4.4)$$

This indicates a refinement of Conjecture 4.19, namely that the following two statements hold:

- (L1) $\ell_{\text{dR}}^{i,(a)} \leq e \cdot \ell_{\text{dR}}^i$,
- (L2) $\ell_{\text{dR}}^{i,(n)}$ is a non-decreasing quantity in n .

Remark 4.25. One interesting and non-obvious consequence of Proposition 4.24 and (4.4) is that the quantity $\ell_{\text{dR}}^{i,(n)}$ is constant for $n \geq 0$.

To make the preceding statements more tractable, we would like to instead formulate them in terms of the highly structured u^∞ -torsion in Breuil–Kisin cohomology. In the following, for a Breuil–Kisin module $\mathfrak{M}$ (see [BMS18, Section 4]), we shall write $\mathfrak{M}[u^\infty]$ for its u^∞ -torsion submodule and $\overline{\mathfrak{M}} := \mathfrak{M}^{\vee\vee}/\mathfrak{M}_{\text{tf}}$ (see Section A.1 for details). Our main conjecture concerns the preceding modules arising from Breuil–Kisin cohomology. Additionally, for a (complex of) $\mathfrak{S}$ -module(s) $\mathfrak{M}$ we shorten the notation $\phi_{\mathfrak{S}}^{*n}\mathfrak{M}$ to $\mathfrak{M}^{(n)}$.

Conjecture 4.26. *Set $\mathfrak{M}^i := H_{\mathfrak{S}}^i(X/\mathcal{O}_K)$ and let $f(n)$ denote either $\ell_{\mathcal{O}_K}(\mathfrak{M}^i[u^\infty]^{(n+1)}[E])$ or $\ell_{\mathcal{O}_K}(\overline{\mathfrak{M}^i}^{(n+1)}[E])$. Then, f satisfies the following:*

- (1) $f(a) \leq e \cdot f(0)$,
- (2) $f(n) \leq f(n+1)$, for all n .

Remark 4.27. A neater conjecture would be that if we set $\mathfrak{M}_m^i := H^i(\text{R}\Gamma_{\mathfrak{S}}(X/\mathcal{O}_K)/p^m)$ and $f_m(n) := \ell_{\mathcal{O}_K}(\mathfrak{M}_m^i[u^\infty]^{(n+1)})$, then $f_m(n) \leq f_m(n+1)$ for every m . The precise relationship between this statement and Conjecture 4.26 is unclear, but it seems somewhat likely that the former is stronger than the later. Corollary A.24 and its proof may be seen as evidence for this.

Conjecture 4.26 is not sufficient to obtain **(L1)** and **(L2)**, and so a further hypothesis is required. To state this, observe that as $H_{\text{dR}}^{i,(n)}(X/\mathcal{O}_K)$ is defined as the i^{th} -cohomology of $\text{R}\Gamma_{\mathfrak{S}}(X/\mathcal{O}_K)^{(n)}/E$, one may apply [SP, Tag 061Z] to obtain the exact sequence,

$$0 \longrightarrow H_{\mathfrak{S}}^i(X/\mathcal{O}_K)^{(n+1)}/E \longrightarrow H_{\text{dR}}^{i,(n)}(X/\mathcal{O}_K) \longrightarrow H_{\mathfrak{S}}^{i+1}(X/\mathcal{O}_K)^{(n+1)}[E] \longrightarrow 0, \quad (4.5)$$

which may fail to be exact on the right when one passes to p^∞ -torsion submodules.

Hypothesis 4.28. *Set $\mathfrak{Q}^{i,(n)}$ to be the image of $H_{\text{dR}}^{i,(n)}(X/\mathcal{O}_K)[p^\infty]$ in $H_{\mathfrak{S}}^{i+1}(X/\mathcal{O}_K)^{(n+1)}[E]$. Then,*

$$\ell_{\mathcal{O}_K}(\mathfrak{Q}^{i,(a)}) \geq \ell_{\mathcal{O}_K}(\mathfrak{Q}^{i,(0)}), \quad \text{and} \quad \ell_{\mathcal{O}_K}(\mathfrak{Q}^{i,(n+1)}) \geq \ell_{\mathcal{O}_K}(\mathfrak{Q}^{i,(n)}) \text{ for all } n \geq 0.$$

Remark 4.29. We have chosen not to state Hypothesis 4.28 as a conjecture due to a lack of substantial evidence. That said, it is reasonable to believe that it holds in all cases. In fact, the authors suspect that $\mathfrak{Q}^{i,(n)} = H_{\mathfrak{S}}^{i+1}(X/\mathcal{O}_K)^{(n+1)}[E]$ for all sufficiently large n , which would allow one to reduce the verification of Hypothesis 4.28 to the verification of Conjecture 4.26.

The precise relationship between Conjecture 4.19 and Conjecture 4.26 can now be made precise. The proof of the following proposition makes use of several routine calculations concerning Breuil–Kisin modules done in Appendix A and so we postpone its proof until Section A.3.1.

Proposition 4.30. *Under Hypothesis 4.28, Conjecture 4.26 implies Conjecture 4.19.*

As remarked in the introduction and explained via Proposition 4.30, we hope that Conjecture 4.26 will be useful in proving Conjecture 4.19 as the u^∞ -torsion in Breuil–Kisin cohomology has incredibly rich structure (e.g. see [LL23] and [GL25]). To illustrate, we exploit this structure to prove Conjecture 4.26 in Appendix A in a small number of non-trivial cases (see Theorem A.20).

Remark 4.31. When $\mathcal{O}_K = W[\zeta_{p^m}]$, it is also reasonable to use the q -de Rham prism and its variants (see [BS22, Section 16], [BL22a, Section 2.6] and [Abh24, Section 2]) instead of the Breuil–Kisin prism, in order to better understand torsion in de Rham and crystalline cohomologies. This would require a robust understanding of the structure of torsion in q -de Rham cohomology, analogous to the u^∞ -torsion in Breuil–Kisin cohomology. We intend to explore these ideas and their consequences in a future work.

We end by giving an interesting non-trivial example where Conjecture 4.26 holds.

