Crystalline part of the Galois cohomology of crystalline representations

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Abstract. For $p \geq 3$ and an unramified extension $F/\mathbb{Q}_p$ with perfect residue field, we define a syntomic complex with coefficients in a Wach module over a certain period ring for F. We show that our complex computes the crystalline part of the Galois cohomology (in the sense of Bloch and Kato) of the associated crystalline representation of the absolute Galois group of F. Furthermore, we establish that Wach modules of Berger naturally descend over to a smaller period ring studied by Fontaine and Wach. This enables us to define another syntomic complex with coefficients and we show that its cohomology also computes the crystalline part of the Galois cohomology of the associated representation.

1. Introduction

Let F be an unramified extension of $\mathbb{Q}_p$ with perfect residue field and let G_F denote the absolute Galois group of F. One of the main goals of p-adic Hodge theory is to classify p-adic representations of G_F arising from geometry, for example, *crystalline*, semistable, de Rham etc. The notion of p-adic crystalline representations, defined by Fontaine in [Fon82], is meant to capture the idea of "good reduction" of algebraic varieties defined over F.

To understand p-adic representations more explicitly, Fontaine intiated different programs aiming to describe p-adic representations in terms of certain semilinear algebraic objects. In [CF00] Colmez and Fontaine showed that the category of p-adic crystalline representations of G_F is equivalent to the category of weakly admissible filtered φ-modules over F (see Subsection 2.2). On the other hand, in [Fon90], Fontaine showed that the category of all p-adic representations of G_F is equivalent to the category of étale (φ, Γ_F)-modules, where $\Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$. Fontaine's equivalence was further refined to establish an equivalence between the category of p-adic crystalline representations of G_F and the category of Wach modules, where a Wach module is a certain lattice inside the étale (φ, Γ_F)-module associated to the representation (see [Fon90; Wac96; Col99; Ber04]). All preceding categorical equivalences are exact and it is natural to ask the following:

Question. *Let V be a p-adic crystalline representation of G_F. Can one (partially) compute the continuous Galois cohomology of V in terms of the associated filtered φ-module, resp. the étale (φ, Γ_F)-module, resp. the Wach module?*

In [BK90], Bloch and Kato defined a complex using the filtered φ-module $D_{\text{cris}}(V)$ associated to V and showed that their complex computes the *crystalline part* of the continuous Galois cohomology of V (see Theorem 1.1). Moreover, in [Her98], Herr defined a complex in terms of the étale (φ, Γ_F)-module associated to V (not necessarily crystalline) and showed that his complex computes the continuous Galois cohomology of V (see Subsection 2.3). The main objective of this article is to answer the *open question* above for Wach modules, i.e. we will define a syntomic complex in terms of the Wach module associated to V and show that our complex computes the *crystalline part* of the continuous Galois cohomology of V (see Theorem 1.3 and Theorem 1.9). Our results are related to the results of [BK90] (see Remark 4.3 and Remark 4.14) and [Her98] (see Remark 1.5), however, our constructions and proofs are direct and independent of [BK90; Her98].

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1.1. Bloch–Kato Selmer groups. Let us begin by recalling the result of Bloch and Kato. In [BK90], the authors defined Bloch–Kato Selmer groups of V as a subspace inside the continuous G_F-cohomology of V, i.e. $H^k_f(G_F, V) \subset H^k(G_F, V)$, for $k \in \mathbb{N}$. Morally, Bloch–Kato Selmer group picks out the *crystalline part* of the Galois cohomology of V (see Remark 2.4). More precisely, from [BK90] we have the following:

Theorem 1.1 (Corollary 2.6)**.** *Let V be a p-adic crystalline representation of G_F. Then the complex*

$$\mathscr{D}^\bullet(D_{\mathrm{cris}}(V)) : \mathrm{Fil}^0 D_{\mathrm{cris}}(V) \xrightarrow{1-\varphi} D_{\mathrm{cris}}(V),$$

computes the crystalline part of the Galois cohomology of V, i.e. we have natural isomorphisms $H^k(\mathscr{D}^\bullet(D_{\mathrm{cris}}(V))) \xrightarrow{\sim} H^k_f(G_F, V)$ for each $k \in \mathbb{N}$.

1.2. Syntomic complexes and Galois cohomology. Let $p \geq 3$ and set $F_\infty := \cup_n F(\mu_{p^n})$ with $\Gamma_F := \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times$, via the p-adic cycolotomic character χ. Note that Γ_F fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1,$$

where $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ and $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$, and the projection map in (2.1) admits a section $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$, where the second map is given as $a \mapsto [a]$, the Teichmüller lift of a. We also fix a topological generator γ of Γ_0.

Let q be an indeterminate and set $A_F^+ := O_F[\![q-1]\!]$, equipped with a Frobenius endomorphism φ extending the Frobenius on O_F by setting $\varphi(q) = q^p$, and an O_F-linear and continuous action of Γ_F defined by setting $g(q) = q^{\chi(g)}$ for any g in Γ_F (see Subsection 2.1). Set $\mu := q-1$ and fix the following elements inside A_F^+:

$$[p]_q := \tfrac{q^p-1}{q-1}, \quad \mu_0 := -p + \textstyle\sum_{a \in \mathbb{F}_p} q^{[a]}, \quad \tilde{p} := \mu_0 + p.$$

Now, we set $S := O_F[\![\mu_0]\!] = (A_F^+)^{\mathbb{F}_p^\times} \subset A_F^+$, which is stable under the action of φ and Γ_0; we equip S with the induced (φ, Γ_0)-action. Our goal is to define syntomic complexes with coefficients in Wach modules over A_F^+ (resp. S).

1.3. Syntomic complex over A_F^+. Let $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_F and let $(\varphi, \Gamma)\text{-}\mathrm{Mod}^{[p]_q}_{A_F^+}$ denote the category of Wach modules over A_F^+ (see Definition 3.1). Then by [Fon90; Wac96; Col99; Ber04], we have an equivalence of categories $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{A_F^+}$, by sending $T \mapsto N_F(T)$ (see Theorem 3.9). Moreover, after inverting p, i.e. upon passing to associated isogeny categories, the Wach module functor induces an exact equivalence of categories (see Remark 3.11).

Let T be a $\mathbb{Z}_p$-representation of G_F such that $V := T[1/p]$ is crystalline. Let $N := N_F(T)$ denote the Wach module over A_F^+ associated to T by Theorem 3.9. Define a decreasing filtration on N called the *Nygaard filtration* as $\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}$, for $k \in \mathbb{Z}$. Define an operator on N as $\nabla_q := \frac{\gamma-1}{q-1} : N \to N$. Then for each $k \in \mathbb{Z}$ we have that $\nabla_q(\mathrm{Fil}^k N) \subset \mathrm{Fil}^{k-1} N$ (see Remark 3.17).

Definition 1.2. Define the *syntomic complex* with coeffcients in N as

$$\mathcal{S}^\bullet(N) : \mathrm{Fil}^0 N \xrightarrow{(\nabla_q, 1-\varphi)} \mathrm{Fil}^{-1} N \oplus N \xrightarrow{(1-[p]_q\varphi, \nabla_q)^\intercal} N,$$

where the first map is $x \mapsto (\nabla_q(x), (1-\varphi)x)$ and the second map is $(x, y) \mapsto (1-[p]_q\varphi)x - \nabla_q(y)$.

Our first main result is as follows:

Theorem 1.3 (Theorem 4.2)**.** *For each $k \in \mathbb{N}$, we have a natural isomorphism*

$$H^k(\mathcal{S}^\bullet(N))[1/p] \xrightarrow{\sim} H^k_f(G_F, V).$$

The proof of Theorem 1.3 is subtle. For computing $H^0(\mathcal{S}^\bullet(N))$, we first show that Wach modules over A_F^+ canonically descend to Wach modules over S (see Theorem 1.7), and then we study the action of Γ_0 on N (see Lemma 4.4). To prove the claim for H^1, we show that our complex computes the extension classes of A_F^+ by N in the category of Wach modules over A_F^+, and after inverting p, each class gives rise to a crystalline extension class of $\mathbb{Q}_p$ by V, and vice versa (see Proposition 4.5). Finally, by explicitly studying the action on N, of the generator γ of Γ_0, we show that $H^2(\mathcal{S}^\bullet(N))[1/p]$ vanishes (see Proposition 4.6).

Remark 1.4. Note that $(N/\mu N)[1/p]$ is a φ-module over F since $[p]_q = p \bmod \mu A_F^+$ and $N/\mu N$ is equipped with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $(N/\mu N)[1/p]$ with the induced filtration $\mathrm{Fil}^k((N/\mu N)[1/p]) := \mathrm{Fil}^k(N/\mu N)[1/p]$, and note that it is a filtered φ-module over F. Then, from [Ber04, Théorème III.4.4] and [Abh23a, Theorem 1.7 & Remark 1.8] we have that $(N/\mu N)[1/p] \xrightarrow{\sim} D_{\mathrm{cris}}(V)$ as filtered φ-modules over F (see Theorem 3.15). The preceding comparison enables us to define a morphism of complexes $\mathcal{S}^\bullet(N)[1/p] \to \mathscr{D}^\bullet(D_{\mathrm{cris}}(V))$, which induces a quasi-isomorphism (see Remark 4.3). In particular, the complex $\mathcal{S}^\bullet(N)$ may be regarded as a "lifting to A_F^+" of the complex $\mathscr{D}^\bullet(D_{\mathrm{cris}}(V))$.

Remark 1.5. Definiton 1.2 can be modified (upto isomorphism) to obtain a subcomplex of the Fontaine–Herr complex from [Her98] (see Remark 4.3 and Remark 4.14). Note that the Fontaine–Herr complex computes the Galois cohomology of a representation, while the complex in Definition 1.2 or Remark 4.3 is concerned with capturing the crystalline part of the Galois cohomology. Complexes similar to the modified complex in Remark 4.3 were studied in [Abh23c] and named syntomic complexes. Hence, we refer to the complex in Definition 1.2 as the syntomic complex with coefficients in N.

Remark 1.6. In [Bha23, Chapter 6], Bhatt and Lurie have defined syntomic cohomology of prismatic F-gauges on the stack $\mathbb{Z}_p^{\mathrm{syn}}$ and, in case of reflexive F-gauges, compared it to the Bloch–Kato Selmer groups of the associated crystalline representation of $\mathrm{Gal}(\overline{F}/\mathbb{Q}_p)$ (see [Bha23, Proposition 6.7.3]). In the light of Theorem 1.3 and the prismatic interpretation of Wach modules (see [Abh24]), a natural and interesting question is to ask for a direct (integral) relationship between Definition 1.2 and the definition of [Bha23]. The aforementioned question and generalisation of the theory above to the relative case, i.e. Definition 1.2 and its relationship with Galois cohomology, will be investigated in a future work.

1.4. Syntomic complex over S.

Let $(\varphi, \Gamma_0)\text{-}\mathrm{Mod}_S^{\tilde{p}}$ denote the category of Wach modules over S (see Definition 3.4). Our second main result establishes the following descent statement for Wach modules:

Theorem 1.7. *The following natural functor induces an exact equivalence of $\otimes$-categories,*

$$\begin{aligned} (\varphi, \Gamma_0)\text{-}\mathrm{Mod}_S^{\tilde{p}} &\xrightarrow{\sim} (\varphi, \Gamma_F)\text{-}\mathrm{Mod}_{A_F^+}^{[p]_q} \\ M &\longmapsto A_F^+ \otimes_S M. \end{aligned} \tag{1.1}$$

with an exact $\otimes$-compatible quasi-inverse functor given as $N \mapsto N^{\mathbb{F}_p^\times}$.

By combining Theorem 1.7 and Theorem 3.13, we obtain a natural equivalence of categories $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_0)\text{-}\mathrm{Mod}_S^{\tilde{p}}$, by sending $T \mapsto M_F(T) := N_F(T)^{\mathbb{F}_p^\times}$ (see Theorem 3.13).

Let T be a $\mathbb{Z}_p$-representation of G_F such that $V := T[1/p]$ is crystalline. Let $M := M_F(T)$ denote the Wach module over S associated to T by Theorem 3.13. Define a decreasing filtration on M called the *Nygaard filtration* as $\mathrm{Fil}^k M := \{x \in M \text{ such that } \varphi(x) \in \tilde{p}^k M\}$, for $k \in \mathbb{Z}$. Define an operator on M as $\nabla_0 := \frac{\gamma - 1}{\mu_0} : M \to M$. Then for each $k \in \mathbb{Z}$ we have $\nabla_0(\mathrm{Fil}^k M) \subset \mathrm{Fil}^{k-p+1} M$ (see Remark 3.25).

Definition 1.8. Define the *syntomic complex* with coeffcients in M as

$$\mathcal{S}^\bullet(M) : \mathrm{Fil}^0 M \xrightarrow{(\nabla_0, 1-\varphi)} \mathrm{Fil}^{-p+1} M \oplus M \xrightarrow{(1-\tilde{p}^{p-1}\varphi, \nabla_0)^\intercal} M,$$

where the first map is $x \mapsto (\nabla_0(x), (1-\varphi)x)$ and the second map is $(x, y) \mapsto (1 - \tilde{p}^{p-1}\varphi)x - \nabla_0(y)$.

Our third main result is as follows:

Theorem 1.9 (Theorem 4.13)**.** *For each $k \in \mathbb{N}$, we have a natural isomorphism*

$$H^k(\mathcal{S}^\bullet(M))[1/p] \xrightarrow{\sim} H^k_f(G_F, V).$$

To prove Theorem 1.9, we first define a morphism of complexes $\mathcal{S}^\bullet(M) \to \mathcal{S}^\bullet(N)$, where $N = A_F^+ \otimes_S M$ is a Wach module over A_F^+ (see Theorem 1.7). Then using the $\mathbb{F}_p^\times$-decomposition of N (see Remark 2.1), we show that the natural map on cohomology $H^k(\mathcal{S}^\bullet(M)) \to H^k(\mathcal{S}^\bullet(N))$ is bijective for $k = 0, 1$ and injective for $k = 2$ (see Proposition 1.7). Combining this with Theorem 1.3, yields the claim.

