Prismatic F-crystals and Wach modules

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Abstract. We show a direct equivalence between the category of analytic/completed prismatic F-crystals on the absolute prismatic site of a small (unramified at p) base ring and the category of relative Wach modules from the theory of (φ, Γ)-modules. The result is obtained by showing that the data of the Galois action on a Wach module is equivalent to the data of a prismatic stratification on the underlying φ-module.

1. Introduction

Recent groundbreaking advances in integral p-adic Hodge theory have been brought on by the seminal works of Bhatt, Morrow and Scholze on A_{inf}-cohomology in [BMS18; BMS19], and of Bhatt and Scholze on prismatic cohomology in [BS22]. In the latter theory, the study of prismatic F-crystals has led to exciting applications towards the classification of p-divisible groups [AL23], and more generally, of all p-adic crystalline local systems over smooth p-adic formal schemes [BS23; DLMS24; GR24]. However, similar to the crystals appearing in the theory of crystalline cohomology of Grothendieck and Berthelot, the prismatic F-crystals are mysterious objects.

To unravel these objects, a common idea is to describe them is terms of certain equivalent and computable data, for example, in the crystalline cohomology theory, one usually replaces crystals with modules equipped with a flat connection as in [Ber74]. In the prismatic theory, several successful attempts have been made to understand prismatic crystals in terms of more explicit data in various settings, for example, crystals on the relative prismatic site in terms of generalised representations, q-de Rham complexes and q-Higgs fields in [MT20; Tsu24], crystals on the relative/absolute prismatic site in terms of twisted/absolute differential calculus in [GLQ22; GLQ23], Hodge-Tate crystals in terms of Higgs fields in [Tia23] and prismatic (Laurent) F-crystals in terms of étale (φ, Γ)-modules and similar objects in [Wu21; DL21; MW21].

The aim of this article is to describe analytic/completed prismatic F-crystals of [DLMS24; GR24] in terms of more explicit data. More precisely, our main result provides a direct equivalence between the category of prismatic F-crystals over the absolute prismatic (ringed) site of a small (unramified at p) base ring and the category of relative Wach modules (certain (φ, Γ)-modules) introduced in [Abh23b] (see Theorem 1.1). Morally, our result is obtained by showing that the data of the Galois action (i.e. Γ-action) on a Wach module is equivalent to the data of a prismatic stratification on the underlying φ-module (see Theorem 1.3). Note that the proof of the preceding equivalence is highly non-trivial and constitutes the heart of this article. Moreover, our approach is different and independent of all previous methods and (categorical equivalence) results in [BS23; DLMS24; GR24] and [Wu21; DL21; MW21].

Let us note that in the case that the base ring is a complete discrete valuation ring with perfect residue field, Wach modules were studied in [Fon90], [Wac96; Wac97], [Col99] and [Ber04]. In this case, using our methods, we also show that the classical Wach modules from [Wac96; Ber04] descend to a smaller ring, beyond the Fontaine-Laffaille case treated in [Wac97] (see Theorem 1.5 and Remark 1.6). Furthermore, let us note that the theory of Wach modules (in the context of (φ, Γ)-modules) and its relationship with crystalline representations in different settings was studied in [Abh21; Abh23a; Abh23b].

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Besides being an explicit description of prismatic F-crystals, the usefulness of Wach modules stems from its applications towards the computation of p-adic vanishing cycles via syntomic complexes. Indeed, in order to generalise the computation of p-adic vanishing cycles via crystalline syntomic complexes of [CN17], to the case of crystalline coefficients, in [Abh23c] we used the theory of Wach modules from [Abh21] as an important ingredient. However, the results obtained in [Abh21] and [Abh23c] were restrictive. Furthermore, note that in [BMS19], the authors defined a prismatic syntomic complex for smooth p-adic formal schemes and compared it to the complex of p-adic vanishing cycles integrally. Beyond the smooth case, similar comparison results have also been obtained in [AMMN22] and [BM23], where the latter uses the theory of prismatic cohomology. The preceding results were obtained for the case of constant coefficients and it is natural to ask the following: is it possible to generalise [BMS19, Theorem 10.1] to arbitrary crystalline coefficients, i.e. can one define a prismatic syntomic complex with coefficients in a prismatic F-crystal and compare it to the complex of p-adic vanishing cycles? In our approach to providing an answer to the preceding question, the prismatic interpretation of Wach modules from the current paper and the relationship between Wach modules and crystalline representations from [Abh23b] will serve as crucial inputs.

1.1. A categorical equivalence. Let p be a fixed prime, κ a perfect field of characteristic p and $O_F := W(\kappa)$, the ring of p-typical Witt vectors with coefficients in κ and equipped with the natural Witt vector Frobenius endomorphism. Let R denote the p-adic completion of an étale algebra over the p-adically complete Laurent polynomial ring $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ such that its special fiber $\mathrm{Spec}\,(R/pR)$ is connected. We take $X := \mathrm{Spf}\ R$ to be an affine p-adic formal scheme and consider its absolute prismatic ringed site $(X_{\Delta}, \mathcal{O}_{\Delta})$ in the sense of [BS22] (see Section 2.1). Let $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta})$ denote the category of analytic prismatic F-crystals from [GR24] (note that one could also work with the equivalent notion of completed prismatic F-crystals from [DLMS24], see Subsection 2.3 for definitions and explanations).