Example 4.32. Let $G \rightarrow H$ be a map of finite flat $\mathcal{O}_K$ -group schemes which is generically an isomorphism, and let $A/\mathcal{O}_K$ be an abelian scheme with $H \subseteq A$ (see [BBM82, Théorème 3.1.1]).

Write $\mathfrak{M}(\bullet)$ for the Breuil–Kisin Dieudonné functor of Kisin (see [Kis06, Theorem 2.3.5]). Then, a forthcoming result of Kubrak–Li–Mondal implies that setting $X = [A/G]$, a smooth proper formal $\mathcal{O}_K$ -stack, and $A' = A/H$, an abelian scheme over $\mathcal{O}_K$, then we have that

$$H_{\mathfrak{S}}^i(X/\mathcal{O}_K) \simeq H_{\mathfrak{S}}^i(X/\mathcal{O}_K)[u^\infty] \oplus H_{\mathfrak{S}}^i(A'/\mathcal{O}_K), \quad \text{for } i \geq 0,$$

$$H_{\mathfrak{S}}^1(X/\mathcal{O}_K)_{\text{tor}} = 0 \quad \text{and} \quad H_{\mathfrak{S}}^2(X/\mathcal{O}_K)_{\text{tor}} \simeq \text{coker}(\mathfrak{M}(H) \rightarrow \mathfrak{M}(G)) =: \mathfrak{M}.$$

So, now assume that K contains $K_0[\zeta_{p^m}]$, and consider the map $\mathbb{Z}/p^m \rightarrow \mu_{p^m}$ sending 1 to $\zeta_{p^m} \in \mu_{p^m}(\mathcal{O}_K)$, which clearly satisfies the desired conditions. In this case, we have that $\mathfrak{M} \simeq k$ for which Conjecture 4.26 clearly holds.

APPENDIX A. SOME CALCULATIONS INVOLVING BREUIL–KISIN MODULES

We continue to use the notation from global notation and conventions, Notation 2.1 and the ones discussed in Section 2.3. In addition, we often make use of the following pieces of notation.

Notation A.1. For a $\mathfrak{S}$ -module $\mathfrak{M}$ and integer $n \geq 0$, we write $\mathfrak{M}^{(n)}$ as a shorthand for $\mathfrak{M} \otimes_{\mathfrak{S}, \phi_{\mathfrak{S}}^n} \mathfrak{S}$. For any subscript $?$, one should interpret $\mathfrak{M}_?^{(n)}$ as $(\mathfrak{M}_?)^{(n)}$. Finally, as we use it so frequently, we also note the notation from Proposition A.3 below.

A.1. Preliminaries on Breuil–Kisin modules. In this section, we will collect some basic results on Breuil–Kisin modules studied in [BMS18, LL20].

A.1.1. Generalised Breuil–Kisin modules. Recall that an *effective φ -module* over $\mathfrak{S}$ is a module $\mathfrak{M}$ equipped with an $\mathfrak{S}$ -linear map $\varphi_{\mathfrak{M}}: \mathfrak{M}^{(1)} \rightarrow \mathfrak{M}$.

Definition A.2. A *generalised Breuil–Kisin module of height $i \geq 0$* is a finite type effective φ -module over $\mathfrak{S}$ such that there exists an $\mathfrak{S}$ -linear map $\psi_{\mathfrak{M}}: \mathfrak{M} \rightarrow \mathfrak{M}^{(1)}$ satisfying the following:

- (1) $\psi_{\mathfrak{M}} \circ \varphi_{\mathfrak{M}} = E^i \cdot \text{id}_{\mathfrak{M}^{(1)}}$,
- (2) $\varphi_{\mathfrak{M}} \circ \psi_{\mathfrak{M}} = E^i \cdot \text{id}_{\mathfrak{M}}$.

Morphisms between generalised Breuil–Kisin modules are $\mathfrak{S}$ -linear, Frobenius equivariant maps.

Proposition A.3 ([BMS18, Proposition 4.3]). *Let $\mathfrak{M}$ be a generalised Breuil–Kisin module. Then, there exist canonical short exact sequences of generalised Breuil–Kisin modules:*

$$0 \longrightarrow \mathfrak{M}_{\text{tor}} \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}_{\text{tf}} \longrightarrow 0 \quad (\text{A.1})$$

$$0 \longrightarrow \mathfrak{M}_{\text{tf}} \longrightarrow \mathfrak{M}_{\text{free}} \longrightarrow \overline{\mathfrak{M}} \longrightarrow 0, \quad (\text{A.2})$$

where $\mathfrak{M}_{\text{tor}} \subset \mathfrak{M}$ is the torsion submodule and is killed by a power of p , the module $\mathfrak{M}_{\text{tf}}$ is torsion-free, the module $\mathfrak{M}_{\text{free}}$ is free over $\mathfrak{S}$ and given as the reflexive hull of $\mathfrak{M}_{\text{tf}}$ over $\mathfrak{S}$, and the module $\overline{\mathfrak{M}}$ is a torsion $\mathfrak{S}$ -module and killed by a power of (p, u) .

Let $\mathfrak{M}[u^\infty]$ denote the u -power torsion submodule of $\mathfrak{M}$, a generalised Breuil–Kisin module height i , and observe that we have,

$$\varphi_{\mathfrak{M}}(\mathfrak{M}[u^\infty]^{(1)}) \subset \mathfrak{M}[u^\infty], \quad \text{and} \quad \psi_{\mathfrak{M}}(\mathfrak{M}[u^\infty]) \subset \mathfrak{M}[u^\infty]^{(1)}.$$

Lemma A.4 ([LL20, Lemma 6.2]). *The following is a short exact sequence in $\text{Mod}_{\mathfrak{S}}^{\varphi, i}$:*

$$0 \longrightarrow \mathfrak{M}[u^\infty] \longrightarrow \mathfrak{M} \longrightarrow \mathfrak{M}/\mathfrak{M}[u^\infty] \longrightarrow 0. \quad (\text{A.3})$$

Lemma A.5 ([LL20, Lemma 6.3]). *Let $\mathfrak{M}$ be a generalised Breuil–Kisin module of height i and assume that it is killed by p^n for some $n \geq 0$. Then, the following are equivalent:*

- (1) *The module $\mathfrak{M}$ is u -torsion free,*
- (2) *$\mathfrak{M} = \mathfrak{N}/\mathfrak{N}'$, where $\mathfrak{N}' \subset \mathfrak{N}$ and both are finite free Breuil–Kisin modules of height i ,*
- (3) *the module $\mathfrak{M}$ may be written as a successive extension of finite free $k[[u]]$ -modules $\mathfrak{M}_j$ such that each $\mathfrak{M}_j$ is a generalised Breuil–Kisin module of height i .*

Finally, we note a structural result on generalised Breuil–Kisin modules modulo powers of p .