Remark 1.10. Note that $(M/\mu_0 M)[1/p]$ is a φ-module over F since $\tilde{p} = p \bmod \mu_0 S$ and $M/\mu_0 M$ is equipped with a filtration $\mathrm{Fil}^k(M/\mu_0 M)$ given as the image of $\mathrm{Fil}^k M$ under the surjection $M \twoheadrightarrow M/\mu_0 M$. We equip $(M/\mu_0 M)[1/p]$ with the induced filtration $\mathrm{Fil}^k((M/\mu_0 M)[1/p]) := \mathrm{Fil}^k(M/\mu_0 M)[1/p]$, and note that it is a filtered φ-module over F. Then, in Theorem 3.23 we show that $(M/\mu_0 M)[1/p] \xrightarrow{\sim} D_{\mathrm{cris}}(V)$ as filtered φ-modules over F. The preceding comparison enables us to define a morphism of complexes $\mathcal{S}^\bullet(M)[1/p] \to \mathscr{D}^\bullet(D_{\mathrm{cris}}(V))$, which induces a quasi-isomorphism (see Remark 4.14). In particular, the complex $\mathcal{S}^\bullet(M)$ may be regarded as a "lifting to S" of the complex $\mathscr{D}^\bullet(D_{\mathrm{cris}}(V))$.

1.5. Outline of the paper.

This article consists of three main sections. In Section 2, we quickly recall the necessary definitions and results on period rings, p-adic representations and Galois cohomology. We begin by fixing some notations and recalling several period rings in Subsection 2.1 and the theory of p-adic crystalline representations and (φ, Γ)-modules in Subsection 2.2. In Subsection 2.3, we recall the Fontaine–Herr complex and its relationship with the Galois cohomology of p-adic representations; in Subsection 2.4 we recall the definition of Bloch–Kato Selmer groups and the construction of the Bloch–Kato complex from Theorem 1.1. Section 3 is devoted to the study of Wach modules and the Nygaard filtration on Wach modules. In Subsection 3.1, we first recall the definition of Wach modules over A_F^+ and S, and then prove the descent claim of Theorem 1.7. In Subsection 3.2, we recall the relationship between Wach modules and crystalline representations, and in Subsection 3.3 we define the Nygaard filtration on Wach modules and prove several properties of the filtration that are to be used later. In Section 4, we arrive at the main results on the computation of Galois cohomology. We begin Subsection 4.1 by defining the syntomic complex over A_F^+ (see Definition 1.2) and then state and prove Theorem 1.3. The proof for each cohomological degree $k = 0, 1, 2$ is carried out separately (see Lemma 4.4, Proposition 4.5 and Proposition 4.6). Finally, in Subsection 4.2 we define the syntomic complex over S (see Definition 1.8) and prove Theorem 1.3.

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2. p-adic representations

Let $p \geq 3$ be a fixed prime number and let κ denote a perfect field of characteristic p; set $O_F := W(\kappa)$ to be the ring of p-typical Witt vectors with coefficients in κ and $F := \mathrm{Frac}(O_F)$. Let $\overline{F}$ denote a fixed algebraic closure of F, let $\mathbb{C}_p := \widehat{\overline{F}}$ denote its p-adic completion and $G_F := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. Moreover, let $F_\infty := \cup_n F(\mu_{p^n})$, and set $\Gamma_F := \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times$ and $H_F := \mathrm{Gal}(\overline{F}/F_\infty)$. Note that the isomorphism $\chi : \Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$ is given via the p-adic cyclotomic character, and therefore, Γ_F fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1, \tag{2.1}$$

where we have that $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ and $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$, and the projection map in (2.1) admits a section $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \to \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$, where the second map is given as $a \mapsto [a]$, the Teichmüller lift of a. We fix a topological generator γ of Γ_0.

Remark 2.1. Let N be a compact $\mathbb{Z}_p$-module admitting a continuous action of Γ_F. Then from [Iwa59, Section 3], the module N admits an $\mathbb{F}_p^\times$-decomposition $N = \bigoplus_{i=0}^{p-1} N_i$, where $N_0 = N^{\mathbb{F}_p^\times}$. Moreover, since Γ_F is commutative, therefore, each N_i is equipped with an induced continuous action of Γ_0.

2.1. Period rings. In this subsection we will quickly recall the period rings to be used later (see [Fon90; Fon94] for details). Let $O_{\overline{F}}$ (resp. O_{F_∞}) denote the ring of integers of $\overline{F}$ (resp. F_∞) and let $O_{\overline{F}}^\flat := \lim_{x\mapsto x^p} O_{\overline{F}}/p$ (resp. $O_{F_\infty}^\flat := \lim_{x\mapsto x^p} O_{F_\infty}/p$) denote its tilt (see [Fon94]). Let us set $A_{\text{inf}}(O_{F_\infty}) := W(O_{F_\infty}^\flat)$ and $A_{\text{inf}}(O_{\overline{F}}) := W(O_{\overline{F}}^\flat)$ admitting the natural Frobenius on Witt vectors and continuous (for the weak topology) actions of Γ_F and G_F, respectively. We fix $\overline{\mu} := \varepsilon - 1$, where $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots)$ is in $O_{F_\infty}^\flat$, and let $q = [\varepsilon]$, $\mu := q-1$ and $\xi := \mu/\varphi^{-1}(\mu)$ in $A_{\text{inf}}(O_{F_\infty})$. Then, for any g in G_F, we have that $g(1+\mu) = (1+\mu)^{\chi(g)}$, where χ is the p-adic cyclotomic character. Moreover, note that we have a G_F-equivariant surjective homomorphism of rings $\theta : A_{\text{inf}}(O_{\overline{F}}) \to \mathbb{C}_p$ with $\text{Ker}\theta = \xi A_{\text{inf}}(O_{\overline{F}})$. The map θ restricts to a Γ_F-equivariant surjective homomorphism of rings $\theta : A_{\text{inf}}(O_{F_\infty}) \to O_{\widehat{F}_\infty}$ with $\text{Ker}\theta = \xi A_{\text{inf}}(O_{F_\infty})$.

We set $A_{\text{cris}}(O_{F_\infty}) := A_{\text{inf}}(O_{F_\infty})\langle \xi^k/k!, k \in \mathbb{N}\rangle$ and note that $t := \log(1+\mu) = \sum_{k\in\mathbb{N}}(-1)^k \frac{\mu^{k+1}}{k+1}$ converges in $A_{\text{cris}}(O_{F_\infty})$. The ring $A_{\text{cris}}(O_{F_\infty})$ is p-torsion free and t-torsion free, and we set $B_{\text{cris}}^+(O_{F_\infty}) := A_{\text{cris}}(O_{F_\infty})[1/p]$ and $B_{\text{cris}}(O_{F_\infty}) := B_{\text{cris}}^+(O_{F_\infty})[1/t]$. These rings are equipped with a Frobenius endomorphism φ, a continuous Γ_F-action and a decreasing filtration; the ring $A_{\text{cris}}^+(O_{F_\infty})$ and $B_{\text{cris}}^+(O_{F_\infty})$ are further equipped with an appropriate extension of the map θ. Next, let us define $B_{\text{dR}}^+(O_{F_\infty}) := \lim_n (A_{\text{inf}}(O_{F_\infty})[1/p])/(\text{Ker}\theta)^n$ and $B_{\text{dR}}(O_{F_\infty}) := B_{\text{dR}}^+(O_{F_\infty})[1/t]$. These rings are equipped with a Γ_F-action and a decreasing filtration; the ring $B_{\text{dR}}^+(O_{F_\infty})$ is further equipped with an appropriate extension of the map θ. Moreover, we have (φ, Γ_F)-equivariant and filtration compatible injective homomorphism of F-algebras $B_{\text{cris}}^+(O_{F_\infty}) \to B_{\text{dR}}^+(O_{F_\infty})$ and $B_{\text{cris}}(O_{F_\infty}) \to B_{\text{dR}}(O_{F_\infty})$. One can define variations of all the aforementioned rings and their properties over $O_{\overline{F}}$ as well. Furthermore, from [Fon94, Théorème 5.3.7] we have the following (φ, G_F)-equivariant *fundamental exact sequence*:

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \text{Fil}^0 B_{\text{cris}}(O_{\overline{F}}) \xrightarrow{1-\varphi} B_{\text{cris}}(O_{\overline{F}}) \longrightarrow 0. \tag{2.2}$$

Let us set $A_F^+ := O_F[\![\mu]\!]$ and we equip it with a Frobenius endomorphism φ, given as the Witt vector Frobenius on O_F and by setting $\varphi(\mu) = (1+\mu)^p - 1$, and an O_F-linear and continuous action of Γ_F, given as $g(\mu) = (1+\mu)^{\chi(g)} - 1$, for any g in Γ_F. Note that we have a natural embedding $A_F^+ \subset A_{\text{inf}}(O_{F_\infty})$ compatible with Frobenius and Γ_F-action. Set $A_F := A_F^+[1/\mu]^\wedge$, where $^\wedge$ denotes the p-adic completion. The Frobenius endomorphism φ and the continuous action of Γ_F on A_F^+ naturally extend to respective actions on A_F. Let $W(\mathbb{C}_p^\flat)$ denote the ring of p-typical Witt vectors with coefficients in $\mathbb{C}_p^\flat := \text{Frac}(O_{\overline{F}}^\flat)$, and note that $W(\mathbb{C}_p^\flat)$ is equipped with the natural Frobenius on Witt vectors and a continuous (for the weak topology) action of G_F. The the natural (φ, Γ_F)-equivariant embedding $A_F^+ \subset A_{\text{inf}}(O_{F_\infty})$ extends to a natural (φ, Γ_F)-equivariant embedding $A_F \subset W(\mathbb{C}_p^\flat)^{H_F} = W(F_\infty^\flat)$.

Next, we recall some definitions and observations from [Abh24, Subsection 3.1] by fixing the following elements inside A_F^+:

$$\begin{aligned} &[p]_q := \tfrac{q^p-1}{q-1} = \tfrac{\varphi(\mu)}{\mu}, \\ &\mu_0 := \textstyle\sum_{a\in\mathbb{F}_p^\times}((1+\mu)^{[a]} - 1) = -p + \sum_{a\in\mathbb{F}_p}(1+\mu)^{[a]}, \\ &\tilde{p} := \mu_0 + p = \textstyle\sum_{a\in\mathbb{F}_p}(1+\mu)^{[a]}. \end{aligned} \tag{2.3}$$

From loc. cit. note that the element $\tilde{p}$ is the product of $[p]_q$ with a unit in A_F and the element μ_0 is the product of μ^{p-1} with a unit in A_F^+. Now, we consider the subring $S := O_F[\![\mu_0]\!] = (A_F^+)^{\mathbb{F}_p^\times} \subset A_F^+$, which is stable under the action of φ and Γ_0; we equip S with the induced (φ, Γ_0)-action. Moreover, from loc. cit., for any g in Γ_0, we have that $(g-1)\mu_0$ is an element of $\tilde{p}\mu_0 S$ and we can write $\varphi(\mu_0) = u\mu_0\tilde{p}^{p-1}$, for some unit u in S. Furthermore, we have that $\mu_0 S = (\mu A_F^+)^{\mathbb{F}_p^\times}$ and the natural (φ, Γ_F)-equivariant injective map $S \to A_F^+$ is faithfully flat and finite of degree $p-1$. Finally, we set $A_{F,0} := S[1/\mu_0]^\wedge$, where $^\wedge$ denotes the p-adic completion, and note that $A_{F,0}$ is equipped with an induced action of φ and Γ_0.

Remark 2.2. For consistency, one should denote the ring $O_F[\![\mu_0]\!]$ as $A_{F,0}^+$. However, this ring will appear frequently in the rest of the text and the aforementioned notation is too clunky. So we have chosen to denote the ring $O_F[\![\mu_0]\!]$ simply by S.

2.2. p-adic representations. Let T be a finite free $\mathbb{Z}_p$-representation of G_F. By the theory of étale (φ, Γ_F)-modules (see [Fon90]), we can functorially associate to T a finite free étale (φ, Γ_F)-module $D_F(T)$ over A_F of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$. Moreover, by taking Γ_{tor}-invariants of $D_F(T)$, we obtain a finite free étale (φ, Γ_0)-module $D_{F,0}(T) := D_F(T)^{\Gamma_{\mathrm{tor}}}$ over $A_{F,0}$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$. These constructions are functorial in $\mathbb{Z}_p$-representations and induce an exact equivalence of $\otimes$-categories

$$\begin{aligned} \mathrm{Rep}_{\mathbb{Z}_p}(G_F) &\xrightarrow{\sim} (\varphi, \Gamma_F)\text{-}\mathrm{Mod}_{A_F}^{\text{ét}}, \\ \mathrm{Rep}_{\mathbb{Z}_p}(G_F) &\xrightarrow{\sim} (\varphi, \Gamma_0)\text{-}\mathrm{Mod}_{A_{F,0}}^{\text{ét}}. \end{aligned} \tag{2.4}$$

with an exact $\otimes$-compatible quasi-inverse given as $T_F(D) := (W(\mathbb{C}^\flat) \otimes_{A_F} D)^{\varphi=1}$ and $T_F(D_0) := (W(\mathbb{C}^\flat) \otimes_{A_{F,0}} D_0)^{\varphi=1}$, respectively. Similar statements are also true for p-adic representations of G_F.