Let R_∞ denote the R-algebra obtained by adjoining to R all p-power roots of unity and p-power roots of X_i, for all $1 \leq i \leq d$. Then $R_\infty[1/p]$ is Galois over $R[1/p]$ with the Galois group $\Gamma_R := \mathrm{Gal}(R_\infty[1/p]/R[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$. Let $A_{\mathrm{inf}}(R_\infty) := W(R_\infty^\flat)$, where $R_\infty^\flat$ denotes the tilt of R_∞ (see Subsection 1.6). Note that $A_{\mathrm{inf}}(R_\infty)$ is equipped with a Witt vector Frobenius endomorphism φ and a continuous action of Γ_R (see Subsection 3.1). Moreover, we have a subring $A_R \subset A_{\mathrm{inf}}(R_\infty)$ which is stable under the the action of (φ, Γ_R) on the latter (see Subsection 3.1.1); we equip A_R with the induced structures. Let $\varepsilon := (1, \zeta_p, \ldots)$ denote a compatible system of p-power roots of unity in $R_\infty^\flat$ and let $q := [\varepsilon]$ denote its Teichmüller lift in $A_{\mathrm{inf}}(R_\infty)$; set $\mu := q-1$ and $[p]_q := (q^p-1)/(q-1)$ in $A_{\mathrm{inf}}(R_\infty)$. We denote the category of Wach modules over A_R (see Definition 4.1) as $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{[p]_q}$.

Now, let us note that the pair $(A_R, [p]_q)$ is a prism, an object of X_{Δ} (see Lemma 3.10) and a cover of the final object of the topos $\mathrm{Shv}(X_{\Delta})$ (see Lemma 3.11). Moreover, the action of Γ_R on A_R induces automorphisms of $(A_R, [p]_q)$ in X_{Δ} (see Lemma 3.12). Evaluating an analytic prismatic F-crystal on the prism $(A_R, [p]_q)$ gives a Wach module (see Proposition 5.1), and we have a well-defined evaluation functor,

$$\begin{aligned} \mathrm{ev}_{A_R}^{\Delta} : \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) &\longrightarrow (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{[p]_q} \\ \mathcal{F} &\longmapsto \mathcal{F}(A_R, [p]_q). \end{aligned} \tag{1.1}$$

Our main result is the following claim:

Theorem 1.1 (Theorem 5.5)**.** *The evaluation functor in* (1.1) *induces an equivalence of categories* $\mathrm{ev}_{A_R}^{\Delta} : \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{[p]_q}$.

Our proof of Theorem 1.1 is direct, in particular, we do not assume the equivalence between crystalline $\mathbb{Z}_p$-representations and Wach modules over A_R from [Abh23b], or the catgeorical equivalence results from [BS23; GR24; DLMS24]. As mentioned earlier, we show that "the Galois action on a Wach module, i.e. the action of Γ_R, is equivalent to a prismatic stratification on the underlying φ-module". Our approach is inspired by the work of [MT20], where the authors study coefficients for relative prismatic cohomology. However, the results obtained and the techniques employed in our proofs are vastly diffferent from loc. cit. This stems largely from the fact that our respective settings are quite different: we work over the absolute prismatic site of R and consider analytic prismatic F-crystals, whereas in loc. cit., the authors work over the relative prismatic site and consider prismatic F-crystals of vector bundles.

1.2. Crystals as modules with stratification. In order to prove Theorem 1.1, our first course of action is to bring the source of the functor $\mathrm{ev}_{A_R}^{\Delta}$ in (1.1) on an equal footing with the target of that functor. This is achieved by interpreting analytic prismatic F-crystals crystals as certain modules with stratification.

More precisely, let us note that $(A_R, [p]_q)$ is a cover of the final object of the topos $\mathrm{Shv}(X_{\Delta})$ (see Lemma 3.11). We set $A_R(\bullet)$ to be the cosimplicial ring obtained by taking the prismatic Čech nerve $(A_R(\bullet), I(\bullet))$ of $(A_R, [p]_q)$ in X_{Δ}. Then, it is possible to describe the n^{th}-term of $A_R(\bullet)$ in the site X_{Δ} (see Construction 3.13), and in case $n = 1, 2$, we can describe $A_R(n)$ very explicity (see the discussion after Lemma 3.14). Now, let $\mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet))$ denote the category of "analytic" φ-modules over A_R equipped with a stratification with respect to $A_R(\bullet)$ (see Definition 5.6). Then by the general theory of crystals, evaluation of analytic F-crystals on the simplicial object $(A_R(\bullet), I(\bullet))$ induces the following natural equivalence of categories (see Construction 5.7 and Proposition 5.8).

$$\mathrm{ev}_{A_R(\bullet)}^{\Delta} : \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)). \tag{1.2}$$

1.3. Prismatic stratifications and Galois action on Wach modules. Our next course of action is to relate φ-modules over A_R equipped with a stratification to Wach modules over A_R. So let (N, ε) denote an object of the category $\mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet))$, where N is a finitely generated A_R-module satisfying certain conditions and ε is a stratification on N with respect to $A_R(\bullet)$. Moreover, let us note that the action of Γ_R on $(A_R, [p]_q)$ induces a natural action of $\Gamma_R^{\times(n+1)}$ on $A_R(n)$, for each $n \in \mathbb{N}$. Then, by using the action of Γ_R^2 on $A_R(1)$ and the stratification ε, we equip N with a continuous action of Γ_R satisfying the properties of a Wach module over A_R from Definition 4.1 (see Construction 5.10). In particular, we have a well-defined natural functor,

$$\mathrm{ev}_{A_R}^{\mathrm{Strat}} : \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)) \longrightarrow (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{[p]_q}, \tag{1.3}$$

and we show the following:

Proposition 1.2 (Proposition 5.11)**.** *The following diagram is commutative up to canonical isomorphisms:*

$$\begin{tikzcd}
\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) \arrow[r, "\mathrm{ev}_{A_R(\bullet)}^{\Delta}(1.2)", "\sim"'] \arrow[dr, "\mathrm{ev}_{A_R}^{\Delta}(1.1)"'] & \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)) \arrow[d, "(1.3)"', "\mathrm{ev}_{A_R}^{\mathrm{Strat}}"] \\
 & (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{[p]_q}.
\end{tikzcd}$$

From the diagram in Proposition 1.2, we see that in order to show that (1.1) induces a categorical equivalence, it is enough to show that (1.3) induces a categorical equivalence. Therefore, we show the following:

Theorem 1.3 (Theorem 5.12)**.** *The functor in* (1.3) *induces a natural equivalence of categories*

$$\mathrm{ev}_{A_R}^{\mathrm{Strat}} : \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{[p]_q}.$$

In order to prove Theorem 1.3, we define a quasi-inverse to the functor in (1.3). More precisely, given a Wach module N over A_R, we use the action of Γ_R on N to build a stratification on N with respect to $A_R(\bullet)$ and obtain a natural quasi-inverse to the functor in (1.3),

$$\mathrm{Strat}_{A_R(\bullet)} : (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{[p]_q} \longrightarrow \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)). \tag{1.4}$$

The construction of the stratification on a Wach module, using the action of Γ_R on it, is the *main technical heart* of the paper. Our construction is given via a "3-step" argument. The three steps correspond to the three subgroups of different nature making up the Galois group Γ_R. More precisely, recall that $\Gamma_R \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$. Furthermore, $\mathbb{Z}_p^\times$ fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \mathbb{Z}_p^\times \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1,$$

where, for $p \geq 3$, we have that $\Gamma_0 \xrightarrow{\sim} 1+p\mathbb{Z}_p$ and for $p = 2$, we have that $\Gamma_0 \xrightarrow{\sim} 1+4\mathbb{Z}_2$. Moreover, for $p \geq 3$, we have that $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$, and for $p = 2$, we have that $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \{\pm 1\}$, as groups. Then the three steps of our argument correspond to the groups $\mathbb{Z}_p(1)^d$, Γ_{tor} and Γ_0. The case $p = 2$ is different from the cases $p \geq 3$ and requires entirely different and quite technical arguments. While the arguments involving $\mathbb{Z}_p(1)^d$ and Γ_0 are "prismatic/crystalline" in nature and feature q-de Rham complexes, the arguments involving the group Γ_{tor} feature some techniques commonly used in Iwasawa theory (see Appendix A.2). We refer the reader to Subsection 5.2 for precise details on the construction of the functor in (1.4).

1.4. Some descent results for Wach modules. The most important input for the construction of the functor in (1.4) is the following comparison result (notations are explained after the statement):

Theorem 1.4 (Theorem 4.5)**.** *Let N be a Wach module over A_R and consider the R-module $M := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\Gamma_R}$, equipped with the tensor product Frobenius. Then M is a finitely generated p-torsion free R-module and we have a natural (φ, Γ_R)-equivariant isomorphism*

$$A_R(1)/p_1(\mu) \otimes_{p_1, R} M \xrightarrow{\sim} A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N.$$

Moreover, the preceding isomorphism induces a φ-equivariant (after inverting p) isomorphism of R-modules $M \xrightarrow{\sim} N/\mu N$.

In Theorem 1.4, note that $(A_R(1), [p]_q)$ denotes the self product of $(A_R, [p]_q)$ in $X_{\mathbb{\Delta}}$ and $p_1, p_2 : A_R \to A_R(1)$ are the two projection maps to the two components (see Construction 3.13).

The statement in Theorem 1.4 is a descent statement for the action of Γ_R and involves the ring $A_R(1)/p_1(\mu)$. Therefore, in order to prove the statement, one requires an understanding of the action of Γ_R on the objects appearing in the statement and an explicit description of the ring $A_R(1)/p_1(\mu)$. The latter is achieved in Proposition 3.25, where we use some computations of Tsuji on prismatic envelopes and divided power rings (see Appendix B). Next, similar to the discussion in the previous subsection, note that the descent for the action of Γ_R is quite technical. Our strategy is to prove the statement via a "3-step descent" argument. The case $p = 2$ is different from the cases $p \geq 3$ and requires entirely different and quite technical arguments. Note that while the arguments involving $\mathbb{Z}_p(1)^d$ and Γ_0 are "crystalline" in nature and feature de Rham complexes, the arguments involving the group Γ_{tor} feature some techniques commonly used in Iwasawa theory. We refer the reader to Subsections 4.2 and 4.3 for precise details on the descent isomorphism.