Lemma A.6 ([GL25, Lemma 3.5]). *Let $\mathfrak{M}$ be a generalised Breuil–Kisin module of height i and $n \geq 0$ be an integer such that p^n kills both $\mathfrak{M}_{\text{tor}}$ and $\overline{\mathfrak{M}}$. Then, for $N \geq 2n$, the sequence*

$$0 \longrightarrow \mathfrak{M}_{\text{tor}} \oplus \overline{\mathfrak{M}} \longrightarrow \mathfrak{M}/p^N \mathfrak{M} \longrightarrow \mathfrak{M}_{\text{free}}/p^N \mathfrak{M}_{\text{free}} \longrightarrow \overline{\mathfrak{M}} \longrightarrow 0, \quad (\text{A.4})$$

is exact. In particular, we have an isomorphism of $\mathfrak{S}$ -modules $(\mathfrak{M}/p^N \mathfrak{M})[u^\infty] \xrightarrow{\sim} \mathfrak{M}[u^\infty] \oplus \overline{\mathfrak{M}}$.

A.1.2. Structure of the u^∞ -torsion in Breuil–Kisin cohomology.

Definition A.7 ([LL23, Definition 3.1]). A *quasi-filtered Breuil–Kisin module of height $i \geq 1$* consists of the following set of data:

- Two $\mathfrak{S}$ -modules $\mathfrak{M}$ and $\mathfrak{N}$;
- Four $\mathfrak{S}$ -linear maps $f: \mathfrak{M}^{(1)} \rightarrow \mathfrak{N}$, $g: \mathfrak{N} \rightarrow \mathfrak{M}^{(1)}$, $h: \mathfrak{N} \rightarrow \mathfrak{M}$ and $h': \mathfrak{M} \rightarrow \mathfrak{N}$;

satisfying the following conditions:

- (1) The map h is injective.
- (2) We have $g \circ f = E^{i-1} \cdot \text{id}_{\mathfrak{M}^{(1)}}$, $f \circ g = E^{i-1} \cdot \text{id}_{\mathfrak{N}}$, $h' \circ h = E \cdot \text{id}_{\mathfrak{N}}$ and $h \circ h' = E \cdot \text{id}_{\mathfrak{M}}$.

Note that in Definition A.7, $\mathfrak{M}$ is Breuil–Kisin module of height i with $\varphi_{\mathfrak{M}} := h \circ f$ and $\psi_{\mathfrak{M}} = g \circ h'$. When the extra structure of a quasi-filtered Breuil–Kisin module beyond this is unimportant, we abuse notation and just refer to $\mathfrak{M}$ as the quasi-filtered Breuil–Kisin module.

Proposition A.8 ([LL23, Proposition 3.3 & Corollary 3.4]). *Let $\mathfrak{M}$ be a quasi-filtered Breuil–Kisin module of height i . Then, the annihilator ideal of $\mathfrak{M}$ admits the following restriction:*

$$E^{i-1} \cdot \text{Ann}(\mathfrak{M}) \subset \text{Ann}(\mathfrak{M}^{(1)}) = \text{Ann}(\mathfrak{M})^{(1)}. \quad (\text{A.5})$$

Moreover, if there exists an integer $\alpha \geq 0$ such that $\text{Ann}(\mathfrak{M}) + (p) = (p, u^\alpha)$, then $\alpha \leq \lfloor \frac{e(i-1)}{p-1} \rfloor$.

In what follows, we denote quantity α appearing in Proposition A.8 by $\alpha(\mathfrak{M})$.

Theorem A.9 ([LL23, Theorem 3.6] and [GL25, Theorem 3.1]). *Let $\mathfrak{M}$ be a u^∞ -torsion quasi-filtered Breuil–Kisin module. Then, we have the following:*

- (1) if $e(i-1) < p-1$, then $\mathfrak{M} = 0$,
- (2) if $e(i-1) = p-1$, then $\text{Ann}(\mathfrak{M}) = (p, u)$,
- (3) if $e(i-1) < 2(p-1)$, then $\text{Ann}(\mathfrak{M}) + (u) \supset (p^{i-1}, u)$,
- (4) if $i \leq 2$ and $e < p(p-1)$, then $\text{Ann}(\mathfrak{M}) + (u) = (p, u)$.

Using this, we are able to show that for small height and ramification, the annihilator of $\mathfrak{M}$ must contain an element of a particularly simple form.

Proposition A.10. *Let $\mathfrak{M}$ be a u^∞ -torsion quasi-filtered Breuil–Kisin module of height $i \leq 2$, and assume $e < p(p-1)$ and write $\alpha = \alpha(\mathfrak{M})$. Then, $1 \leq \alpha \leq \lceil \frac{e}{p-1} \rceil < p$, and either*

- $\mathfrak{M}$ is p -torsion,
- or $\text{Ann}(\mathfrak{M})$ contains an element of the form $u^\alpha + px$ with x in $\mathfrak{S}^\times$.

Proof. For $i = 0, 1$, Theorem A.9 (1) implies that $\mathfrak{M} = 0$. So, we assume that $i = 2$. Moreover, the inequality concerning α follows from Proposition A.8. So, we focus only on the last claim.