Next, from the p-adic Hodge theory of G_F (see [Fon82]), one can attach to a p-adic representation V, a filtered φ-module over F of rank $\leq \dim_{\mathbb{Q}_p} V$, given as $D_{\mathrm{cris}}(V) := (B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)^{G_F}$. The representation V is said to be *crystalline* if the natural map $B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_F D_{\mathrm{cris}}(V) \to B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V$ is an isomorphism, or equivalently, $\dim_F D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p} V$. Restricting D_{cris} to the category of crystalline representations of G_F and writing $\mathrm{MF}_F^{\mathrm{wa}}(\varphi)$ for the category of weakly admissible filtered φ-modules over F (see [CF00]), we obtain an exact equivalence of $\otimes$-categories (see [CF00, Théorème A]):

$$D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_F) \xrightarrow{\sim} \mathrm{MF}_F^{\mathrm{wa}}(\varphi), \tag{2.5}$$

with an exact $\otimes$-compatible quasi-inverse given as $V_{\mathrm{cris}}(D) := (\mathrm{Fil}^0(B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_F D))^{\varphi=1}$.

2.3. Galois cohomology and Fontaine–Herr complex. Let T be a $\mathbb{Z}_p$-representation of G_F, and let $D := D_{F,0}(T)$ denote the associated étale (φ, Γ_F)-module over $A_{F,0}$. In [Her98], Herr defined a three term complex in terms of D, which computes the continuous G_F-cohomology of T. More precisely, recall that γ is a generator of Γ_0 and consider the following complex:

$$\mathscr{C}^\bullet(D) : D \xrightarrow{(\gamma-1, 1-\varphi)} D \oplus D \xrightarrow{(1-\varphi, \gamma-1)^\intercal} D, \tag{2.6}$$

where the first map is $x \mapsto ((\gamma-1)x, (1-\varphi)y)$ and the second map is $(x, y) \mapsto (1-\varphi)x - (\gamma-1)y$. Then the complex $\mathscr{C}^\bullet(D)$ computes the continuous G_F-cohomology of T in each cohomological degree, i.e. for each $k \in \mathbb{N}$, we have natural (in T) isomorphims $H^k(\mathscr{C}^\bullet(D)) \xrightarrow{\sim} H^k_{\mathrm{cont}}(G_F, T)$. From the complex it is clear that $H^k_{\mathrm{cont}}(G_F, T) = 0$, for $k \geq 3$. To ease notations, from now onwards we will write $H^k(G_F, T)$ instead of $H^k_{\mathrm{cont}}(G_F, T)$.

Note that for a $\mathbb{Z}_p$-representation T of G_F, the space $H^1(G_F, T)$ classifies all extension classes of $\mathbb{Z}_p$ by T in the category of $\mathbb{Z}_p$-representations of G_F. Similarly, for an étale (φ, Γ_0)-module D, the space $H^1(\mathscr{C}^\bullet(D))$ classifies all extension classes of $A_{F,0}$ by D in the category of étale (φ, Γ_0)-modules over $A_{F,0}$. Therefore, by the exact equivalence in (2.4), we have natural isomorphisms

$$H^1(G_F, T) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Z}_p}(G_F)}(\mathbb{Z}_p, T) \xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi, \Gamma_F)\text{-}\mathrm{Mod}^{\text{ét}}_{A_{F,0}}}(A_{F,0}, D) \xleftarrow{\sim} H^1(\mathscr{C}^\bullet(D)).$$

2.4. Bloch–Kato Selmer groups. In this subsection, we will recall the definition of Bloch–Kato Selmer groups from [BK90]. Let V be a p-adic crystalline representation of G_F. Then we have a natural G_F-equivariant morphism of $\mathbb{Q}_p$-vector spaces $V \to B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V$, sending $x \mapsto 1 \otimes x$. By considering the continuous G_F-cohomology groups, we obtain natural homomorphisms $H^k(G_F, V) \to H^k(G_F, B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)$, for each $k \in \mathbb{N}$.

Definition 2.3. Define the Bloch–Kato Selmer groups of V denoted $H^k_f(G_F, V) \subset H^k(G_F, V)$, for $k \in \mathbb{N}$, by setting,

$$H^k_f(G_F, V) := \begin{cases} H^0(G_F, V), & \text{if } k = 0 \\ \mathrm{Ker}\,(H^1(G_F, V) \to H^1(G_F, B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)), & \text{if } k = 1 \\ 0, & \text{if } k \geq 2. \end{cases}$$

Remark 2.4. For $k \in \mathbb{N}$, the subspace $H^k_f(G_F, V) \subset H^k(G_F, V)$ are also referred to as the *crystalline part of the Galois cohomology of* V. Notably, the subspace $H^1_f(G_F, V) \subset H^1(G_F, V)$ classifies all crystalline extension classes of $\mathbb{Q}_p$ by V, i.e. we have natural isomorphisms

$$H^1_f(G_F, V) \xrightarrow{\sim} \operatorname{Ext}^1_{\operatorname{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F)}(\mathbb{Q}_p, V) \xrightarrow{\sim} \operatorname{Ext}^1_{\mathrm{MF}^{\mathrm{wa}}_F(\varphi)}(F, D_{\mathrm{cris}}(V)),$$

where the last isomorphism follows from the exactness of D_{cris} and V_{cris} (see Subsection 2.2).

Note that we have a natural $\mathbb{Q}_p$-linear and G_F-equivariant morphism $V \to \operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V$, sending $x \mapsto 1 \otimes x$, and it induces a natural homomorphism $H^1(G_F, V) \to H^1(G_F, \operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)$.

Proposition 2.5. *The following natural map is an isomorphism:*

$$\operatorname{Ker}\big(H^1(G_F, V) \to H^1(G_F, \operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)\big) \xrightarrow{\sim} H^1_f(G_F, V).$$

Proof. By the naturality of the action of G_F, we have the following commutative diagram:

$$\begin{array}{ccc} H^1(G_F, V) & \longrightarrow & H^1(G_F, \operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V) \\ & \searrow & \downarrow \\ & & H^1(G_F, B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V). \end{array} \tag{2.7}$$

To show the claim, it is enough to show that the right vertical arrow is injective. Now consider the following exact sequence:

$$0 \longrightarrow \operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \longrightarrow B_{\mathrm{cris}}(O_{\overline{F}}) \longrightarrow B_{\mathrm{cris}}(O_{\overline{F}})/\operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \longrightarrow 0.$$

Upon tensoring this exact sequence with V and taking continuous G_F-cohomology, we obtain an injective map of F-vector spaces:

$$\alpha : D_{\mathrm{cris}}(V)/\operatorname{Fil}^0 D_{\mathrm{cris}}(V) \longrightarrow \big(B_{\mathrm{cris}}(O_{\overline{F}})/\operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V\big)^{G_F}, \tag{2.8}$$

and it is clear that the vertical map in (2.7) is injective if and only if (2.8) is bijective. Now, as we have $B_{\mathrm{dR}}(O_{\overline{F}}) = \operatorname{Fil}^0 B_{\mathrm{dR}}(O_{\overline{F}}) + B_{\mathrm{cris}}(O_{\overline{F}})^{\varphi=1}$ (see [BK90, Proposition 1.17]), therefore, we obtain G_F-equivariant isomorphisms

$$B_{\mathrm{cris}}(O_{\overline{F}})/\operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \xrightarrow{\sim} B_{\mathrm{dR}}(O_{\overline{F}})/\operatorname{Fil}^0 B_{\mathrm{dR}}(O_{\overline{F}}) \xrightarrow{\sim} \oplus_{k<0} \mathbb{C}_p \cdot t^k,$$

where the last isomorphism follows from [Fon94, Subsection 1.5.5]. Therefore, the codomain of the map (2.8) can be written as $\big(B_{\mathrm{cris}}(O_{\overline{F}})/\operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V\big)^{G_F} = (\oplus_{k<0} t^k \mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{G_F} = \oplus_{k<0} \operatorname{gr}^k D_{\mathrm{cris}}(V)$. Counting dimensions, we note that we have

$$\dim_F(\operatorname{Fil}^0 D_{\mathrm{cris}}(V)) + \dim_F(\oplus_{k<0} \operatorname{gr}^k D_{\mathrm{cris}}(V)) = \dim_F D_{\mathrm{cris}}(V),$$

so the domain and the codomain of the F-linear injective map in (2.8) have the same dimension. Hence, (2.8) is bijective allowing us to conclude. ∎

Corollary 2.6. *Let V be a p-adic crystalline representation of G_F. Then the following complex*

$$\mathscr{D}^\bullet(D_{\mathrm{cris}}(V)) : \operatorname{Fil}^0 D_{\mathrm{cris}}(V) \xrightarrow{1-\varphi} D_{\mathrm{cris}}(V), \tag{2.9}$$

computes the crystalline part of the Galois cohomology of V, i.e. $H^k(\mathscr{D}^\bullet(D_{\mathrm{cris}}(V))) \xrightarrow{\sim} H^k_f(G_F, V)$ for each $k \in \mathbb{N}$.

Proof. Tensoring the fundamental exact sequence in (2.2) with V, we obtain a G_F-equivariant exact sequence

$$0 \longrightarrow V \longrightarrow \operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V \xrightarrow{1-\varphi} B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V \longrightarrow 0.$$

By computing the continuous Galois cohomoloy, we obtain the following long exact sequence:

$$\begin{aligned} 0 \longrightarrow H^0(G_F, V) \longrightarrow \operatorname{Fil}^0 D_{\mathrm{cris}}(V) \xrightarrow{1-\varphi} D_{\mathrm{cris}}(V) \longrightarrow H^1(G_F, V) \longrightarrow \\ \longrightarrow H^1(G_F, \operatorname{Fil}^0 B_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V). \end{aligned} \tag{2.10}$$

The claim now follows from Proposition 2.5. ∎

3. Wach modules

In this section, we will recall the definition of Wach modules, their relationship with p-adic crystalline representations and prove some results on the Nygaard filtration on Wach modules. From Subsection 2.1 recall that we have the ring $A_F^+ = O_F[\![\mu]\!]$ equipped with a Frobenius endomorphism φ and a continuous action of Γ_F. Moreover, we fixed $\mu = q-1$ and $[p]_q = \varphi(\mu)/\mu$ in A_F^+.

Definition 3.1. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over A_F^+ with weights in the interval $[a, b]$ is a finite free A_F^+-module N equipped with a continuous and semilinear action of Γ_F satisfying the following:

(1) Γ_F acts trivially on $N/\mu N$.

(2) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \rightarrow N[1/\varphi(\mu)]$ commuting with the action of Γ_F such that $\varphi(\mu^b N) \subset \mu^b N$ and the cokernel of the induced injective map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \rightarrow \mu^b N$ is killed by $[p]_q^{b-a}$.

Say that N is *effective* if one can take $b = 0$ and $a \leq 0$. Denote the category of Wach modules over A_F^+ as $(\varphi, \Gamma)\text{-Mod}_{A_F^+}^{[p]_q}$ with morphisms between objects being A_F^+-linear, Γ_F-equivariant and φ-equivariant (after inverting μ) morphisms.

Remark 3.2. Let N be a finitely generated A_F^+-module. Then from [Abh23b, Lemma 3.10] we have that the condition (2) of Definition 3.1 is equivalent to giving an A_F^+-linear and Γ_F-equivariant isomorphism $\varphi_N : (\varphi^* N)[1/[p]_q] = (A_F^+ \otimes_{\varphi, A_F^+} N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$.

Remark 3.3. Extending scalars along $A_F^+ \rightarrow A_F$ induces a fully faithful functor $(\varphi, \Gamma_F)\text{-Mod}_{A_F^+}^{[p]_q} \rightarrow (\varphi, \Gamma_F)\text{-Mod}_{A_F}^{\text{ét}}$ (see [Abh23a, Proposition 3.3]).

3.1. Descent of Wach modules.

In this subsection, we will show that Wach modules over A_F^+ descend to a certain subring of A_F^+ (see [Fon90; Wac97]). From Subsection 2.1 recall that we have the ring $S = O_F[\![\mu_0]\!]$ equipped with a Frobenius endomorphism φ and a continuous action of Γ_0. Moreover, we fixed the elements $\mu_0 = \sum_{a \in \mathbb{F}_p^\times}((1+\mu)^{[a]} - 1)$ and $\tilde{p} = \mu_0 + p$ in S.

Definition 3.4. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over S with weights in the interval $[a, b]$ is a finite free S-module M equipped with a continuous and semilinear action of Γ_0 satisfying the following:

(1) Γ_0 acts trivially on $M/\mu_0 M$.

(2) There is a Frobenius-semilinear operator $\varphi : M[1/\mu_0] \rightarrow M[1/\varphi(\mu_0)]$ commuting with the action of Γ_0 such that $\varphi(\mu_0^b M) \subset \mu_0^b M$ and the cokernel of the induced injective map $(1 \otimes \varphi) : \varphi^*(\mu_0^b M) \rightarrow \mu_0^b M$ is killed by $\tilde{p}^{b-a}$.

Say that M is *effective* if one can take $b = 0$ and $a \leq 0$. Denote the category of Wach modules over S as $(\varphi, \Gamma)\text{-Mod}_S^{\tilde{p}}$ with morphisms between objects being S-linear, Γ_0-equivariant and φ-equivariant (after inverting μ_0) morphisms.

Lemma 3.5. *Let M be a finitely generated S-module. Then (2) of Definition 3.4 is equivalent to giving an S-linear and Γ_0-equivariant isomorphism $\varphi_M : (\varphi^* M)[1/\tilde{p}] = (S \otimes_{\varphi, S} M)[1/\tilde{p}] \xrightarrow{\sim} M[1/\tilde{p}]$.*

Proof. Suppose M satisfes condition (2) of Definition 3.4. Then, the map $1 \otimes \varphi : \varphi^*(\mu_0^b M) \rightarrow \mu_0^b M$ induces an isomorphism $1 \otimes \varphi : (\mu_0^b \varphi^* M)[1/\tilde{p}] \xrightarrow{\sim} (\mu_0^b M)[1/\tilde{p}]$. Hence, we obtain an isomorphism

$$\varphi_M : (\varphi^* M)[1/\tilde{p}] \xrightarrow[\sim]{\mu_0^b} (\mu_0^b \varphi^* M)[1/\tilde{p}] \xrightarrow[\sim]{1 \otimes \varphi} (\mu_0^b M)[1/\tilde{p}] \xleftarrow[\sim]{\mu_0^b} M[1/\tilde{p}].$$

Since, $1 \otimes \varphi$ commutes with the action of Γ_0, we deduce that φ_M is Γ_0-equivariant.