Now, let us assume that $p \geq 3$ and note that as a consequence of the proof of Theorem 1.4 (namely, the descent step for the action of $\Gamma_{\mathrm{tor}} = \mathbb{F}_p^\times$), we obtain a descent statement for (classical) Wach modules (see Remark 1.6). More precisely, note that we have $A_R \xrightarrow{\sim} R[\![\mu]\!]$ as rings (see Subsection 3.1.1), and by transport of structrure, we equip the target with a Frobenius endomorphism and an R-linear action of Γ_F. Let N be a Wach module over A_R, i.e. a finitely generated module over $R[\![\mu]\!]$, equipped with an $R[\![\mu]\!]$-linear Frobenius isomorphism $\varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ and an R-linear and continous action of Γ_F commuting with Frobenius and such that the action of Γ_F is trivial on $N/\mu N$. Let us set,

$$\mu_0 := -p + \textstyle\sum_{a \in \mathbb{F}_p} (1+\mu)^{[a]} \quad \text{and} \quad \tilde{p} := \mu_0 + p,$$

as elements of $R[\![\mu]\!]^{\mathbb{F}_p^\times}$. Then, from Lemma 3.4, we have a (φ, Γ_0)-equivariant isomorphism of rings $R[\![\mu_0]\!] \xrightarrow{\sim} R[\![\mu]\!]^{\mathbb{F}_p^\times}$ and we show the following:

Theorem 1.5 (Proposition 4.19)**.** *Let N be a Wach module over A_R. Then $N_0 := N^{\mathbb{F}_p^\times}$ is a finitely generated $R[\![\mu_0]\!]$-module, equipped with a continuous and R-linear action of Γ_0 such that the action of Γ_0 is trivial on $N_0/\mu_0 N_0 \xrightarrow{\sim} N/\mu N$, and we have a natural (φ, Γ_F)-equivariant isomorphism of $R[\![\mu]\!]$-modules $R[\![\mu]\!] \otimes_{R[\![\mu_0]\!]} N_0 \xrightarrow{\sim} N$. Moreover, N_0 is equipped with an $R[\![\mu_0]\!]$-linear isomorphism $\varphi^*(N_0)[1/\tilde{p}] \xrightarrow{\sim} N_0[1/\tilde{p}]$, compatible with the action of Γ_0.*

Remark 1.6. In Theorem 1.5, setting $R = O_F$ shows that classical Wach modules over $A_F = O_F[\![\mu]\!]$ canonically descend to Wach modules over $O_F[\![\mu_0]\!] \xrightarrow{\sim} A_F^{\mathbb{F}_p^\times}$. Using somewhat different arguments, this claim has also been proven in [Abh24, Theorem 1.7].

1.5. Relation to previous works.

The theory of prismatic F-crystals was introduced in [BS23] and its generalisation to analytic/completed prismatic F-crystals was studied in [DLMS24; GR24]. The theory of classical Wach modules was introduced and developed in [Fon90; Wac96; Col99; Ber04] and its generalisation to the relative case was introduced and developed in [Abh21; Abh23a; Abh23b]. Now, let us consider the following diagram (notations explained immediately afterwards):

$$\begin{tikzcd}
\mathrm{Vect}^{\varphi,\mathrm{an}}(X_{\Delta}) \arrow[rrr, hook, "\mathcal{E} \mapsto \mathcal{E}[1/\mathcal{I}_{\Delta}] \otimes_{\mathcal{O}[1/\mathcal{I}_{\Delta}]} \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p"] \arrow[dr, "T_{\text{ét}}^{\text{an}}", "\sim"'] \arrow[dd, "\mathrm{ev}^{\Delta}_{A_R}(1.1)"', "\wr"] & & & \mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p) \arrow[dl, "T_{\text{ét}}"', "\sim"] \arrow[dd, "\mathrm{ev}^{\Delta}_{A_R}", "\wr"'] \\
& \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_R) \arrow[r, hook] \arrow[dl, "\mathbf{N}_R", "\sim"'] & \mathrm{Rep}_{\mathbb{Z}_p}(G_R) \arrow[dr, "\mathbf{D}_R"', "\sim"] & \\
(\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{A_R} \arrow[rrr, hook, "N \mapsto N \otimes_{A_R} B_R"] & & & (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{\text{ét}}_{B_R}.
\end{tikzcd} \tag{1.5}$$