From Theorem A.9 and the definition of α we have that,

$$\text{Ann}(\mathfrak{M}) + (p) = (p, u^\alpha), \quad \text{and} \quad \text{Ann}(\mathfrak{M}) + (u) = (p, u).$$

So, let us write $f = u^\alpha + pb$ and $g = p + uc$, for some f and g in $\text{Ann}(\mathfrak{M}[u^\infty])$ with b and c in $\mathfrak{S}$. Moreover, we may write $c = \sum_{i \geq 0} c_i u^i$, with each c_i in W . If for each $i \geq 0$, we have that $c_i = pd_i$ with d_i in W , then observe that $g = p(1 + u \sum_{i \geq 0} d_i u^i)$, in particular, p is in $\text{Ann}(\mathfrak{M})$. Otherwise, we may assume that there exists a minimal $r \geq 1$ such that c_{r-1} is in $W^\times$.

Let us write $c_i = pd_i$ with d_i in W and $0 \leq i \leq r-2$, and observe that

$$g = p(1 + d_1 u + \cdots + d_{r-2} u^{r-1}) + u^r \sum_{i \geq 0} c_{i+r-1} u^i \in \text{Ann}(\mathfrak{M}). \quad (\text{A.6})$$

If $r \geq \alpha + 1$, then substituting $u^\alpha = f - pb$ in (A.6), we obtain that

$$p(1 + d_1 u + \cdots + d_{r-2} u^{r-1}) - pb u^{r-\alpha} \sum_{i \geq 0} c_{i+r-1} u^i \in \text{Ann}(\mathfrak{M}).$$

In particular, if $r \geq \alpha + 1$, then it follows that p is in $\text{Ann}(\mathfrak{M})$. On the other hand, if $1 \leq r \leq \alpha$, then using (A.6), it easily follows that $g = pd + u^r a'$, with d and a' in $\mathfrak{S}^\times$, is in $\text{Ann}(\mathfrak{M})$. As α is minimal, it follows that $1 \leq r = \alpha$. This allows us to conclude. $\square$

We end this section with the following simple observation.

Lemma A.11. *Let $\mathfrak{N}$ be a finite $\mathfrak{S}$ -module. Assume that there exists $f = u^r + px$ in $\text{Ann}(\mathfrak{N})$ with x in $\mathfrak{S}^\times$. If E is in $\text{Ann}(\mathfrak{N})$ and $e \neq r$, then $\mathfrak{N}$ is killed by p .*

A.1.3. E -torsion and reduction modulo E of Frobenius twists. If $\mathfrak{M}$ is a generalised Breuil–Kisin module and $n \geq 0$, then (A.1) and (A.2), give exact sequences:

$$0 \longrightarrow \mathfrak{M}_{\text{tor}}^{(n)} \longrightarrow \mathfrak{M}^{(n)} \longrightarrow \mathfrak{M}_{\text{tf}}^{(n)} \longrightarrow 0 \quad (\text{A.7})$$

$$0 \longrightarrow \mathfrak{M}_{\text{tf}}^{(n)} \longrightarrow \mathfrak{M}_{\text{free}}^{(n)} \longrightarrow \overline{\mathfrak{M}}^{(n)} \longrightarrow 0, \quad (\text{A.8})$$

Reducing (A.7) modulo E , yields the following long exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathfrak{M}_{\text{tor}}^{(n)}[E] \longrightarrow \mathfrak{M}^{(n)}[E] \longrightarrow \mathfrak{M}_{\text{tf}}^{(n)}[E] \longrightarrow \\ \longrightarrow \mathfrak{M}_{\text{tor}}^{(n)}/E \longrightarrow \mathfrak{M}^{(n)}/E \longrightarrow \mathfrak{M}_{\text{tf}}^{(n)}/E \longrightarrow 0. \end{aligned} \quad (\text{A.9})$$

Lemma A.12. Equation (A.9) yields the following isomorphism of finite torsion $\mathcal{O}_K$ -modules:

$$\mathfrak{M}_{\text{tor}}^{(n)}[E] \xrightarrow{\sim} \mathfrak{M}^{(n)}[E] = (\mathfrak{M}^{(n)}[E])[p^\infty]. \quad (\text{A.10})$$

Proof. In the long exact sequence (A.9), let us note that $\mathfrak{M}_{\text{tf}}^{(n)}[E] = 0$. As $\mathfrak{M}_{\text{tor}}^{(n)}$ is a p -power torsion $\mathfrak{S}$ -module, because $\mathfrak{M}_{\text{tor}}$ is, the isomorphism in (A.10) follows. $\square$

Lemma A.13. The following sequence of finite torsion $\mathcal{O}_K$ -modules is exact:

$$0 \longrightarrow \mathfrak{M}_{\text{tor}}^{(n)}/E \longrightarrow \mathfrak{M}^{(n)}/E \longrightarrow \mathfrak{M}_{\text{tf}}^{(n)}/E \longrightarrow 0. \quad (\text{A.11})$$

Additionally, we have the following short exact sequence of finite torsion $\mathcal{O}_K$ -modules:

$$0 \longrightarrow \mathfrak{M}_{\text{tor}}^{(n)}/E \longrightarrow (\mathfrak{M}^{(n)}/E)[p^\infty] \longrightarrow (\mathfrak{M}_{\text{tf}}^{(n)}/E)[p^\infty] \longrightarrow 0. \quad (\text{A.12})$$

Proof. By combining (A.9) and Lemma A.10, we obtain the short exact sequence in (A.11). Next, in the sequence (A.12), the only non-obvious part is the surjectivity of the following map:

$$(\mathfrak{M}^{(n)}/E)[p^\infty] \longrightarrow (\mathfrak{M}_{\text{tf}}^{(n)}/E)[p^\infty].$$

But, this easily follows from the fact that $\mathfrak{M}_{\text{tor}}^{(n)}$, and thus $\mathfrak{M}_{\text{tor}}^{(n)}/E$, is p^∞ -torsion. $\square$

Next, let us note that we have $\mathfrak{M}[u^\infty] = \mathfrak{M}_{\text{tor}}[u^\infty]$ and consider the following short exact sequence of p^∞ -torsion $\mathfrak{S}$ -modules:

$$0 \longrightarrow \mathfrak{M}[u^\infty] \longrightarrow \mathfrak{M}_{\text{tor}} \longrightarrow \mathfrak{M}_{\text{tor}}/\mathfrak{M}[u^\infty] \longrightarrow 0. \quad (\text{A.13})$$

Then, $\mathfrak{M}_{\text{tor}, u\text{-tf}} := \mathfrak{M}_{\text{tor}}/\mathfrak{M}[u^\infty]$ is u -torsion free.