Conversely, suppose that M is equipped with an S-linear and Γ_0-equivariant isomorphism $\varphi_M : (\varphi^* M)[1/\tilde{p}] \xrightarrow{\sim} M[1/\tilde{p}]$. Then note that for some $a, b \in \mathbb{Z}$ with $b \geq a$ we can write $\tilde{p}^b \varphi_M(\varphi^* M) \subset M \subset$

$\tilde{p}^a\varphi_M(\varphi^*M)$. So we get an S-semilinear and Γ_0-equivariant map as the composition $\varphi : \mu_0^b M \xrightarrow{\text{can}} \varphi^*(\mu_0^b M) \xrightarrow{\varphi_M} \mu_0^b M$. This extends to an S-semilinear and Γ_0-equivariant map $\varphi : M[1/\mu_0] \to M[1/\varphi(\mu_0)]$, and we have,

$$\varphi_M(\varphi^*(\mu_0^b M)) = \mu_0^b \tilde{p}^{(p-1)b}\varphi_M(\varphi^*M) \subset \mu_0^b \tilde{p}^{(p-1)(b-1)}M \subset \tilde{p}^{a-b}\varphi_M(\varphi^*(\mu_0^b M))$$

Then it follows that $1 \otimes \varphi = \varphi_M : \varphi^*(\mu_0^b M) \to \mu_0^b M$ is injective, its cokernel is killed by $\tilde{p}^{b-a}$ and it commutes with the action of Γ_0. Hence, M satisfies condition (2) of Definition 3.4. ■

Remark 3.6. Similar to Remark 3.3, it is easy to see that extending scalars along $S \to A_{F,0}$ induces a fully faithful functor $(\varphi, \Gamma_0)\text{-Mod}_S^{\tilde{p}} \to (\varphi, \Gamma_0)\text{-Mod}_{A_{F,0}}^{\text{ét}}$.

Next, we will compare the two notions of Wach modules over S and A_F^+, respectively. We start with the following observation:

Proposition 3.7. *Let N be a Wach module over A_F^+. Then $M := N^{\mathbb{F}_p^\times}$ is a Wach module over S, and A_F^+-linearly extending the S-linear inclusion $M \subset N$, induces a natural isomorphism*

$$A_F^+ \otimes_S M \xrightarrow{\sim} N, \tag{3.1}$$

of Wach modules over A_F^+. Moreover, the isomorphism in (3.1) induces a natural isomorphism of O_F-modules $M/\mu_0 M \xrightarrow{\sim} N/\mu N$ compatible with the respective Frobenii.

Proof. The claim follows from [Abh24, Theorem 1.5]. However, we will give another self-contained proof. Let $A := A_F^+$ and from Remark 2.1, note that for the A-module N we have an $\mathbb{F}_p^\times$-decomposition $N = \bigoplus_{i=0}^{p-2} N_i$, where $M = N^{\mathbb{F}_p^\times} = N_0$ and each N_i is a (p, μ_0)-adically complete S-module equipped with a continuous and semilinear action of Γ_0. Moreover, recall that A is flat and finite of degree $p-1$ over the noetherian ring S, so it follows that N is finite free over S, and the S-submodule $M \subset N$ is finitely generated. Now, let us consider the following natural commutative diagram:

$$\begin{array}{ccc} A \otimes_S M & \longrightarrow & N \\ \downarrow & & \downarrow \\ A \otimes_S M[1/\tilde{p}] & \longrightarrow & N[1/\tilde{p}], \end{array} \tag{3.2}$$

where the right vertical arrow is the natural inclusion, the left vertical arrow is injective because M is $\tilde{p}$-torsion free (since the same is true for N) and the natural map $S \to A$ is flat.

We claim that the top horizontal arrow of (3.2) is bijective. First, note that to show the injectivity of the top horizontal arrow, it is enough to show that the injectivity of the bottom horizontal arrow in (3.2). The module $N[1/\tilde{p}]$ is finite free over $A[1/\tilde{p}]$ and the map $S \to A$ is faithfully flat and finite of degree $p-1$, so we see that $N[1/\tilde{p}]$ is finite free over $S[1/\tilde{p}]$ as well. As $N[1/\tilde{p}]$ is equipped with an induced action of Γ_F, therefore, from the decomposition in Remark 2.1, it follows that $M[1/\tilde{p}] \xrightarrow{\sim} (N[1/\tilde{p}])_0$ as $S[1/\tilde{p}]$-modules, in particular, $M[1/\tilde{p}]$ is finite projective over $S[1/\tilde{p}]$, hence finite free because $S[1/\tilde{p}]$ is a principal ideal domain. As the natural map $S \to A$ is flat, it follows that the bottom horizontal arrow in (3.2) is injective. Next, let us show that the top horizontal arrow in (3.2) is surjective. Note that we have an O_F-linear and Γ_F-equivariant surjection $N \to N/\mu N$ and using the decomposition in Remark 2.1, it can be rewritten as a Γ_F-equivariant surjection $\bigoplus_{i=0}^{p-2} N_i \to \bigoplus_{i=0}^{p-2}(N/\mu N)_i$. In particular, the latter map is surjective on each term, i.e. the induced O_F-linear map $N_i \to (N/\mu N)_i$ is surjective, for each $0 \leq i \leq p-2$. However, recall that Γ_F acts trivially on $N/\mu N$, therefore, we see that $N/\mu N = \bigoplus_{i=0}^{p-2}(N/\mu N)_i = (N/\mu N)_0$. In particular, the natural O_F-linear map $M \to N/\mu N$ is surjective. Since μ belongs to the Jacobson radical of A, therefore, Nakayama Lemma implies that the natural (φ, Γ_F)-equivariant map $A \otimes_S M \to N$ is also surjective. Hence, it follows that the A-linear extension of the S-linear inclusion $M \subset N$ induces the natural (φ, Γ_F)-equivariant isomorphism in (3.1), i.e. $A \otimes_S M \xrightarrow{\sim} N$.

Next, we will compute the kernel of the surjective map $M \to N/\mu N$, which is given as $\mu N \cap M$ inside N. Using the decomposition in Remark 2.1, we get that $\mu N \cap M = (\mu M)_0$, and we claim that

$\mu_0 M \xrightarrow{\sim} (\mu N)_0$ as S-modules. As the natural map $\mu_0 M \rightarrow (\mu N)_0$ is injective, we need to show that it is surjective as well. Note that we can write,

$$(\mu N)_0 = \sum_{i+j=0 \bmod p-1} (\mu A)_i \otimes_S N_j.$$

Moreover, using the (φ, Γ_F)-equivariant isomorphism $A \otimes_S M \xrightarrow{\sim} N$, note that we have $A_j \otimes_S M \xrightarrow{\sim} N_j$, for each $0 \leq j \leq p-2$. Therefore, for each pair i, j such that $i + j = 0 \bmod p - 1$, we see that

$$(\mu A)_i \otimes_S N_j \xleftarrow{\sim} (\mu A)_i \otimes_S A_j \otimes_S N_0 \subset (\mu A)_0 \otimes_S N_0.$$

Now, recall that we have, $\mu_0 S = (\mu A)^{\mathbb{F}_p^\times} = (\mu A)_0$. Hence, from the preceding discussion, we get that $\mu_0 M \xrightarrow{\sim} (\mu N)_0$. Subsequently, we have also obtained that $M/\mu_0 M \xrightarrow{\sim} N/\mu N$ as O_F-modules, and therefore, the action of Γ_0 is trivial on $M/\mu_0 M$. Moreover, note that the S-module M is p-torsion free and μ_0-torsion free and $M/\mu_0 M \xrightarrow{\sim} N/\mu N$ is p-torsion free. Therefore, from [Abh23b, Lemma 3.5] and [Fon90, Proposition B.1.2.4], it follows that M is finite free over A.

Finally, we will show the Frobenius condition on M (see Definition 3.4 and Lemma 3.5). Recall that $\tilde{p}$ is the product of $[p]_q$ with a unit in A, so the Frobenius on N can also be given as an A-linear isomorphism $\varphi^*(N)[1/\tilde{p}] \xrightarrow{\sim} N[1/\tilde{p}]$ (see Definition 3.1 and Remark 3.2). Moreover, the Frobenius on N commutes with the action of Γ_F. Therefore, by taking invariants under the action of $\mathbb{F}_p^\times$, of the isomorphism $\varphi^*(N)[1/\tilde{p}] \xrightarrow{\sim} N[1/\tilde{p}]$, and using the (φ, Γ_F)-equivariant isomorphism in (3.1), we conclude that M is equipped with an S-linear isomorphism $\varphi^*(M)[1/\tilde{p}] \xrightarrow{\sim} M[1/\tilde{p}]$, compatible with the natural action of Γ_0. This allows us to conclude. ■

Now, let M be a Wach module over S and let $N := A_F^+ \otimes_S M$ be a finite free module over A_F^+. Then using the natural (φ, Γ_0)-action on M, we see that N is naturally equipped with a semilinear and continuous action of Γ_F and an A_F^+-linear isomorphism $\varphi_N : (\varphi^* N)[1/\tilde{p}] \xrightarrow{\sim} N[1/\tilde{p}]$. As $\tilde{p}$ is the product of $[p]_q$ with a unit in A_F^+, therefore, using Remark 3.2 we conclude that N is a Wach module over A_F^+. More generally, we have that,

Theorem 3.8. *The following natural functor induces an exact equivalence of $\otimes$-categories,*

$$\begin{aligned} (\varphi, \Gamma_0)\text{-Mod}_S^{\tilde{p}} &\xrightarrow{\sim} (\varphi, \Gamma_F)\text{-Mod}_{A_F^+}^{[p]_q} \\ M &\longmapsto A_F^+ \otimes_S M. \end{aligned} \tag{3.3}$$

with an exact $\otimes$-compatible quasi-inverse functor given as $N \mapsto N^{\mathbb{F}_p^\times}$.

Proof. Note that the natural (φ, Γ_F)-equivariant map $S \rightarrow A_F^+$ is faithfully flat and finite of degree $p-1$, in particular, the functor in (3.3) is exact and fully faithful. Moreover, by Proposition 3.7 we see that (3.3) is essentially surjective and its compatibility with tensor products is obvious. It remains to show that the quasi-inverse functor is exact and compatible with $\otimes$-products. So, consider an A_F^+-linear and (φ, Γ_F)-equivariant exact sequence of Wach modules over A_F^+ as

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0.$$

Setting $M_i := N_i^{\mathbb{F}_p^\times}$, for each $i = 1, 2, 3$, and considering the associated long exact exact sequence for the cohomology of the $\mathbb{F}_p^\times$-action and noting that $H^1(\mathbb{F}_p^\times, N_1) = 0$, since $p - 1$ is invertible in $\mathbb{Z}_p$, we obtain an S-linear and (φ, Γ_F)-equivariant exact sequence of Wach modules over S as

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0.$$

Finally, for any two Wach modules N_1 and N_2 over A_F^+, set $M_i := N_i^{\mathbb{F}_p^\times}$, for each $i = 1, 2$, and using Proposition 3.7, note that we have

$$(N_1 \otimes_{A_F^+} N_2)^{\mathbb{F}_p^\times} = ((A_F^+ \otimes_S M_1) \otimes_{A_F^+} (A_F^+ \otimes_S M_2))^{\mathbb{F}_p^\times} = M_1 \otimes_S M_2.$$

This allows us to conclude. ■

3.2. Wach modules and crystalline representations. Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_F. To any T in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$, Berger functorially associated a unique Wach module $N_F(T)$ over A_F^+ in [Ber04]. More generally, we have the following:

Theorem 3.9 ([Fon90; Wac96; Col99; Ber04])**.** *The Wach module functor induces an equivalence of $\otimes$-catgeories*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_F)\text{-}\mathrm{Mod}_{A_F^+}^{[p]_q}, \qquad T \longmapsto N_F(T),$$

with a $\otimes$-compatible quasi-inverse given as $N \mapsto T_F(N) = \big(W(\mathbb{C}_p^\flat) \otimes_{A_F^+} N\big)^{\varphi=1}$.

Remark 3.10. Let us recall an important comparison result from [Ber04, Théorème III.3.1], between a Wach module N and its associated $\mathbb{Z}_p$-representation $T_F(N)$, which will be useful later. To recall the result, we need to introduce some notations. Let A denote the p-adic completion of the maximal unramified extension of A_F inside $W(\mathbb{C}_p^\flat)$. The ring A is stable under the (φ, G_F)-action on $W(\mathbb{C}_p^\flat)$, and we equip it with induced structures. Then, we have that $A^{H_F} = A_F$ and $A^{\varphi=1} = \mathbb{Z}_p$. Moreover, we set $A^+ := A_{\mathrm{inf}}(O_{\overline{F}}) \cap A \subset W(\mathbb{C}_p^\flat)$, which is stable under the (φ, G_F)-action on A, and we have that $(A^+)^{H_F} = A_F^+$ and $(A^+)^{\varphi=1} = \mathbb{Z}_p$. Now, the result in loc. cit. states that we have a natural $A^+[1/\mu]$-linear and (φ, G_F)-equivariant comparison isomorphism $A^+[1/\mu] \otimes_{A_F^+} N \xrightarrow{\sim} A^+[1/\mu] \otimes_{\mathbb{Z}_p} T_F(N)$.