In top row of the diagram (1.5), the top right corner denotes the category of Laurent F-crystals over X_{Δ} (see [BS23] or Definition 1.5); in the middle row, G_R denotes the étale fundatmental group of $R[1/p]$ (see Subsection 1.6), the right-hand-term denotes the category of $\mathbb{Z}_p$-representations of G_R and the left-hand-term denotes the category of lattices inside p-adic crystalline representations of G_R (see [Bri08]); in the bottom row we have $B_R := A_R[1/\mu]^{\wedge}_p$ equipped with an induced action of (φ, Γ_R) and the bottom right corner denotes the category of étale (φ, Γ_R)-modules over B_R (see [And06]). The top horizontal arrow is a natural embedding from [GR24], the middle horizontal arrow is easily seen to be a natural embedding and the bottom horizontal arrow is a natural embedding from [Abh23b, Proposition 3.15]. From the first to the second row, the slanted arrow $T_{\text{ét}}$ is the natural étale realisation functor and an equivalence from [BS23] (see Lemma 2.13), and the slanted arrow $T_{\text{ét}}^{\text{an}}$ is an equivalence from [GR24] (see Definition 2.25 and Remark 2.27); the upper square commutes using loc. cit. From the middle row to the bottom row, the slanted arrow $\mathbf{D}_R$ is the natural étale (φ, Γ)-module functor and an equivalence from [And06], and the slanted arrow $\mathbf{N}_R$ is the natural Wach module functor and an equivalence from [Abh23b, Theorem 1.7]; commutativity of the bottom square follows from the compatibility between the results of [Abh23b] and [And06]. Next, in the diagram (1.5), the leftmost and the rightmost vertical arrows are evaluation functors, i.e. evaluation of an analytic (resp. Laurent) prismatic F-crystal over the prism $(A_R, [p]_q)$; the left vertical arrow is an equivalence from Theorem 1.1. Note that the left-hand-side triangle commutes by comparing the explicit formulas for the slanted arrow $T_{\text{ét}}^{\text{an}}$ in Definition 2.25 and the composition of the left vertical arrow $\mathrm{ev}^{\Delta}_{A_R}$ in (1.1) with the quasi-inverse of the slanted arrow $\mathbf{N}_R$ described in [Abh23b, Theorem 1.7], and similarly, for the right-hand-side triangle. In particular, we obtain that the right vertical arrow is an equivalence (also see [Wu21; MW21]) and the outer square commutes by the definition of the arrows. Hence, it follows that the results of this article are compatible with previous constructions.

As mentioned earlier, our proof of Theorem 1.1 is direct and we do not assume the equivalence between crystalline $\mathbb{Z}_p$-representations and Wach modules over A_R from [Abh23b], or the catgeorical equivalence results from [BS23; GR24; DLMS24] and [Wu21; MW21; DL21]. Furthermore, our approach is inspired by the work of [MT20], however, the results obtained and the techniques employed in our proofs are different: we work over the absolute prismatic site of R and consider analytic prismatic F-crystals, whereas in loc. cit., the authors work over the relative prismatic site and consider prismatic F-crystals of vector bundles, in particular, in this article we work with a non-commutative group Γ_R which has an "arithmetic" part (see Γ_F in Subsection 1.6), in contrast with the commutative and "geometric" Γ considered in loc. cit.

1.6. Setup and notations. In this subsection we will describe the setup for this paper and fix some notations which are essentially the same as in [Abh23b, Subsection 1.4]. We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers.

Let p be a fixed prime number, κ a perfect field of characteristic p, let $O_F := W(\kappa)$ denote the ring of p-typical Witt vectors with coefficients in κ and $F := O_F[1/p]$, the fraction field of O_F. In particular, F is an unramified extension of $\mathbb{Q}_p$ with ring of integers O_F. Let $\overline{F}$ denote a fixed algebraic closure of F so that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of κ. Further, we denote by $G_F := \mathrm{Gal}(\overline{F}/F)$, the absolute Galois group of F.

Let $Z = (Z_1, \ldots, Z_s)$ denote a set of indeterminates and $\mathbf{k} = (k_1, \ldots, k_s)$ in $\mathbb{N}^s$ be a multi-index and set $Z^{\mathbf{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$. We will write $\mathbf{k} \to +\infty$ to denote $\sum k_i \to +\infty$. For a topological algebra S, define

$$S\langle Z\rangle := \big\{ \textstyle\sum_{\mathbf{k}\in\mathbb{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}, \text{ where } a_{\mathbf{k}} \in S \text{ and } a_{\mathbf{k}} \to 0 \text{ as } \mathbf{k} \to +\infty\big\}.$$

Fix $d \in \mathbb{N}$ and let $X_1, X_2, \ldots, X_d$ be some indeterminates. Let R be the p-adic completion of an étale algebra over $R^\square = O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ with non-empty and connected special fiber. We fix an algebraic closure $\overline{\mathrm{Frac}(R)}$ of $\mathrm{Frac}(R)$ containing $\overline{F}$. Let $\overline{R}$ denote the union of finite R-subalgebras $R' \subset \overline{\mathrm{Frac}(R)}$, such that $R'[1/p]$ is étale over $R[1/p]$. Let $\overline{\eta}$ denote the fixed geometric point of the generic fiber $\mathrm{Spec}\, R[1/p]$ (defined by $\overline{\mathrm{Frac}(R)}$) and let G_R denote the étale fundamental group $\pi_1^{\text{ét}}(\mathrm{Spec}\, R[1/p], \overline{\eta})$. We can write this étale fundamental group as the Galois group (of the fraction field of $\overline{R}[1/p]$ over the fraction field of $R[1/p]$), i.e.

$$G_R = \pi_1^{\text{ét}}(\mathrm{Spec}\,(R[1/p]), \overline{\eta}) = \mathrm{Gal}(\overline{R}[1/p]/R[1/p]).$$

Next, set $F_\infty := F(\mu_{p^\infty})$, $R_\infty := \cup_{i=1}^d R[\mu_{p^\infty}, X_i^{1/p^\infty}] \subset \overline{R}$ and we will consider the following groups (see [Abh21, Subsections 2 & 3] and [Abh23a, Section 2]),

$$\begin{aligned}
&H_R := \mathrm{Gal}(\overline{R}[1/p]/R_\infty[1/p]),\ H_F := \mathrm{Gal}(\overline{F}/F_\infty)\\
&\Gamma_R := G_R/H_R = \mathrm{Gal}(R_\infty[1/p]/R[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times,\ \Gamma_F := \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times\\
&\Gamma_R' := \mathrm{Gal}(R_\infty[1/p]/R(\mu_{p^\infty})[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d,\ \mathrm{Gal}(R(\mu_{p^\infty})[1/p]/R[1/p]) = \Gamma_R/\Gamma_R' \xrightarrow{\sim} \Gamma_F.
\end{aligned}$$

Note that the isomorphism $\chi : \Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$ is given via the p-adic cyclotomic character, and therefore, Γ_F fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1, \tag{1.6}$$

where, for $p \geq 3$, we have $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ and for $p = 2$, we have $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$. Moreover, for $p \geq 3$, we have $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$, and the projection map in (1.6) admits a section $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \to \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$, where the second map is given as $a \mapsto [a]$, the Teichmüller lift of a. Furthermore, note that for $p = 2$, we have $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \{\pm 1\}$, as groups.