Lemma A.14. For any $n \geq 0$, we have that $\mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n)}[E] = 0$.

Proof. Note that $\mathfrak{M}_{\text{tor}, u\text{-tf}}$ is a p^∞ -torsion and u -torsion-free Breuil–Kisin module. So, from Lemma A.5, there exist finite free Breuil–Kisin modules $\mathfrak{N}$ and $\mathfrak{N}'$ with $\mathfrak{N}' \subset \mathfrak{N}$ and $\mathfrak{M}_{\text{tor}, u\text{-tf}} \xrightarrow{\sim} \mathfrak{N}'/\mathfrak{N}$. As $\phi_{\mathfrak{S}}$ is flat, we may twist by $\phi_{\mathfrak{S}}$ and reduce modulo E to obtain the exact sequence,

$$0 \longrightarrow \mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n)}[E] \longrightarrow \mathfrak{N}'^{(n)}/E \longrightarrow \mathfrak{N}^{(n)}/E.$$

As $\mathfrak{N}'^{(n)}/E$ is a free $\mathcal{O}_K$ -module, the claim follows. $\square$

Now, Frobenius twisting the short exact sequence (A.13) and reducing modulo E yields the following long exact sequence of $\mathfrak{S}$ -modules:

$$\begin{aligned} 0 \longrightarrow \mathfrak{M}[u^\infty]^{(n)}[E] \longrightarrow \mathfrak{M}_{\text{tor}}^{(n)}[E] \longrightarrow \mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n)}[E] \longrightarrow \\ \longrightarrow \mathfrak{M}[u^\infty]^{(n)}/E \longrightarrow \mathfrak{M}_{\text{tor}}^{(n)}/E \longrightarrow \mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n)}/E \longrightarrow 0. \end{aligned} \quad (\text{A.14})$$

By combining Equation (A.14) and Lemma A.14, we obtain the following:

Lemma A.15. For a generalised Breuil–Kisin module $\mathfrak{M}$, there is an isomorphism of $\mathcal{O}_K$ -modules:

$$\mathfrak{M}[u^\infty]^{(n)}[E] \xrightarrow{\sim} \mathfrak{M}_{\text{tor}}^{(n)}[E]. \quad (\text{A.15})$$

Moreover, the following sequence of finite torsion $\mathcal{O}_K$ -modules is exact:

$$0 \longrightarrow \mathfrak{M}[u^\infty]^{(n)}/E \longrightarrow \mathfrak{M}_{\text{tor}}^{(n)}/E \longrightarrow \mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n)}/E \longrightarrow 0. \quad (\text{A.16})$$

Finally, note that by reducing the short exact sequence (A.8) modulo E one obtains the following long exact sequence:

$$\begin{aligned} 0 \longrightarrow \mathfrak{M}_{\text{tf}}^{(n)}[E] \longrightarrow \mathfrak{M}_{\text{free}}^{(n)}[E] \longrightarrow \overline{\mathfrak{M}}^{(n)}[E] \longrightarrow \\ \longrightarrow \mathfrak{M}_{\text{tf}}^{(n)}/E \longrightarrow \mathfrak{M}_{\text{free}}^{(n)}/E \longrightarrow \overline{\mathfrak{M}}^{(n)}/E \longrightarrow 0. \end{aligned} \quad (\text{A.17})$$

Using this, we obtain the following.

Lemma A.16. For any $n \geq 0$, there is an isomorphism of $\mathcal{O}_K$ -modules:

$$(\overline{\mathfrak{M}}^{(n)}[E])[p^\infty] = \overline{\mathfrak{M}}^{(n)}[E] \xrightarrow{\sim} (\mathfrak{M}_{\text{tf}}^{(n)}/E)[p^\infty]. \quad (\text{A.18})$$

A.2. Length calculations for generalised Breuil–Kisin modules. In this section, we collect some results about lengths of Breuil–Kisin modules, their Frobenius twists, and their E -torsion and reduction modulo E .

Lemma A.17. *Let $\mathfrak{M}$ be a generalised Breuil–Kisin module. Then, for any $n \geq 0$:*

$$\ell_{\mathcal{O}_K}(\mathfrak{M}^{(n)}[E]) = \ell_{\mathcal{O}_K}(\mathfrak{M}[u^\infty]^{(n)}[E]), \quad (\text{A.19})$$

$$\ell_{\mathcal{O}_K}((\mathfrak{M}^{(n)}/E)[p^\infty]) = \ell_{\mathcal{O}_K}(\mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n)}/E) + \ell_{\mathcal{O}_K}(\mathfrak{M}[u^\infty]^{(n)}/E) + \ell_{\mathcal{O}_K}(\overline{\mathfrak{M}}^{(n)}[E]). \quad (\text{A.20})$$

Proof. The claim in (A.19) follows by combining the isomorphism of $\mathcal{O}_K$ -modules in (A.10) of Lemma A.12 and (A.15) of Lemma A.15. Next, for (A.20), let us first note that we have the following equality using (A.12) of Lemma A.11:

$$\ell_{\mathcal{O}_K}((\mathfrak{M}^{(n)}/E)[p^\infty]) = \ell_{\mathcal{O}_K}(\phi^{*n}\mathfrak{M}_{\text{tor}}^{(n)}/E) + \ell_{\mathcal{O}_K}((\mathfrak{M}_{\text{tf}}^{(n)}/E)[p^\infty]).$$