Remark 3.11. In Theorem 3.9, note that we do not expect the functor N_F to be exact. However, by passing to the associated isogeny categories, the Wach module functor induces an exact equivalence of $\otimes$-categories $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_F) \xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{A_F^+[1/p]}^{[p]_q}$, via $T[1/p] \mapsto N_F(T)[1/p]$ and with an exact $\otimes$-compatible quasi-inverse given as $N[1/p] \mapsto V_F(N[1/p]) = \big(W(\mathbb{C}_p^\flat) \otimes_{A_F^+} N[1/p]\big)^{\varphi=1}$ (see [Abh23a, Corollary 4.3]).

Remark 3.12. Let N be a Wach module over A_F^+ and $T = T_F(N)$ the associated $\mathbb{Z}_p$-representation of G_F from Theorem 3.9. Then for each $r \in \mathbb{Z}$, it is straightforward to verify that $\mu^{-r}N(r)$ is a Wach module over A_F^+ and $T_F(\mu^{-r}N(r)) \xrightarrow{\sim} T(r)$, where (r) denotes a twist by χ^r.

By combining Theorem 3.8 and Theorem 3.9, we obtain the following:

Theorem 3.13. *The following functor induces an equivalence of $\otimes$-catgeories*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_0)\text{-}\mathrm{Mod}_S^{\tilde{p}}, \qquad T \longmapsto M_F(T) := N_F(T)^{\mathbb{F}_p^\times},$$

with a $\otimes$-compatible quasi-inverse given as $M \mapsto T_F(M) = \big(W(\mathbb{C}_p^\flat) \otimes_S M\big)^{\varphi=1}$.

3.3. Nygaard filtration on Wach modules. In this subsection, we will study the Nygaard filtration on Wach modules over A_F^+ and over S. We begin with the former case.

3.3.1. Filtration on Wach modules over A_F^+. Let N be a Wach module over A_F^+. We equip N with a decreasing filtration called the *Nygaard filtration* as,

$$\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}, \text{ for } k \in \mathbb{Z}. \tag{3.4}$$

From the definition it is clear that N is effective if and only if $\mathrm{Fil}^0 N = N$. Similarly, we equip the $A_F^+[1/p]$-module $N[1/p]$ with a Nygaard filtration and it is easy to see that $\mathrm{Fil}^k(N[1/p]) = (\mathrm{Fil}^k N)[1/p]$.

Lemma 3.14 ([Abh23c, Lemma 3.3 & Lemma 3.4])**.** *Let N be a Wach module A_F^+.*

(1) *For any $k, r \in \mathbb{Z}$, and the Wach module $\mu^{-r}N(r)$ over A_F^+, we have that $\mathrm{Fil}^k(\mu^{-r}N(r)) = \mu^{-r}(\mathrm{Fil}^{r+k} N)(r)$.*

(2) *For all* $j, k \in \mathbb{Z}$, *we have that* $\mu^{-j}\mathrm{Fil}^k N \cap \mu^{-j+1}N = \mu^{-j+1}\mathrm{Fil}^{k-1}N \subset N[1/\mu]$. *A similar statement is true for the* $A_F^+[1/p]$*-module* $N[1/p]$.

Proof. For the claim (1), note that the inclusion $\mu^{-r}(\mathrm{Fil}^{r+k}N)(r) \subset \mathrm{Fil}^k(\mu^{-r}N(r))$ is obvious. For the converse, let $\mu^{-r}x \otimes \epsilon^{\otimes r}$ be an element of $\mathrm{Fil}^k(\mu^{-r}N(r))$, with x in N and $\epsilon^{\otimes r}$ a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(r)$. By assumption, we have that $\varphi(\mu^{-r}x \otimes \epsilon^{\otimes r}) = ([p]_q\mu)^{-r}\varphi(x) \otimes \epsilon^{\otimes r}$ belongs to $[p]_q^k\mu^{-r}N(r)$. Therefore, we see that $\varphi(x)$ belongs to $[p]_q^{r+k}N$, i.e. x is in $\mathrm{Fil}^{r+k}N$.

For (2), note that it is enough to show the claim for $j = 0$, i.e. $\mathrm{Fil}^k N \cap \mu N = \mu\mathrm{Fil}^{k-1}N \subset N$. Now, using (1) we can assume that N is effective. The claim is obvious if $\mathrm{Fil}^{k-1}N = N$. So, we may further assume that $\mathrm{Fil}^{k-1}N \subsetneq N$, i.e. $k \geq 2$. Let x be an element of $\mathrm{Fil}^k N \cap \mu N$ and write $x = \mu y$, for some y in N. We claim that y is in $\mathrm{Fil}^{k-1}N$. Indeed, note that $\varphi(x)$ is in $[p]_q^k N$, therefore, we get that $\mu\varphi(y)$ is in $[p]_q^{k-1}N$, i.e. $\mu\varphi(y) = [p]_q^{k-1}z$, for some z in N. In particular, $[p]_q^{r-1}z = p^{r-1}z = 0 \bmod \mu N$. But, we have that $N/\mu N$ is p-torsion free, so it follows that $z = 0 \bmod \mu N$, i.e. y belongs to $\mathrm{Fil}^{k-1}N$. The other inclusion is obvious, as we have that $\mu\mathrm{Fil}^{k-1}N \subset \mathrm{Fil}^k N$. This concludes our proof. ∎

Next, we note that $(N/\mu N)[1/p]$ is a φ-module over F since $[p]_q = p \bmod \mu A_F^+$, and $N/\mu N$ is equipped with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $(N/\mu N)[1/p]$ with the induced filtration $\mathrm{Fil}^k((N/\mu N)[1/p]) := \mathrm{Fil}^k(N/\mu N)[1/p]$, and note that it is a filtered φ-module over F. From [Ber04, Théorème III.4.4] and [Abh23a, Theorem 1.7 & Remark 1.8] we have the following:

Theorem 3.15. *Let* N *be a Wach module over* A_F^+ *and* $V := T_F(N)[1/p]$ *the associated crystalline representation of* G_F *from Theorem 3.9. Then we have that* $(N/\mu N)[1/p] \xrightarrow{\sim} D_{\mathrm{cris}}(V)$ *as filtered* φ*-modules over* F*.*

From Theorem 3.15 we have a surjection $\mathrm{Fil}^k N[1/p] \twoheadrightarrow \mathrm{Fil}^k D_{\mathrm{cris}}(V)$ and we would like to determine its kernel.

Lemma 3.16. *Let* N *be a Wach module over* A_F^+*. Then, for any* $k \in \mathbb{Z}$*, the following sequence is exact:*

$$0 \longrightarrow \mathrm{Fil}^{k-1}N \xrightarrow{\mu} \mathrm{Fil}^k N \longrightarrow \mathrm{Fil}^k(N/\mu N) \longrightarrow 0. \tag{3.5}$$

In particular, we have that $\mathrm{Ker}(\mathrm{Fil}^k N[1/p] \twoheadrightarrow \mathrm{Fil}^k D_{\mathrm{cris}}(V)) = \mu\mathrm{Fil}^{k-1}N[1/p]$*. Moreover, by taking the associated graded pieces, we get that* $\mathrm{gr}^k N \xrightarrow{\sim} \mathrm{gr}^k(N/\mu N)$ *and* $\mathrm{gr}^k N[1/p] \xrightarrow{\sim} \mathrm{gr}^k D_{\mathrm{cris}}(V)$*.*

Proof. Exactness of (3.5) easily follows from Lemma 3.14 (2). Then, by taking the associated graded pieces, we obtain the following exact sequence:

$$0 \longrightarrow \mathrm{gr}^{k-1}N \xrightarrow{\mu} \mathrm{gr}^k N \longrightarrow \mathrm{gr}^k(N/\mu N) \longrightarrow 0.$$

It is clear that the map $\mathrm{gr}^{k-1}N \xrightarrow{\mu} \mathrm{gr}^k N$ is trivial, i.e. $\mathrm{gr}^k N \xrightarrow{\sim} \mathrm{gr}^k(N/\mu N)$. Rest is obvious. ∎

Remark 3.17. The Nygaard filtration on a Wach module N over A_F^+ is stable under the action of Γ_F. Therefore, for any g in Γ_F and $k \in \mathbb{Z}$, using Lemma 3.14 (2) we see that $(g-1)\mathrm{Fil}^k N \subset (\mathrm{Fil}^k N) \cap \mu N = \mu\mathrm{Fil}^{k-1}N$.

Finally, we will check the compatibility of the Nygaard filtration with exact sequences of Wach modules over A_F^+. So consider the following A_F^+-linear and (φ, Γ_F)-equivariant exact sequence of Wach modules over A_F^+:

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0. \tag{3.6}$$

Lemma 3.18. *For* $k \in \mathbb{Z}$*, we have that* $N_1 \cap \mathrm{Fil}^k N_2 = \mathrm{Fil}^k N_1$*.*

Proof. Let $D_i := A_F \otimes_{A_F^+} N_i$, for $i = 1, 2$. Note that we have $N_1 := D_1 \cap N_2 \subset D_2$. So, if x is in $N_1 \cap \mathrm{Fil}^k N_2$, then we see that $\varphi(x)$ is in $D_1 \cap [p]_q^k N_2$, i.e. $[p]_q^{-k}\varphi(x)$ is in $D_1 \cap N_2 = N_1$. Hence, we get that x in $\mathrm{Fil}^k N_1$. ∎

Remark 3.19. For any $j, k \in \mathbb{Z}$, we have that $N_1 \cap \mu^j \mathrm{Fil}^k N_2 = \mu^j \mathrm{Fil}^k N_1$. Indeed, using the same notation as in the proof of Lemma 3.18, we note that if x is in $N_1 \cap \mu^j N_2$, then we can write $x = \mu^j y$ for some y in N_2 and we see that $y = \mu^{-j} x$ is in $D_1 \cap N_2 = N_1$, i.e. x is in $\mu^j N_1$. Combining this with Lemma 3.18 we get the claim.

The statement of Lemma 3.18 can be strengthened after inverting p. More precisely, we have,

Lemma 3.20. *The following sequence is exact for each $k \in \mathbb{Z}$:*

$$0 \longrightarrow \mathrm{Fil}^k N_1[1/p] \longrightarrow \mathrm{Fil}^k N_2[1/p] \longrightarrow \mathrm{Fil}^k N_3[1/p] \longrightarrow 0. \tag{3.7}$$

Proof. For each $i = 1, 2, 3$ and $r \in \mathbb{Z}$, from Remark 3.12 note that $\mu^{-r} N_i(r)$, where (r) denotes a twist by χ^r, is again a Wach module over A_F^+ and (3.6) is exact if and only if the following is exact

$$0 \longrightarrow \mu^{-r} N_1(r) \longrightarrow \mu^{-r} N_2(r) \longrightarrow \mu^{-r} N_3(r) \longrightarrow 0.$$

Now, let $P_i := N_i[1/p]$ for $i = 1, 2, 3$, and by using Lemma 3.14 (1), note that we have $\mathrm{Fil}^{k-r}(\mu^{-r} P_i(r)) = \mu^{-r} \mathrm{Fil}^k P_i(r)$. Therefore, we see that (3.7) is exact if and only if the following is exact:

$$0 \longrightarrow \mathrm{Fil}^{k-r}(\mu^{-r} P_1(r)) \longrightarrow \mathrm{Fil}^{k-r}(\mu^{-r} P_2(r)) \longrightarrow \mathrm{Fil}^{k-r}(\mu^{-r} P_2(r)) \longrightarrow 0.$$

In particular, without loss of generality we may assume that each N_i is an effective Wach module over A_F^+, for $i = 1, 2, 3$, in particular, $\mathrm{Fil}^0 N_i = N_i$ and $\mathrm{Fil}^0 P_i = P_i$. We will prove the claim by induction on $k \in \mathbb{N}$. So let us assume the claim for $k-1$ and consider the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu \mathrm{Fil}^{k-1} P_1 & \longrightarrow & \mathrm{Fil}^k P_1 & \longrightarrow & \mathrm{Fil}^k(P_1/\mu P_1) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu \mathrm{Fil}^{k-1} P_2 & \longrightarrow & \mathrm{Fil}^k P_2 & \longrightarrow & \mathrm{Fil}^k(P_2/\mu P_2) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mu \mathrm{Fil}^{k-1} P_3 & \longrightarrow & (\mathrm{Fil}^k P_2)/(\mathrm{Fil}^k P_1) & \longrightarrow & \mathrm{Fil}^k(P_3/\mu P_3) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array} \tag{3.8}$$

In (3.8), note that the first and the second rows are exact by Lemma 3.16 and the first column is exact by the induction assumption. In the second column, using that $P_1 = (A_F \otimes_{A_F^+} P_1) \cap P_2 \subset A_F \otimes_{A_F^+} P_2$, it easily follows that $\mathrm{Fil}^k P_1 \subset \mathrm{Fil}^k P_2$. Now, let $V_i := V_F(P_i)$, for each $i = 1, 2, 3$ (see Remark 3.11). Then, from Theorem 3.15, we have filtered isomorphisms $\mathrm{Fil}^k(P_i/\mu P_i) \xrightarrow{\sim} \mathrm{Fil}^k D_{\mathrm{cris}}(V_i)$. Recall that D_{cris} is an exact functor and in the category $\mathrm{MF}_F^{\mathrm{wa}}(\varphi)$ (see Subsection 2.2) exact sequences are compatible with filtration. So we get that the rightmost column in (3.8) is also exact. Hence, it follows that the last row in (3.8) is exact and from Lemma 3.16 we conclude that $(\mathrm{Fil}^k P_2)/(\mathrm{Fil}^k P_1) \xrightarrow{\sim} \mathrm{Fil}^k P_3$, proving the claim. ∎

3.3.2. Filtration on Wach modules over S. Let M be a Wach module over S. We equip M with a decreasing filtration called the *Nygaard filtration* as,

$$\mathrm{Fil}^k M := \{x \in N \text{ such that } \varphi(x) \in \tilde{p}^k M\}, \text{ for } k \in \mathbb{Z}. \tag{3.9}$$

From the definition it is clear that M is effective if and only if $\mathrm{Fil}^0 M = M$. Now, let $N := A_F^+ \otimes_S M$ and note that the natural S-linear map $M \to N$ is injective and (φ, Γ_F)-equivariant. Moreover, it is easy to see that $\mathrm{Fil}^k M = M \cap \mathrm{Fil}^k N \subset N$, where the Nygaard filtration on N was defined in (3.4). Similar to Lemma 3.14, we claim the following:

Lemma 3.21. *For all $k \in \mathbb{Z}$, we have that* $\mathrm{Fil}^k M \cap \mu_0 M = \mu_0 \mathrm{Fil}^{k-p+1} M \subset M[1/\mu_0]$.