Let $\varphi : R^\square \to R^\square$ denote a morphism extending the natural Witt vector Frobenius on O_F by setting $\varphi(X_i) = X_i^p$, for all $1 \leq i \leq d$. The endomorphism φ of $R^\square$ is flat by [Bri08, Lemma 7.1.5] and faithfully flat since $\varphi(\mathfrak{m}) \subset \mathfrak{m}$, for any maximal ideal $\mathfrak{m} \subset R^\square$. Moreover, it is finite of degree p^d using

Nakayama Lemma and the fact that φ modulo p is evidently of degree p^d. Recall that the O_F-algebra R is given as the p-adic completion of an étale algebra $R^\square$, therefore, the Frobenius endomorphism φ on $R^\square$ admits a unique extension $\varphi : R \to R$ such that the induced map $\varphi : R/p \to R/p$ is the absolute Frobenius $x \mapsto x^p$ (see [CN17, Proposition 2.1]). Similar to above, we again note that the endomorphism $\varphi : R \to R$ is finite and faithfully flat of degree p^d. Next, for $k \in \mathbb{N}$, let Ω^k_R denote the p-adic completion of module of k-differentials of R relative to $\mathbb{Z}$. Then, we have that $\Omega^1_R = \oplus_{i=1}^d R\, d\log X_i$ and $\Omega^k_R = \wedge^k_R \Omega^1_R$.

Finally, let $A = R_\infty$ or $\overline{R}$. Then the tilt of A is defined as $A^\flat = \lim_\varphi A/p$ and the tilt of $A[1/p]$ is defined as $A[1/p]^\flat = A^\flat[1/p^\flat]$, where $p^\flat = (1, p^{1/p}, \dots)$ in $A^\flat$ (see [Fon77, Chapitre V, Subsection 1.4]). Finally, let B be a $\mathbb{Z}_p$-algebra equipped with a Frobenius endomorphism φ lifting the absolute Frobenius on B/p, then for any B-module M, we set $\varphi^*(M) := B \otimes_{\varphi,B} M$.

1.7. Outline of the paper. In Section 2 we recall some basic definitions and results from the prismatic theory. In particular, in Subsection 2.1, we recall the definition of prismatic site and prismatic crystals from [BS22; BS23], and in Subsections 2.2 and 2.3 we recall the notion of prismatic F-crystals and its variations from [BS23; DLMS24; GR24], as well as, describe the étale realisation functors. Section 3 is devoted to the study of the prism $(A_R, [p]_q)$ in detail. In Subsection 3.1, we describe the ring A_R and study the structural properties of some of its subrings. In Subsection 3.2, we show that $(A_R, [p]_q)$ is an object of $(\mathrm{Spf}\ R)_{\Delta}$, explicitly compute first few terms of its prismatic Čech nerve $(A_R(\bullet), I(\bullet))$ in $(\mathrm{Spf}\ R)_{\Delta}$, and then study the action of $\Gamma_R^{\times(n+1)}$ on $A_R(n)$. Then in Subsection 3.3 we provide an explicit description of the ring $A_R(1)/p_1(\mu)$, where $p_1 : A \to A(1)$ is the first projection map, and study the action of $1 \times \Gamma_R \subset \Gamma_R^2$ on it. Finally, in Subsection 3.4 we study the action of Γ_R^2 on $A_R(1)$ and carry out the "3-step" argument, described after Theorem 1.3, for constant coefficients, i.e. the ring A_R, in order to compute various rings that will appear at various stages of the proof of Theorem 1.3 in Section 5.

The goal of Section 4 is to state and prove Theorem 1.4. We begin by recalling the definition of Wach modules and some of its properties from [Abh23b] and in Subsection 1.4 we describe the q-connection on a Wach module and its scalar extensions originating from the natural action of Γ_R. Then, in Subsection 4.2, we carry out the first step of the proof of Theorem 1.4, i.e. the descent for the action of Γ'_R, and in Subsection 4.3 we show the second and third steps, i.e. the descent for the action of $\Gamma_F \xrightarrow{\sim} \Gamma_0 \rtimes \Gamma_{\mathrm{tor}}$. Note that the Subsection 4.3 is divided into two parts: the first part deals with the case $p \geq 3$ where we also prove Theorem 1.5, and the second part deals with $p = 2$ which requires completely different arguments. Finally, in Subsection 4.4 we put everything together to prove Theorem 1.4.