Moreover, from (A.16) in Lemma A.15, we have that,

$$\ell_{\mathcal{O}_K}(\mathfrak{M}_{\text{tor}}^{(n)}/E) = \ell_{\mathcal{O}_K}(\mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n)}/E) + \ell_{\mathcal{O}_K}(\mathfrak{M}[u^\infty]^{(n)}/E),$$

and from (A.18) in Lemma A.16, we have that $\overline{\mathfrak{M}}^{(n)}[E] \xrightarrow{\sim} (\mathfrak{M}_{\text{tf}}^{(n)}/E)[p^\infty]$ as $\mathcal{O}_K$ -modules. Hence, the equality in (A.20) follows. $\square$

Lemma A.18. *Let $\mathfrak{M}$ be a generalised Breuil–Kisin module of height i . Then, for any $n \geq 0$:*

$$\ell_{\mathcal{O}_K}(\mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n+1)}/E) = \ell_{\mathcal{O}_K}(\mathfrak{M}_{\text{tor}, u\text{-tf}}^{(n)}/E). \quad (\text{A.21})$$

Proof. From Lemma A.5 (3), note that there exists some $N \geq 0$ such that we may write $\mathfrak{M}_{\text{tor}, u\text{-tf}}$ as a successive extension of finite free $k[[u]]$ -modules. Using this, one quickly reduces to the case when $\mathfrak{M} = \mathfrak{M}_{\text{tor}, u\text{-tf}}$ is a finite free $k[[u]]$ -module. But, writing $\mathfrak{M} \xrightarrow{\sim} k[[u]]^{\oplus r}$, one computes that $\mathfrak{M}^{(n)}/E \xrightarrow{\sim} (k[u]/u^e k[u])^{\oplus r}$ as $\mathcal{O}_K$ -modules and for any $m \geq 0$, so the claim follows. $\square$

Lemma A.19. *For all $n \geq 0$, we have that $\ell_{\mathcal{O}_K}(\mathfrak{M}[u^\infty]^{(n)}[E]) = \ell_{\mathcal{O}_K}(\mathfrak{M}[u^\infty]^{(n)}/E)$.*

Proof. Note that $\mathfrak{N} := \mathfrak{M}[u^\infty]$ is killed by p^m , for some $m \geq 0$, and we proceed by induction on m . So, assume that $\mathfrak{N}$ is p -torsion. As $\mathfrak{S}/p = k[[u]]$ is a complete DVR, we have a decomposition

$$\mathfrak{N} \simeq \bigoplus_{j=1}^s k[u]/u^{r_j},$$

for some r_j and s in $\mathbb{N}$. Then, we see that,

$$\mathfrak{N}^{(n)} \simeq \bigoplus_{j=1}^s k[u]/u^{p^n r_j},$$

and noting that $E = u^e \bmod p$, it is easy to compute that

$$\ell_{\mathcal{O}_K}(\mathfrak{N}^{(n)}[E]) = \ell_{\mathcal{O}_K}(\mathfrak{N}^{(n)}/E) = \sum_{j=1}^s \min\{e, p^n r_j\}.$$

Assume now that $\mathfrak{N}$ is p^m -torsion, and consider the exact sequence

$$0 \longrightarrow p\mathfrak{N}^{(n)} \longrightarrow \mathfrak{N}^{(n)} \longrightarrow \mathfrak{N}^{(n)}/p \longrightarrow 0.$$

The first term is p^{m-1} -torsion, and the last term in p -torsion. Reducing the preceding exact sequence modulo E , one obtains a long exact sequence of torsion $\mathcal{O}_K$ -modules,

$$0 \longrightarrow (p\mathfrak{N}^{(n)})[E] \longrightarrow \mathfrak{N}^{(n)}[E] \longrightarrow (\mathfrak{N}^{(n)}/p)[E] \longrightarrow (p\mathfrak{N}^{(n)})/E \longrightarrow \mathfrak{N}^{(n)}/E \longrightarrow (\mathfrak{N}^{(n)}/p)/E \longrightarrow 0,$$

and the claim follows from the induction hypothesis by the additivity of length. $\square$

A.3. Applications to cohomology. We now apply the abstract statements on Breuil–Kisin modules to prove some results concerning Breuil–Kisin cohomology of smooth formal $\mathcal{O}_K$ -schemes.

A.3.1. Proof of Proposition 4.30. We shall in fact show the stronger claim that under Hypothesis 4.28, Conjecture 4.26 implies **(L1)** and **(L2)** from Section 4.4 (see the discussion after Proposition 4.24), i.e. that $\ell_{\text{dR}}^{i,(a)} \leq e \cdot \ell_{\text{dR}}^i$ and that $\ell_{\text{dR}}^{i,(n)}$ is non-decreasing in n . By (4.5), we have the following equality:

$$\ell_{\text{dR}}^{i,(n)} = \ell_{\mathcal{O}_K}((\mathfrak{M}^{i,(n)}/E)[p^\infty]) + \ell_{\mathcal{O}_K}(\mathfrak{Q}^{i,(n)}). \quad (\text{A.22})$$

Using (A.20) in Lemma A.17, and Lemmas A.18 and A.19, we see that statement (2) in Conjecture 4.26 implies that the first term on the right hand side of (A.22) is non-decreasing in n . Moreover, the second inequality in Hypothesis 4.28 implies that the second term on the right hand side of (A.22) is non-decreasing in n . A similar analysis, using statement (1) in Conjecture 4.26 and the first inequality in Hypothesis 4.28, implies the inequality $\ell_{\text{crys}}^i \leq \ell_{\text{dR}}^i$ in Conjecture 4.19.

A.3.2. Verification of Conjecture 4.26 in some small cases. We finally aim to use the results in Sections A.1 and A.2 to prove the following.

Theorem A.20. *Conjecture 4.26 holds when $i \leq 2$ and $p \cdot \alpha(\mathfrak{M}^i) \neq e < p(p-1)$.*

The key to proving this result is the following length estimates.