Proof. From Lemma 3.21, note that we have $\mathrm{Fil}^k N \cap \mu N = \mu \mathrm{Fil}^{k-1} N$. Moreover, recall that μ_0 is given as the product of μ^{p-1} with a unit in A_F^+. Therefore, it follows that $\mathrm{Fil}^k N \cap \mu_0 N = \mathrm{Fil}^k N \cap \mu^{p-1} N = \mu^{p-1}\mathrm{Fil}^{k-p+1} N = \mu_0 \mathrm{Fil}^{k-p+1} N$. Hence, we obtain that $\mathrm{Fil}^k M \cap \mu_0 M = \mathrm{Fil}^k N \cap \mu_0 M = \mu_0 \mathrm{Fil}^{k-p+1} N \cap \mu_0 M = \mu_0 \mathrm{Fil}^{k-p+1} M$. ∎

Next, we equip the O_F-module $M/\mu_0 M$ with a filtration $\mathrm{Fil}^k(M/\mu_0 M)$, for $k \in \mathbb{Z}$ and given as the image of $\mathrm{Fil}^k M$ under the surjection $M \twoheadrightarrow M/\mu_0 M$. Moreover, we equip the O_F-module $N/\mu N$ with a filtration $\mathrm{Fil}^k(N/\mu N)$, for $k \in \mathbb{Z}$ and given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. From Proposition 3.7, we have that the natural S-linear and (φ, Γ_F)-equivariant map $M \to N$ induces an O_F-linear isomorphism $M/\mu_0 M \xrightarrow{\sim} N/\mu N$ compatible with the respective Frobenii. We claim the following:

Lemma 3.22. *The natural isomorphism $M/\mu_0 M \xrightarrow{\sim} N/\mu N$ induces an O_F-*linear *isomorphism* $\mathrm{Fil}^k(M/\mu_0 M) \xrightarrow{\sim} \mathrm{Fil}^k(N/\mu N)$, *for each $k \in \mathbb{Z}$.*

Proof. Consider the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mu_0 \mathrm{Fil}^{k-p+1} M & \longrightarrow & \mathrm{Fil}^k M & \longrightarrow & \mathrm{Fil}^k(M/\mu_0 M) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mu \mathrm{Fil}^{k-1} N & \longrightarrow & \mathrm{Fil}^k N & \longrightarrow & \mathrm{Fil}^k(N/\mu N) & \longrightarrow & 0, \end{array} \tag{3.10}$$

where the top row is exact by Lemma 3.21 and the bottom row is the exact sequence (3.5) in Lemma 3.16. Now, it is clear that $\mu_0 \mathrm{Fil}^{k-p+1} M \subset \mu \mathrm{Fil}^{k-1} N \cap \mathrm{Fil}^k M$ (since μ_0 is the product of μ^{p-1} with a unit in A_F^+). Conversely, note that we have $\mu \mathrm{Fil}^{k-1} N \cap \mathrm{Fil}^k M \subset \mu N \cap M \cap \mathrm{Fil}^k M = \mu_0 M \cap \mathrm{Fil}^k M = \mu_0 \mathrm{Fil}^{k-p+1} M$. Therefore, it follows that the right vertical arrow in (3.10) is injective and we claim that it is surjective as well. Indeed, let x be an element of $\mathrm{Fil}^k(N/\mu N)$ and let y in M be a lift of x, under the composition $M \twoheadrightarrow M/\mu_0 M \xrightarrow{\sim} N/\mu N$. Then via the natural S-linear and (φ, Γ_F)-equivariant injective map $M \to N$, we see that y is in N and a lift of x. In particular, we get that y is in $\mathrm{Fil}^k N \cap M = \mathrm{Fil}^k M$. Taking the image of y under the map $\mathrm{Fil}^k M \to \mathrm{Fil}^k(M/\mu_0 M)$ gives a lifting of x under the right vertical map of (3.10). Hence, we obtain that $\mathrm{Fil}^k(M/\mu_0 M) \xrightarrow{\sim} \mathrm{Fil}^k(N/\mu N)$. ∎

Now, note that $(M/\mu_0 M)[1/p]$ is a φ-module over F since $\tilde{p} = p \bmod \mu_0 M$ and $M/\mu_0 M$ is equipped with a filtration $\mathrm{Fil}^k(M/\mu_0 M)$ as above. We equip $(M/\mu_0 M)[1/p]$ with the induced filtration $\mathrm{Fil}^k((M/\mu_0 M)[1/p]) := \mathrm{Fil}^k(M/\mu_0 M)[1/p]$, and note that it is a filtered φ-module over F. By combining Theorem 3.15 and Lemma 3.22, we get the following:

Theorem 3.23. *Let M be a Wach module over S and $V := T_F(M)[1/p]$ the associated crystalline representation of G_F from Theorem 3.13. Then we have that $(M/\mu_0 M)[1/p] \xrightarrow{\sim} D_{\mathrm{cris}}(V)$ as filtered φ-modules over F.*

From Theorem 3.23 we have a surjection $\mathrm{Fil}^k M[1/p] \twoheadrightarrow \mathrm{Fil}^k D_{\mathrm{cris}}(V)$ and we would like to determine its kernel.

Lemma 3.24. *Let M be a Wach module over S. Then, for any $k \in \mathbb{Z}$, the following sequence is exact:*

$$0 \longrightarrow \mathrm{Fil}^{k-p+1} M \xrightarrow{\mu_0} \mathrm{Fil}^k M \longrightarrow \mathrm{Fil}^k(M/\mu_0 M) \longrightarrow 0. \tag{3.11}$$

In particular, we have that $\mathrm{Ker}(\mathrm{Fil}^k M[1/p] \twoheadrightarrow \mathrm{Fil}^k D_{\mathrm{cris}}(V)) = \mu_0 \mathrm{Fil}^{k-p+1} M[1/p]$. *Moreover, by taking the associated graded pieces, we get that* $\mathrm{gr}^k M \xrightarrow{\sim} \mathrm{gr}^k(M/\mu_0 M)$ *and* $\mathrm{gr}^k M[1/p] \xrightarrow{\sim} \mathrm{gr}^k D_{\mathrm{cris}}(V)$.

Proof. Exactness of (3.11) easily follows from Lemma 3.21. Then, by taking the associated graded pieces, we obtain the following exact sequence:

$$0 \longrightarrow \mathrm{gr}^{k-1} M \xrightarrow{\mu_0} \mathrm{gr}^k M \longrightarrow \mathrm{gr}^k(M/\mu M) \longrightarrow 0.$$

It is clear that the map $\mathrm{gr}^{k-1} M \xrightarrow{\mu_0} \mathrm{gr}^k M$ is trivial, i.e. $\mathrm{gr}^k M \xrightarrow{\sim} \mathrm{gr}^k(M/\mu_0 M)$. Rest is obvious. ∎

Remark 3.25. The Nygaard filtration on a Wach module M over S is stable under the action of Γ_0. Therefore, for any g in Γ_0 and $k \in \mathbb{Z}$, using Lemma 3.21 we see that $(g-1)\mathrm{Fil}^k M \subset (\mathrm{Fil}^k M) \cap \mu_0 M = \mu_0 \mathrm{Fil}^{k-p+1} M$.

4. Syntomic complexes and Galois cohomology

In this section, we will define syntomic complexes with coefficients in a Wach module over A_F^+ and S, respectively, and show that, after inverting p, our complexes compute the crystalline part of the Galois cohomology of the associated crystalline representation (see Theorem 4.2 and Theorem 4.13).

4.1. Syntomic complex over A_F^+. Let N be a Wach module over A_F^+ and define an operator $\nabla_q := \frac{\gamma-1}{\mu} : N \to N$. From Remark 3.17, note that we have $\nabla_q(\mathrm{Fil}^k N) \subset \mathrm{Fil}^{k-1} N$, for each $k \in \mathbb{Z}$.

Definition 4.1. Define the *syntomic complex* with coeffcients in N as

$$\mathcal{S}^\bullet(N) : \mathrm{Fil}^0 N \xrightarrow{(\nabla_q, 1-\varphi)} \mathrm{Fil}^{-1} N \oplus N \xrightarrow{(1-[p]_q\varphi, \nabla_q)^\intercal} N, \tag{4.1}$$

where the first map is $x \mapsto (\nabla_q(x), (1-\varphi)x)$ and the second map is $(x, y) \mapsto (1-[p]_q\varphi)x - \nabla_q(y)$.

The goal of this subsection is to show the following claim:

Theorem 4.2. *Let N be a Wach module over A_F^+ and $V = T_F(N)[1/p]$ the associated p-adic crystalline representation of G_F from Theorem 3.9. Then we have a natural isomorphism, for each $k \in \mathbb{N}$,*

$$H^k(\mathcal{S}^\bullet(N))[1/p] \xrightarrow{\sim} H_f^k(G_F, V).$$

Proof. The claim for H_f^0 follows from Lemma 4.4. For H_f^1 recall that from Remark 2.4 we have a natural (in V) isomorphism

$$H_f^1(G_F, V) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F)}(\mathbb{Q}_p, V).$$

Moreover, from Remark 3.11 the functors N_F and its quasi-inverse V_F are exact. Therefore, we have a natural (in N) isomorphism

$$\mathrm{Ext}^1_{(\varphi,\Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{A_F^+[1/p]}}(A_F^+[1/p], N[1/p]) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F)}(\mathbb{Q}_p, V).$$

Combining these observations with Proposition 4.5 and after inverting p, we get a natural (in N) isomorphism

$$H^1(\mathcal{S}^\bullet(N))[1/p] \xrightarrow{\sim} H_f^1(G_F, V).$$

Finally, note that the Wach module N over A_F^+ can always be written as a twist of an effective Wach module over A_F^+, and similarly, the representation $V = T_F(N)[1/p]$, is the twist of the corresponding positive crystalline representation by a power of the cyclotomic character (see Remark 3.12). Therefore, the claim for H_f^2 follows from Proposition 4.6. ∎

Remark 4.3. Let us consider the following diagram of complexes:

$$\begin{array}{ccccc}
\mu\mathrm{Fil}^{-1}N & \xrightarrow{(\gamma-1, 1-\varphi)} & \mu\mathrm{Fil}^{-1}N \oplus \mu N & \xrightarrow{(1-\varphi, \gamma-1)^\intercal} & \mu N \\
\downarrow & & \downarrow & & \| \\
\mathrm{Fil}^0 N & \xrightarrow{(\gamma-1, 1-\varphi)} & \mu\mathrm{Fil}^{-1}N \oplus N & \xrightarrow{(1-\varphi, \gamma-1)^\intercal} & \mu N \\
\downarrow & & \downarrow & & \\
\mathrm{Fil}^0(N/\mu N) & \xrightarrow{1-\varphi} & N/\mu N, & &
\end{array} \tag{4.2}$$

where the complex in the middle row is isomorphic to the complex $\mathcal{S}^\bullet(N)$ in (4.1). Let $V := T_F(N)[1/p]$ from Theorem 3.9. Then, after inverting p and using Theorem 3.15, we see that the complex in the bottom row of (4.2) is the same as the complex $\mathscr{D}^\bullet(D_{\mathrm{cris}}(V))$ in (2.9). Moreover, note that in (4.2), the middle column is exact by Theorem 3.15 and the left-hand side column is exact by Lemma 3.16. Hence, by Theorem 4.2 and Corollary 2.6, it follows that the diagram (4.2) induces a natural quasi-isomorphism of complexes $\mathcal{S}^\bullet(N)[1/p] \simeq \mathscr{D}^\bullet(D_{\mathrm{cris}}(V))$.

In the rest of this subsection, we will compute the cohomology of the complex $\mathcal{S}^\bullet(N)$ from (4.1).

4.1.1. Comparing H^0 and H^1. In this subsubsection, we will compute H^0 and H^1 of the complex $\mathcal{S}^\bullet(N)$.

Lemma 4.4. *Let N be a Wach module over A_F^+ and $T = T_F(N)$ the associated $\mathbb{Z}_p$-representation of G_F from Theorem 3.9 such that $T[1/p]$ is crystalline. Then we have a natural isomorphism*

$$H^0(\mathcal{S}^\bullet(N)) = (\mathrm{Fil}^0 N)^{\varphi=1,\nabla_q=0} \xrightarrow{\sim} T^{G_F}.$$

Proof. Note that a simple computation shows that we have $(\mathrm{Fil}^0 N)^{\varphi=1,\nabla_q=0} = (\mathrm{Fil}^0 N)^{\varphi=1,\gamma=1} = N^{\varphi=1,\gamma=1}$. Now, let $M := N^{\mathbb{F}_p^\times}$ and recall that we have an A_F^+-linear and (φ, Γ_F)-equivariant isomorphism $A_F^+ \otimes_S M \xrightarrow{\sim} N$ (see Proposition 3.7). Therefore, we see that $N^{\varphi=1,\gamma=1} = (A_F^+ \otimes_S M)^{\varphi=1,\gamma=1} = M^{\varphi=1,\gamma=1} = M^{\varphi=1,\Gamma_0} = N^{\varphi=1,\Gamma_F}$, where the second to last equality follows from the continuity of the action of Γ_0 on M. Moreover, note that $(A_F^+[1/\mu])^{\gamma=1} = O_F$, therefore, a similar argument shows that we have $(N[1/\mu])^{\varphi=1,\gamma=1} = (N[1/\mu])^{\varphi=1,\Gamma_F}$. We claim that $N^{\varphi=1,\gamma=1} = (N[1/\mu])^{\varphi=1,\gamma=1}$. Indeed, let (x/μ^k) be in $N[1/\mu]^{\varphi=1,\gamma=1}$, for some x in N and $k \in \mathbb{Z}$. Then it is enough to show that x is in $\mu^k N$. Note that γ is a topological generator of Γ_0 and we have $\gamma(x) = (\gamma(\mu)^k/\mu^k)x$. So, reduction modulo μ gives $\gamma(x) = \chi(\gamma)^k x \bmod \mu N$. Since Γ_F acts trivially on $N/\mu N$ and $\chi(\gamma)^k - 1$ is a unit in $A_F^+[1/p]$, we obtain that x is in $\mu N[1/p] \cap N = \mu N$. Iterating this k times, we obtain that x is in $\mu^k N$, as claimed. In particular, we have that $N^{\varphi=1,\Gamma_F} = (N[1/\mu])^{\varphi=1,\Gamma_F}$ and it remains to compute the latter term.