In Section 5, we state and prove Theorem 1.1. We begin by describing the functor $\mathrm{ev}^{\Delta}_{A_R}$ and then in Subsection 5.1 we describe the relation between analytic/completed prismatic F-crystals and modules with stratification, as well as, prove Proposition 1.2. Subsection 5.2 is dedicated to the construction of stratification on Wach modules using the action of Γ_R via the "3-step" argument. This completes the proof of Theorem 1.1.

In Appendix A.1, we recall some standard definitions that have been used throughout in the text. Then in Appendix A.2 we describe the structure of modules admitting a continuous action of $\mathbb{Z}_p^\times$ by recalling some standard constructions of Iwasawa [Iwa59]. Appendix B has been adapted from some notes of Tsuji and in that section we study the structure of certain δ-rings and their reduction modulo μ, which is a crucial input for determining the structure of $A_R(1)/p_1(\mu)$ is Subsection 3.3.

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2. Prismatic site and Prismatic F-crystals

In this section, we will recall some fundamental definitions and results on prismatic site and prismatic F-crystals from [BS22; BS23] and analytic/completed prismatic F-crystals from [DLMS24] and [GR24]. For some standard definitions used in this section, we refer the reader to Appendix A.1. We start with the following:

Definition 2.1. A δ-ring is a pair (A, δ) where A is a commutative ring and $\delta : A \to A$ is a map of sets with $\delta(0) = \delta(1) = 0$ and satisfying:

$$\begin{aligned} \delta(xy) &= x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y), \\ \delta(x+y) &= \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}. \end{aligned} \tag{2.1}$$

Remark 2.2. Given a δ-ring (A, δ) define $\varphi : A \to A$ by the formula $\varphi(x) = x^p + p\delta(x)$, for any x in A. This determines a lift of the absolute Forbenius on A/pA. Conversely, if A is p-torsion-free then any lift $\varphi : A \to A$ of the absolute Frobenius on A/pA determines a unique δ-structure on A.

Definition 2.3. An element d of a δ-ring A is distinguished if $\delta(d)$ is a unit.

2.1. Prismatic site and crystals.

Let A be a δ-ring, $I \subset A$ an ideal; we will refer to (A, I) as a δ-pair.

Definition 2.4 (Prism, [BS22, Definition 3.2])**.** A δ-pair (A, I) is called a *prism* if $I \subset A$ defines a Cartier divisor on Spec (A) such that A is derived (p, I)-complete and $p \in I + \varphi(I)A$. The category of all prisms is the corresponding full subcategory of all δ-pairs. The prism (A, I) is called perfect if A is a perfect δ-ring, i.e. $\varphi : A \to A$ is bijective. Finally, (A, I) is bounded if A/I has bounded p^∞-torsion. A map $(A, I) \to (B, J)$ of prisms is (faithfully) flat if the map $A \to B$ is (p, I)-completely (faithfully) flat.

Remark 2.5. If (A, I) is a bounded prism then A is classically (p, I)-complete (see [BS22, Lemma 3.7]). A morphism of prisms $(A, I) \to (B, J)$ we induces an isomorphism $I \otimes_A B \xrightarrow{\sim} J$, in particular, we have that $IB = J$ (see [BS22, Lemma 3.5]).

Lemma 2.6 (Absolute prismatic site, [BS22, Corollary 3.12])**.** *Let* Spf $(\mathbb{Z}_p)_{\Delta}$ *denote the category opposite to that of the category of all bounded prisms* (A, I) *and endow it with a topology for which covers are determined by faithfully flat maps of prisms. Then,* Spf $(\mathbb{Z}_p)_{\Delta}$ *forms a site. Moreover, the functor* $\mathcal{O}_{\Delta} :$ Spf $(\mathbb{Z}_p)_{\Delta} \to$ Rings *(resp.* $\mathcal{I}_{\Delta}$*) defined via* $(A, I) \mapsto A$ *(resp.* $(A, I) \mapsto I$*) forms a sheaf for this topology with vanishing higher Čech cohomology.*

Definition 2.7 (Absolute prismatic site of X, [BS23, Definition 2.3])**.** Let X be a p-adic formal scheme. Define the *absolute prismatic site* of X, denoted as X_{Δ}, to be the category opposite to that of bounded prisms (A, I) which are equipped with a map Spf $(A/I) \to X$, and endow X_{Δ} with the topology induced by the flat topology on prisms. We will write $\mathcal{O}_{\Delta}$ for the structure sheaf and denote the ideal sheaf of the Hodge-Tate divisor by $\mathcal{I}_{\Delta} \subset \mathcal{O}_{\Delta}$. Denote by $\mathrm{Shv}(X_{\Delta})$ the ∞-category of sheaves on X_{Δ}.

Proposition 2.8 ([BS23, Proposition 2.7])**.** *Let X be a p-adic formal scheme and let* $\mathrm{Vect}(X_{\Delta}, \mathcal{O}_{\Delta})$ *denote the category of vector bundles of* $\mathcal{O}_{\Delta}$*-modules. Then there is a natural equivalence*

$$\mathrm{Vect}(X_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \lim_{(A,I)\in X_{\Delta}} \mathrm{Vect}(A).$$

Moreover, a similar statement holds after replacing $\mathcal{O}_{\Delta}$ *with* $\mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p$.

Let X be a quasisyntomic p-adic formal scheme. Then one can describe crystals on $(X_{\Delta}, \mathcal{O}_{\Delta})$ using the quasisyntomic site of X.