Proposition A.21. *Let $\mathfrak{M}$ be a u^∞ -torsion quasi-filtered Breuil–Kisin module of height $i \leq 2$. Assume that $p\alpha(\mathfrak{M}) \neq e < p(p-1)$. Then, the following two inequalities hold:*

$$\ell_{\mathcal{O}_K}(\mathfrak{M}^{(n+1)}[E]) \geq \ell_{\mathcal{O}_K}(\mathfrak{M}^{(n)}[E]), \quad (\text{A.23})$$

$$e \cdot \ell_{\mathcal{O}_K}(\mathfrak{M}^{(1)}[E]) \geq \ell_{\mathcal{O}_K}(\mathfrak{M}^{(a+1)}[E]). \quad (\text{A.24})$$

Proof. Using Theorem A.9, the claim is trivial for $i = 0, 1$. For $i = 2$, the claim follows by using Proposition A.10 and Lemmas A.22 and A.23 below. $\square$

Lemma A.22. *Assume that $\mathfrak{M}$ is p -torsion, then the inequalities (A.23) and (A.24) hold.*

Proof. As $\mathfrak{M}$ is a finitely generated u^∞ -torsion module over $\mathfrak{S}/p = k[[u]]$, therefore, from the proof of Lemma A.19 recall that we have,

$$\ell_{\mathcal{O}_K}(\mathfrak{M}^{(n)}[E]) = \ell_{\mathcal{O}_K}(\mathfrak{M}^{(n)}/E) = \sum_{j=1}^s \min\{e, p^n r_j\},$$

for any $n \geq 0$, which implies the claim. $\square$

Lemma A.23. *Write $\alpha = \alpha(\mathfrak{M})$. Assume that $u^\alpha + px$ is in $\text{Ann}(\mathfrak{M})$ with $1 \leq \alpha \leq \lfloor \frac{e}{p-1} \rfloor < p$ and x in $\mathfrak{S}^\times$, and $e \neq p\alpha$. Then, the inequalities (A.23) and (A.24) hold.*

Proof. For $e \leq p-1$, note that $\text{Ann}(\mathfrak{M})$ is p -torsion by Theorem A.9, and the claim follows by Lemma A.22. So, we may assume that $p \leq e \leq p(p-1)$. Consider the element $f := u^\alpha + px$ assumed to be in $\text{Ann}(\mathfrak{M})$. It suffices to consider the following cases:

- if $E \in \text{Ann}(\mathfrak{M})$, then $\mathfrak{M}$ is p -torsion by Lemma A.11 and we may use Lemma A.22;
- if $E \notin \text{Ann}(\mathfrak{M})$ and $E \in \text{Ann}(\mathfrak{M}^{(1)})$, then $\mathfrak{M}^{(1)}$ is p -torsion by Lemma A.11, in particular, $\mathfrak{M}$ is p -torsion and we may use Lemma A.22;
- if $E \notin \text{Ann}(\mathfrak{M})$ and $E \notin \text{Ann}(\mathfrak{M}^{(1)})$, then the claim follows from the discussion below.

Assume that $E \notin \text{Ann}(\mathfrak{M})$ and $E \notin \text{Ann}(\mathfrak{M}^{(1)})$. Let $n \geq 0$ and observe that $f^{(n)}$ belongs to $(p, u) \setminus (p, u)^2$, and thus $\mathfrak{S}/f^{(n)}$ is a regular local ring of dimension one, and so a DVR with uniformiser u . As $\mathfrak{M}$ is a finite, u^∞ -torsion $\mathfrak{S}/f$ -module, we may write

$$\mathfrak{M} \xrightarrow{\sim} \bigoplus_{j=1}^s \mathfrak{S}/(f, u^{r_j}),$$

for some r_j and s in $\mathbb{N}$, and for all $n \geq 0$, we see that

$$\mathfrak{M}^{(n)} \xrightarrow{\sim} \bigoplus_{j=1}^s \mathfrak{S}/(f^{(n)}, u^{p^n r_j}).$$

Recall that u is a uniformiser of $\mathfrak{S}/f^{(n)}$ and let $v_n(-)$ denote the u -adic valuation on $\mathfrak{S}/f^{(n)}$. Then, we compute that $v_n(p) = p^n \alpha$, and therefore,

$$v_n(E) = \begin{cases} \alpha & \text{if } n = 0, \\ \min\{e, p\alpha\} & \text{if } n = 1, \\ e & \text{if } n \geq 2. \end{cases} \quad (\text{A.25})$$

Observe that we have,

$$\ell_{\mathcal{O}_K}(\mathfrak{M}^{(n)}[E]) = \ell_{\mathcal{O}_K}(\mathfrak{M}^{(n)}/E) = \sum_{j=1}^s \min\{v_n(E), p^n r_j\}.$$

Hence, it follows that the inequalities (A.23) and (A.24) also hold when $E \notin \text{Ann}(\mathfrak{M})$ and $E \notin \text{Ann}(\mathfrak{M}^{(1)})$. This allows us to conclude. $\square$

These methods also imply the following, which actually finishes the proof of Theorem A.20, as $H^i(\text{R}\Gamma_{\mathfrak{S}}(X/\mathcal{O}_K)/p^m)$ and its u^∞ -torsion submodule $H^i(\text{R}\Gamma_{\mathfrak{S}}(X/\mathcal{O}_K)/p^m)[u^\infty]$ are quasi-filtered Breuil–Kisin modules of height i by [LL23, Proposition 3.2].

Proposition A.24. *With the assumptions in Theorem A.20, the following inequality holds:*

$$\ell_{\mathcal{O}_K}(\overline{\mathfrak{M}}^{(n+1)}[E]) \geq \ell_{\mathcal{O}_K}(\overline{\mathfrak{M}}^{(n)}[E]). \quad (\text{A.26})$$

Proof. From Lemma A.6 and [SP, Tag 061Z], we have natural inclusions of $\mathfrak{S}$ -modules

$$\overline{\mathfrak{M}}^i \hookrightarrow (\mathfrak{M}^i/p^N)[u^\infty] \hookrightarrow H^i(\text{R}\Gamma_{\mathfrak{S}}(X/\mathcal{O}_K)/p^N)[u^\infty],$$

for some $N \gg 0$. In particular, from Proposition A.8 and Proposition A.10, it follows that either p kills $\overline{\mathfrak{M}}^i$ or $u^\alpha + px$ kills $\overline{\mathfrak{M}}^i$, with $e \neq p\alpha$, $1 \leq \alpha \leq \lfloor \frac{e}{p-1} \rfloor < p$ and x in $\mathfrak{S}^\times$. Hence, the claim follows by the using the same arguments as in Lemmas A.22 and A.23. $\square$