From Remark 3.10, recall that for a certain ring A^+ equipped with a natural action of (φ, G_F), we have an $A^+[1/\mu]$-linear and (φ, G_F)-equivariant comparison isomorphism $A^+[1/\mu] \otimes_{A_F^+} N \xrightarrow{\sim} A^+[1/\mu] \otimes_{\mathbb{Z}_p} T$. Moreover, we have that $(A^+)^{H_F} = A_F^+$ and $(A^+[1/\mu])^{\varphi=1} = \mathbb{Z}_p$ and the action of φ and G_F commute with each other, therefore, by taking the fixed points of the preceding isomorphism under the action of φ and G_F, yields

$$(N[1/\mu])^{\varphi=1,\Gamma_F} = (A^+[1/\mu] \otimes_{A_F^+} N)^{\varphi=1,G_F} \xrightarrow{\sim} (A^+[1/\mu] \otimes_{\mathbb{Z}_p} T)^{\varphi=1,G_F} = T^{G_F}.$$

This allows us to conclude. ■

Proposition 4.5. *Let N be a Wach module over A_F^+. Then, we have a natural (in N) isomorphism*

$$H^1(\mathcal{S}^\bullet(N)) \xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{A_F^+}}(A_F^+, N). \tag{4.3}$$

Proof. We will construct a map

$$\alpha : H^1(\mathcal{S}^\bullet(N)) \longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{A_F^+}}(A_F^+, N),$$

and show that it is bijective by constructing an inverse map. Let (x, y) represent a class in $H^1(\mathcal{S}^\bullet(N))$, i.e. we have x in $\mathrm{Fil}^{-1}N$ and y in N such that $(1 - [p]_q\varphi)x = \nabla_q(y)$. Set $E_1 := N \oplus A_F^+ \cdot e$ with $\gamma(e) = \mu x + e$, $\varphi(e) = y + e$ and $g(e) = e$, for g a generator of $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$. Clearly, E_1 is a Wach module over A_F^+. Moreover, by sending e to the identity element in A_F^+, we obtain an exact sequence of Wach modules over A_F^+

$$0 \longrightarrow N \longrightarrow E_1 \longrightarrow A_F^+ \longrightarrow 0,$$

This represents an extension class of A_F^+ by N in the category $(\varphi, \Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{A_F^+}$ and we set $\alpha[(x, y)] = [E_1]$, where we represent cohomological classes with "[ ]". To show that α is well-defined we need to show that the extension class $[E_1]$ is independent of the choice of the presentation (x, y). Indeed, let (x', y') be another presentation such that $x' - x = \nabla_q(w)$, $y' - y = (1 - \varphi)w$ for some w in $\mathrm{Fil}^0 N$. Then, similar to above note that $E_2 := N \oplus A_F^+ \cdot e'$, with $\gamma(e') = \mu x' + e'$, $g(e') = e'$ and $\varphi(e') = y' + e'$, is a Wach module over A_F^+ and an extension of A_F^+ by N. Let us define an A_F^+-linear map $f : E_1 \to E_2$ given as identity on N and we set $f(e) = e' - w$. Then f is bijective because we can define $f^{-1} : E_2 \to E_1$, given as the identity on N and we set $f^{-1}(e') = e + w$, and it is easy to verify that $f \circ f^{-1} = id$ and $f^{-1} \circ f = id$. From the formulas $x' - x = \nabla_q(w)$ and $y' - y = (1 - \varphi)y$ it follows that f and f^{-1} are (φ, Γ_F)-equivariant. Now consider the following diagram with A_F^+-linear maps and exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E_1 & \longrightarrow & A_F^+ & \longrightarrow & 0 \\ & & \| & & f\downarrow\wr & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & E_2 & \longrightarrow & A_F^+ & \longrightarrow & 0. \end{array}$$

The left square commutes by the definition of f. Moreover, the A_F^+-linear map $E_1 \to A_F^+$ sends $e \mapsto 1$ and the A_F^+-linear map $E_2 \to A_F^+$ sends $e' \mapsto 1$, therefore, it follows that the right square commutes as well. Hence, E_1 and E_2 represent the same extension class of A_F^+ by N, in the category $(\varphi, \Gamma_F)\text{-Mod}_{A_F^+}^{[p]_q}$. In particular, α is well-defined.

Next, we will construct an inverse of α which we will denote by β. Consider an extension of Wach modules over A_F^+ as

$$0 \longrightarrow N \longrightarrow E_1 \longrightarrow A_F^+ \longrightarrow 0.$$

We write $E_1 = N \oplus A_F^+ \cdot e$, where e in E_1 is a lift of the identity element in A_F^+, and we have $(\gamma-1)e = z$ and $(1-\varphi)e = y$ for some y, z in N. But then we have that $\varphi(e) = e - y$ in E_1, i.e. e is in $\text{Fil}^0 E_1$. Therefore, we get that $z = (\gamma-1)e$ is in $N \cap \mu\text{Fil}^{-1}E_1 = \mu\text{Fil}^{-1}N$, where the last equality follows from Remark 3.19. In particular, we obtain that $\nabla_q(e) = \frac{\gamma-1}{\mu}e = x$, for some x in $\text{Fil}^{-1}N$. By the commutativity of the action of φ and γ, we get that $(1-[p]_q\varphi) \circ \nabla_q(e) = \nabla_q \circ (1-\varphi)e$, or equivalently,

$$(1-[p]_q\varphi)x = \nabla_q(y).$$

Therefore, we see that (x, y) represents a cohomological class in $H^1(\mathcal{S}^\bullet(N))$ and we set $\beta([E_1]) = [(x,y)]$. Let us first show that the class $[(x,y)]$ is independent of the lift e in E_1 of the identity element in A_F^+. So let e' in E denote another lift of the identity element in A_F^+. Then, similar to above we have that e' is in $\text{Fil}^0 E$ and there exist x' in $\text{Fil}^{-1}N$ and y' in N such that $\nabla_q(e') = x'$, $(1-\varphi)e' = y'$ and $(1-[p]_q\varphi)x' = \nabla_q(y')$. Moreover, from Lemma 3.18, we note that $w = e' - e$ is in $\text{Fil}^0 E \cap N = \text{Fil}^0 N$, in particular, we get that $x' = x + \nabla_q(w)$ and $y' = y + (1-\varphi)w$. Since $(1-[p]_q\varphi) \circ \nabla_q = \nabla_q \circ (1-\varphi)$, therefore, we conclude that (x, y) and (x', y') represent the same class in $H^1(\mathcal{S}^\bullet(N))$. Now, to show that β is well-defined, we must show that the class $[(x,y)]$ is independent of the presentation E_1 of the extension class $[E_1]$. So let E_2 denote another presentation of the extension class $[E_1]$, i.e. E_2 is a Wach module over A_F^+ and there exists a (φ, Γ_F)-equivariant isomorphism $f : E_1 \xrightarrow{\sim} E_2$ fitting into the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E_1 & \longrightarrow & A_F^+ & \longrightarrow & 0 \\ & & \| & & f\downarrow\wr & & \| & & \\ 0 & \longrightarrow & N & \longrightarrow & E_2 & \longrightarrow & A_F^+ & \longrightarrow & 0. \end{array}$$

Let e'' in E_2 denote a lift of the identity element in A_F^+ and arguing as above we have that e'' is in $\text{Fil}^0 E_2$ and there exist some x'' in $\text{Fil}^{-1}N$ and y'' in N such that $\nabla_q(e'') = x''$, $(1-\varphi)e'' = y''$ and $(1-[p]_q\varphi)x'' = \nabla_q(y'')$. Then, from the commutative diagram above we have that $f^{-1}(e'')$ in E_1 denotes a lift of the identity element in A_F^+, therefore, it follows that $f^{-1}(e'')$ is in $\text{Fil}^0 A_1$ and $\nabla_q(f^{-1}(e'')) = x''$, $(1-\varphi)f^{-1}(e'') = y''$ and $(1-[p]_q\varphi)f^{-1}(x'') = \nabla_q(f^{-1}(y''))$. Using that the extension class $[E_1]$ is independeent of the choice of a lift in E_1 of the identity element in A_F^+, it follows that (x, y) and (x'', y'') represent the same cohomological class in $H^1(\mathcal{S}^\bullet(N))$. In particular, we obtain that β is well-defined.

Finally, it remains to show that the two constructions described above are inverse to each other, i.e. $\alpha \circ \beta = id$ and $\beta \circ \alpha = id$. Note that starting with a class $[(x,y)]$ in $H^1(\mathcal{S}^\bullet(N))$ we can construct an extension E of A_F^+ by N in $(\varphi, \Gamma_F)\text{-Mod}_{A_F^+}^{[p]_q}$, such that $[E] = \alpha[(x,y)]$, i.e. E can be described using the pair (x, y). After applying β we obtain a class $\beta([E]) = [(x', y')]$ in $H^1(\mathcal{S}^\bullet(N))$ with a presentation (x', y') depending on the choice of some lift in E of the identity element in A_F^+. Note that by construction, E admits two descriptions using (x, y) and (x', y'), respectively, depending on the choice of the lift in E of the identity element in A_F^+. As we have shown that the class $[E]$ is independent of this choice, therefore, it follows that $[(x,y)] = [(x', y')] = \beta \circ \alpha[(x,y)]$ in $H^1(\mathcal{S}^\bullet(N))$. Next, starting with an extension E of A_F^+ by N in $(\varphi, \Gamma_F)\text{-Mod}_{A_F^+}^{[p]_q}$, we can construct a class $[(x,y)] = \beta([E])$ in

$H^1(\mathcal{S}^\bullet(N))$. After applying α, we obtain an extension class $[E'] = \alpha[(x,y)]$, where E' is an extension of A_F^+ by N in (φ, Γ_F)-$\mathrm{Mod}_{A_F^+}^{[p]_q}$. By construction, we can write $E = N \oplus A_F^+ \cdot e$, with $\nabla_q(e) = x$, $g(e) = e$ and $(1-\varphi)e = y$, and $E' = N \oplus A_F^+ \cdot e'$ with $\nabla_q(e') = x$, $g(e) = e$ and $(1-\varphi)e' = y$. Now, note that the A_F^+-linear map $f : E \to E'$, defined using identity on N and by setting $f(e) = e'$, is a (φ, Γ_F)-equivariant isomorphism, in particular, we have that $[E] = [E'] = \alpha \circ \beta([E])$. Hence, we have shown that the map α is a natural (in N) isomorphism. ■

4.1.2. Comparing H^2 rationally. For convenience in computations in this subsubsection, we will rephrase our goal. Let V be a p-adic positive crystalline representation of G_F, i.e. all its Hodge-Tate weights ≤ 0, and let $T \subset V$ be a G_F-stable $\mathbb{Z}_p$-lattice. Set $V(r) := V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ and $T(r) := T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$, for any $r \in \mathbb{Z}$. From Theorem 3.9, recall that we have Wach modules $N_F(T)$ and $N_F(T(r)) = \mu^{-r} N_F(T)(r)$ over A_F^+, and $A_F^+[1/p]$-modules $N_F(V) = N_F(T)[1/p]$ and $N_F(V(r)) = \mu^{-r} N_F(V)(r)$. Let us denote the complex $\mathcal{S}^\bullet(\mu^{-r} N_F(T)(r))[1/p]$ by $\mathcal{S}^\bullet(N_F(V), r)$. Then our goal is to show the following claim:

Proposition 4.6. *The cohomology group $H^2(\mathcal{S}^\bullet(N_F(V), r))$ vanishes. In particular, we have that $H^2(\mathcal{S}^\bullet(N_F(V), r)) = H_f^2(G_F, V(r)) = 0$.*

Proof. Let x be in $N_F(V(r))$ and to prove the claim note that it is enough to show that we can write $x = \nabla_q(y) - (1 - [p]_q\varphi)z$, for some y in $N_F(V(r))$ and z in $\mathrm{Fil}^{-1} N_F(V(r))$. Write $x = \frac{x'}{\mu^r} \otimes \epsilon^{\otimes r}$, for some x' in $N_F(V)$ and $\epsilon^{\otimes r}$ a $\mathbb{Q}_p$-basis of $\mathbb{Q}_p(r)$. Then, from Lemma 4.7 there exist y' and z' in $N_F(V)$ satisfying the following:

$$\tfrac{x'}{\mu^r} \otimes \epsilon^{\otimes r} = \nabla_q\big(\tfrac{y'}{\mu^{r-1}} \otimes \epsilon^{\otimes r}\big) - (1 - [p]_q\varphi)(z' \otimes \epsilon^{\otimes r}).$$

Letting $z = z' \otimes \epsilon^{\otimes r}$ and $y = \frac{y'}{\mu^{r-1}} \otimes \epsilon^{\otimes r}$, we see that $x = \nabla_q(y) - (1-[p]_q\varphi)z$, with y in $N_F(V(r))$ and z in $N_F(V)(r) \subset N_F(V(r))$. However, note that $[p]_q\varphi(z) = x + z + \nabla_q(y)$ is in $N_F(V(r))$, in particular, z is in $\mathrm{Fil}^{-1} N_F(V(r))$. Hence, we get the claim. ■

Let $\epsilon^{\otimes r}$ denote a $\mathbb{Q}_p$-basis of $\mathbb{Q}_p(r)$ and note that the following was used in Proposition 4.6:

Lemma 4.7. *Let x in $N_F(V)$, then for $1 \leq k \leq r$ there exist some y and z in $N_F(V)$ such that*

$$\tfrac{x}{\mu^k} \otimes \epsilon^{\otimes r} = \nabla_q\big(\tfrac{y}{\mu^{k-1}} \otimes \epsilon^{\otimes r}\big) - (1 - [p]_q\varphi)(z \otimes \epsilon^{\otimes r}).$$

Proof. Note that γ is a topological generator of Γ_0, in particular, we have that $\chi(\gamma)$ is in $1 + p\mathbb{Z}_p$. Moreover, note that up to multiplying by some power of p we may assume that x is in $N_F(T)$. Therefore, to prove the claim, it is enough to show that for any x in $N_F(T)$ there exist some y and z in $N_F(V)$ such that

$$\tfrac{x}{\mu^k} \otimes \epsilon^{\otimes r} = \tfrac{\gamma - 1}{\mu}\big(\tfrac{y}{\mu^{k-1}} \otimes \epsilon^{\otimes r}\big) - (1 - [p]_q\varphi)(z \otimes \epsilon^{\otimes r}). \tag{4.4}$$

Let $(\gamma - 1)x = \mu x_1$ for some x_1 in $N_F(T)$ and we will prove the claim by induction on k. So let $k = 1$ and consider the following:

$$\tfrac{\gamma-1}{\mu}\big(\tfrac{x}{\chi(\gamma)^r - 1} \otimes \epsilon^{\otimes r}\big) = \big(\tfrac{x}{\mu} + \tfrac{\chi(\gamma)^r x_1}{\chi(\gamma)^r - 1}\big) \otimes \epsilon^{\otimes r} = \big(\tfrac{x}{\mu} + (1 - [p]_q\varphi)z_1\big) \otimes \epsilon^{\otimes r},$$

where z_1 is in $N_F(V)$ following Remark 4.8. Upon rearranging the terms, we see that (4.4) holds for $k = 1$. Now, we write $u = (\chi(\gamma)\mu)/\gamma(\mu)$ in $1 + p\mu A_F^+$, take $1 < k \leq r$ and assume that (4.4) holds for $k - 1$. Then we have that,

$$\begin{aligned}
\tfrac{\gamma-1}{\mu}\big(\tfrac{x}{\mu^{k-1}(\chi(\gamma)^{r-k+1}-1)} \otimes \epsilon^{\otimes r}\big) &= \tfrac{u^{k-1}\chi(\gamma)^{r-k+1}-1}{\mu^k(\chi(\gamma)^{r-k+1}-1)} x \otimes \epsilon^{\otimes r} + \tfrac{u^{k-1}\chi(\gamma)^{r-k+1}}{\mu^{k-1}(\chi(\gamma)^{r-k+1}-1)} x_1 \otimes \epsilon^{\otimes r} \\
&= \big(\tfrac{x}{\mu^k} + \tfrac{x_k}{\mu^{k-1}}\big) \otimes \epsilon^{\otimes r} \\
&= \tfrac{x}{\mu^k} \otimes \epsilon^{\otimes r} + \tfrac{\gamma-1}{\mu}\big(\tfrac{y_k}{\mu^{k-2}} \otimes \epsilon^{\otimes r}\big) - (1 - [p]_q\varphi)(z_k \otimes \epsilon^{\otimes r}),
\end{aligned}$$

for some x_k, y_k and z_k in $N_F(V)$ and note that the last equality follows by the induction hypothesis. By rearranging the terms, we see that (4.4) also holds for any $1 < k \leq r$. This allows us to conclude. ■

Remark 4.8. For any x in $N_F(T)$, there exists some y in $N_F(T)$ such that $(1-[p]_q\varphi)y = x$. Indeed, note that the series $(1+[p]_q\varphi+([p]_q\varphi)^2+\cdots)$ converges as series of operators on $N_F(T)$ since we have that $\prod_{k=0}^{n}\varphi^k([p]_q)$ is in $(p,\mu)^{n+1}$, for each $n \in \mathbb{N}$. In particular, we see that $([p]_q\varphi)^n$ is (p,μ)-adically nilpotent and we can take $y = (1+[p]_q\varphi+([p]_q\varphi)^2+\cdots)x$ in $N_F(T)$. A similar claim is also true for $N_F(V)$.

4.2. Syntomic complex over S. Let M be a Wach module over S and define an operator $\nabla_0 := \frac{\gamma-1}{\mu_0} : M \to M$. From Remark 3.17 note that we have $\nabla_0(\mathrm{Fil}^k M) \subset \mathrm{Fil}^{k-p+1}M$, for each $k \in \mathbb{Z}$.

Definition 4.9. Define the *syntomic complex* with coeffcients in M as

$$\mathcal{S}^\bullet(M) : \mathrm{Fil}^0 M \xrightarrow{(\nabla_0, 1-\varphi)} \mathrm{Fil}^{-p+1}M \oplus M \xrightarrow{(1-\tilde{p}^{p-1}\varphi, \nabla_0)^\intercal} M, \tag{4.5}$$

where the first map is $x \mapsto (\nabla_0(x), (1-\varphi)x)$ and the second map is $(x,y) \mapsto (1-\tilde{p}^{p-1}\varphi)x - \nabla_0(y)$.

Proposition 4.10. *Let M be a Wach module over S. Then, we have a natural (in M) isomorphism*

$$H^1(\mathcal{S}^\bullet(M)) \xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma_0)\text{-}\mathrm{Mod}_S^{\tilde{p}}}(S, M). \tag{4.6}$$

Proof. Note that the arguments given in the proof of Proposition 4.5 easily adapts to our current setting. In particular, in the proof of Proposition 4.5 we can replace A_F^+ with S and N with M by working in the category $(\varphi,\Gamma_0)\text{-}\mathrm{Mod}_S^{\tilde{p}}$ instead of $(\varphi,\Gamma_F)\text{-}\mathrm{Mod}_{A_F^+}^{[p]_q}$. We omit the details to avoid repetition. ∎

Now, recall that $N := A_F^+ \otimes_S M$ is a Wach module over A_F^+ from Theorem 3.8. We will compare the syntomic complexes defined in (4.5) and (4.1) with coefficients in M and N, respectively.

Proposition 4.11. *The natural S-linear and (φ,Γ_0)-equivariant map $M \to N$ induces a natural morphism of complexes $\mathcal{S}^\bullet(M) \to \mathcal{S}^\bullet(N)$. Moreover, the natural map on cohomology $H^k(\mathcal{S}^\bullet(M)) \to H^k(\mathcal{S}^\bullet(N))$ is bijective for $k = 0, 1$ and injective for $k = 2$.*

Proof. Let us consider the following diagram:

$$\begin{array}{ccccc} C^\bullet = \mathrm{Fil}^0 M & \xrightarrow{(\gamma-1, 1-\varphi)} & \mu_0\mathrm{Fil}^{-p+1}M \oplus M & \xrightarrow{(1-\varphi, \gamma-1)^\intercal} & \mu_0 M \\ \downarrow & & \downarrow & & \downarrow \\ D^\bullet = \mathrm{Fil}^0 N & \xrightarrow{(\gamma-1, 1-\varphi)} & \mu\mathrm{Fil}^{-1}N \oplus N & \xrightarrow{(1-\varphi, \gamma-1)^\intercal} & \mu N, \end{array} \tag{4.7}$$

where the top row is isomorphic to the complex $\mathcal{S}^\bullet(M)$ in (4.5), the bottom row is isomorphic to the complex $\mathcal{S}^\bullet(N)$ in (4.1) and the vertical maps are induced by the natural S-linear and (φ,Γ_0)-equivariant map $M \to N$. In particular, we have obtained a natural morphism of complexes $\mathcal{S}^\bullet(M) \to \mathcal{S}^\bullet(N)$.

Now, note that each term of the complex $D^\bullet$ in (4.7) admits an action of $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$, which commutes with the action of φ and Γ_0. So, by Remark 2.1, the complex $D^\bullet$ admits a decomposition as $\oplus_{i=0}^{p-1} D_i^\bullet$, where

$$D_i^\bullet = (\mathrm{Fil}^0 N)_i \xrightarrow{(\gamma-1, 1-\varphi)} (\mu\mathrm{Fil}^{-1}N)_i \oplus N_i \xrightarrow{(1-\varphi, \gamma-1)^\intercal} (\mu N)_i.$$

Using that $M \xrightarrow{\sim} N_0$ as (φ,Γ_0)-modules over S and the description of filtration on M in Subsection 3.3.2, it follows that $C^\bullet \xrightarrow{\sim} D_0^\bullet$. Moreover, it is clear that we have $H^k(D^\bullet) = \oplus_{i=0}^{p-1} H^k(D_i^\bullet)$. In particular, the natural map $H^k(\mathcal{S}^\bullet(M)) \xrightarrow{\sim} H^k(C^\bullet) \to H^k(D^\bullet) \xleftarrow{\sim} H^k(\mathcal{S}^\bullet(N))$ is injective for each $k = 0, 1, 2$. Now, for $k = 0$, note that since we have $\mathrm{Fil}^0 M \xrightarrow{\sim} (\mathrm{Fil}^0 N)^{\mathbb{F}_p^\times}$ as (φ,Γ_0)-modules over S, therefore, by using Lemma 4.4 we get that

$$\begin{aligned} H^0(S^\bullet(M)) \xrightarrow{\sim} (\mathrm{Fil}^0 M)^{\varphi=1,\gamma=1} &\xrightarrow{\sim} (\mathrm{Fil}^0 N)^{\varphi=1,\gamma=1,\mathbb{F}_p^\times} \\ &= N^{\varphi=1,\gamma=1,\mathbb{F}_p^\times} = N^{\varphi=1,\Gamma_F} \xleftarrow{\sim} H^0(\mathcal{S}^\bullet(N)). \end{aligned}$$

Finally, for $k = 1$, let us consider the following diagram:

$$\begin{array}{ccc} H^1(\mathcal{S}^\bullet(M)) & \xrightarrow[\sim]{(4.6)} & \mathrm{Ext}^1_{(\varphi,\Gamma_0)\text{-}\mathrm{Mod}_S^{\tilde{p}}}(S, M) \\ \downarrow & & \downarrow \wr \\ H^1(\mathcal{S}^\bullet(N)) & \xrightarrow[\sim]{(4.3)} & \mathrm{Ext}^1_{(\varphi,\Gamma_F)\text{-}\mathrm{Mod}_{A_F^+}^{[p]_q}}(A_F^+, N), \end{array}$$

where the left vertical arrow is the natural map constructed above and the right vertical arrow is an isomorphism induced by the exact categorical equivalence in (3.3) of Theorem 3.8. The diagram commutes by definition and it follows that the left vertical arrow is an isomorphism. This allows us to conclude. ∎

Remark 4.12. From the injectivity of $H^2(S^\bullet(M)) \to H^2(S^\bullet(N))$, it follows that $H^2(S^\bullet(M))[1/p] = H^2(S^\bullet(N))[1/p] = 0$ (see Proposition 4.6).

Theorem 4.13. *Let M be a Wach module over S and $V = T_F(N)[1/p]$ the associated p-adic crystalline representation of G_F from Theorem 3.13. Then we have a natural isomorphism, for each $k \in \mathbb{N}$,*

$$H^k(\mathcal{S}^\bullet(M))[1/p] \xrightarrow{\sim} H^k_f(G_F, V).$$

Proof. The claim follows by combining Proposition 4.11, Remark 4.12 and Theorem 4.2. ∎

Remark 4.14. Similar to Remark 4.3, let us consider the following diagram of complexes:

$$\begin{array}{ccccc} \mu_0\mathrm{Fil}^{-1}M & \xrightarrow{(\gamma-1,1-\varphi)} & \mu_0\mathrm{Fil}^{-p+1}M \oplus \mu_0 M & \xrightarrow{(1-\varphi,\gamma-1)^\intercal} & \mu_0 M \\ \downarrow & & \downarrow & & \| \\ \mathrm{Fil}^0 M & \xrightarrow{(\gamma-1,1-\varphi)} & \mu_0\mathrm{Fil}^{-p+1}M \oplus M & \xrightarrow{(1-\varphi,\gamma-1)^\intercal} & \mu_0 M \\ \downarrow & & \downarrow & & \\ \mathrm{Fil}^0(M/\mu_0 M) & \xrightarrow{1-\varphi} & M/\mu_0 M, & & \end{array} \tag{4.8}$$

where the complex in the middle row is isomorphic to the complex $\mathcal{S}^\bullet(M)$ in (4.5), and it can be seen as a subcomplex of the Fontaine–Herr complex in (2.6). Now, let $V := T_F(M)[1/p]$ from Theorem 3.13. Then, after inverting p and using Theorem 3.23, we see that the complex in the bottom row of (4.8) is the same as the complex $\mathscr{D}^\bullet(D_{\mathrm{cris}}(V))$ in (2.9). Moreover, note that in (4.8) the middle column is exact by Theorem 3.23 and the left-hand side column is exact by Lemma 3.24. Hence, by Theorem 4.13 and Corollary 2.6, it follows that the diagram (4.8) induces a natural quasi-isomorphism of complexes $\mathcal{S}^\bullet(M)[1/p] \simeq \mathscr{D}^\bullet(D_{\mathrm{cris}}(V))$.

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