REFERENCES

- [Abh24] Abhinandan, Prismatic F -crystals and Wach modules (2024), arXiv:2405.18245. [26](#)
- [BBM82] P. Berthelot, L. Breen and W. Messing, Théorie de Dieudonné cristalline. II, vol. 930 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1982. [26](#)
- [Ber74] P. Berthelot, Cohomologie cristalline des schémas de caractéristique $p > 0$, Lecture Notes in Mathematics, Vol. 407, Springer-Verlag, Berlin-New York, 1974. [2](#)
- [Bha23] B. Bhatt, Prismatic F -gauges, 2023, unpublished course notes. [6](#), [13](#), [20](#), [21](#), [22](#)
- [BL22a] B. Bhatt and J. Lurie, Absolute prismatic cohomology, 2022, [2201.06120](#). [2](#), [3](#), [6](#), [7](#), [14](#), [21](#), [22](#), [26](#)
- [BL22b] B. Bhatt and J. Lurie, The prismatization of p -adic formal schemes, 2022, [2201.06124](#). [6](#), [7](#), [8](#), [21](#)
- [BMS18] B. Bhatt, M. Morrow and P. Scholze, Integral p -adic Hodge theory, Publ. Math. Inst. Hautes Études Sci. 128 (2018), 219–397. [25](#), [26](#), [27](#)
- [BMS19] B. Bhatt, M. Morrow and P. Scholze, Topological Hochschild homology and integral p -adic Hodge theory, Publ. Math. Inst. Hautes Études Sci. 129 (2019), 199–310. [7](#)
- [BO78] P. Berthelot and A. Ogus, Notes on crystalline cohomology, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1978. [2](#), [13](#), [24](#)
- [BO83] P. Berthelot and A. Ogus, F -isocrystals and de Rham cohomology. I, Invent. Math. 72 (1983), no. 2, 159–199. [2](#), [10](#), [12](#), [22](#), [24](#)
- [Bre97] C. Breuil, Représentations p -adiques semi-stables et transversalité de Griffiths, Math. Ann. 307 (1997), no. 2, 191–224. [4](#)
- [BS22] B. Bhatt and P. Scholze, Prisms and prismatic cohomology, Ann. of Math. (2) 196 (2022), no. 3, 1135–1275. [2](#), [6](#), [7](#), [8](#), [9](#), [13](#), [26](#)
- [BS23] B. Bhatt and P. Scholze, Prismatic F -crystals and crystalline Galois representations, Camb. J. Math. 11 (2023), no. 2, 507–562. [6](#)
- [ČK19] K. Česnavičius and T. Koshikawa, The A_{inf} -cohomology in the semistable case, Compos. Math. 155 (2019), no. 11, 2039–2128. [5](#), [24](#)
- [DLMS24] H. Du, T. Liu, Y. S. Moon and K. Shimizu, Completed prismatic F -crystals and crystalline Z_p -local systems, Compos. Math. 160 (2024), no. 5, 1101–1166. [9](#)
- [Dri21] V. Drinfeld, On a notion of ring groupoid, 2021, [2104.07090](#). [6](#)
- [Dri24] V. Drinfeld, Prismatization, 2024, [2005.04746](#). [2](#), [6](#)
- [Fal99] G. Faltings, Integral crystalline cohomology over very ramified valuation rings, J. Amer. Math. Soc. 12 (1999), no. 1, 117–144. [4](#)
- [GL25] O. Gabber and S. Li, Bounding crystalline torsion from étale torsion, 2025. [5](#), [26](#), [27](#), [28](#)
- [GR24] H. Guo and E. Reinecke, A prismatic approach to crystalline local systems, Invent. Math. 236 (2024), no. 1, 17–164. [8](#), [13](#), [20](#)
- [Hau24] M. Hauck, A stacky approach to p -adic Hodge theory, 2024, [2409.10557](#). [21](#)
- [HK94] O. Hyodo and K. Kato, Semi-stable reduction and crystalline cohomology with logarithmic poles, 223, pp. 221–268, 1994, périodes p -adiques (Bures-sur-Yvette, 1988). [4](#)
- [IKY24] N. Imai, H. Kato and A. Youcis, A Tannakian framework for (analytic) prismatic F -crystals, 2024. [8](#)
- [IKY25] N. Imai, H. Kato and A. Youcis, An integral analogue of Fontaine’s crystalline functor, 2025. [3](#), [13](#), [14](#), [15](#)

- [Kat73] N. Katz, Travaux de Dwork, in Séminaire Bourbaki, 24ème année (1971/1972), vol. Vol. 317 of Lecture Notes in Math., pp. Exp. No. 409, pp. 167–200, Springer, Berlin-New York, 1973. [3](#)
- [Kis06] M. Kisin, Crystalline representations and F -crystals, in Algebraic geometry and number theory, vol. 253 of Progr. Math., pp. 459–496, Birkhäuser Boston, Boston, MA, 2006. [4](#), [26](#)
- [LL20] S. Li and T. Liu, Comparison of prismatic cohomology and derived de Rham cohomology (2020), arXiv:2012.14064. [26](#), [27](#)
- [LL23] S. Li and T. Liu, On the u^∞ -torsion submodule of prismatic cohomology, Compos. Math. 159 (2023), no. 8, 1607–1672. [5](#), [25](#), [26](#), [27](#), [28](#), [32](#)
- [Ogu84] A. Ogus, F -isocrystals and de Rham cohomology. II. Convergent isocrystals, Duke Math. J. 51 (1984), no. 4, 765–850. [3](#)
- [SP] Stacks Project Authors, *Stacks Project*, <http://stacks.math.columbia.edu>, 2023. [7](#), [20](#), [22](#), [26](#), [32](#)
- [Tsu99] T. Tsuji, p -adic étale cohomology and crystalline cohomology in the semi-stable reduction case, Invent. Math. 137 (1999), no. 2, 233–411. [4](#)