Definition 2.9 (Quasisyntomic site of X, [BS23, Definition 2.9])**.** Define the *quasisyntomic site* of X, denoted as X_{qsyn}, to be the category opposite to that of quasisyntomic morphisms $\mathrm{Spf}\,(R) \to X$ and endow X_{qsyn} with the topology induced by the quasisyntomic topology (see [BMS19, Definition 4.10 and Variant 4.35]). Define $X_{\mathrm{qrsp}} \subset X_{\mathrm{qsyn}}$ to be the full subcategory spanned by objects $\mathrm{Spf}\,(R) \to X$ with R quasiregular semiperfectoid (see [BMS19, Definition 4.20 and Variant 4.35]).

Remark 2.10. By [BMS19, Proposition 4.31 and Variant 4.35], restriction of sheaves induces an equivalence $\mathrm{Shv}(X_{\mathrm{qsyn}}) \xrightarrow{\sim} \mathrm{Shv}(X_{\mathrm{qrsp}})$. The functor defined by sending any $(\mathrm{Spf}\,(R) \to X)$ in X_{qrsp} to the initial prism Δ_R (see [BS22, Proposition 7.2]) gives a sheaf of rings $\Delta_\bullet$ on X_{qsyn}; we write $I_{\Delta_\bullet} \subset \Delta_\bullet$ for the ideal sheaf of the Hodge-Tate divisor.

Lemma 2.11 ([BS23, Proposition 2.14])**.** *Let X be a quasisyntomic p-adic formal scheme. Then there are natural equivalences* $\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta) \xrightarrow{\sim} \mathrm{Vect}(X_{\mathrm{qsyn}}, \Delta_\bullet) \xrightarrow{\sim} \lim_{R \in X_{\mathrm{qrsp}}} \mathrm{Vect}(\Delta_R)$.

2.2. Laurent F-crystals.

Let X be a bounded p-adic formal scheme and let X_η denote the generic fiber of X, regarded as a presheaf on perfectoid spaces over $\mathbb{Q}_p$.

Definition 2.12 (Laurent F-crystals, [BS23, Definition 3.2])**.** Define the category of Laurent F-crystals of vector bundles on $(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge_p)$ as

$$\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge_p)^{\varphi=1} := \lim_{(A,I)\in X_\Delta} \mathrm{Vect}(A[1/I]^\wedge_p)^{\varphi=1},$$

i.e. any $\mathcal{E}$ in $\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge_p)$ is equipped with an isomorphism $\varphi_{\mathcal{E}} : \varphi^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$.

Lemma 2.13 ([BS23, Corollary 3.8])**.** *Let $\mathrm{Loc}(X_\eta, \mathbb{Z}_p)$ denote the category of étale $\mathbb{Z}_p$-local systems on X_η. Then there is a natural equivalence*

$$T_{\text{ét}} : \mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge_p)^{\varphi=1} \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(X_\eta).$$

Passing to the associated isogeny categories we obtain

$$\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge_p)^{\varphi=1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(X_\eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

2.3. Prismatic F-crystals.

In this subsection we will recall the definition of prismatic F-crystals and its analytic/completed variants.

2.3.1. Vector bundles.

Let X be a p-adic formal scheme and let (A, I) denote a prism with φ its Frobenius endomorphism.

Definition 2.14 (Prismatic F-crystals, [BS23, Definition 4.1])**.** Define the category $\mathrm{Vect}^\varphi(A)$ of prismatic F-crystals of vector bundles on A as follows: an object is a pair (M, φ_M) with M a finite projective A-module equipped with an A-linear isomorphism $\varphi_M : (\varphi^*M)[1/I] \xrightarrow{\sim} M[1/I]$. Morphisms between two objects are given as A-linear maps compatible with φ_M. Say that (M, φ_M) is effective if φ_M carries φ^*M into M.

Define the category $\mathrm{Vect}^\varphi(X_\Delta, \mathcal{O}_\Delta)$ of prismatic F-crystals of vector bundles on X_Δ as follows: an object is a pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ with $\mathcal{E}$ a vector bundle on $(X_\Delta, \mathcal{O}_\Delta)$ equipped with an isomorphism $\varphi_{\mathcal{E}} : (\varphi^*\mathcal{E})[1/\mathcal{I}_\Delta] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_\Delta]$. Morphisms between two objects are maps of vector bundles compatible with $\varphi_{\mathcal{E}}$. Say that $(\mathcal{E}, \varphi_{\mathcal{E}})$ is effective if $\varphi_{\mathcal{E}}$ carries $\varphi^*\mathcal{E}$ into $\mathcal{E}$.

Remark 2.15. From Proposition 2.8 and Lemma 2.11 we have equivalences $\mathrm{Vect}^\varphi(X_\Delta, \mathcal{O}_\Delta) \xrightarrow{\sim} \lim_{(A,I)\in X_\Delta} \mathrm{Vect}^\varphi(A) \xrightarrow{\sim} \lim_{R\in X_{\mathrm{qrsp}}} \mathrm{Vect}^\varphi(\Delta_R)$.

2.3.2. Completed F-crystals. Let O_K be a finite ramified extension of O_F and take π in O_K to be a uniformizer and $E(u)$ in $O_F[u]$ to be its minimal polynomial.

Definition 2.16 (Completed prismatic crystals, [DLMS24, Definition 3.11])**.** Let X be a smooth p-adic formal scheme over O_K. Define the category $\mathrm{CR}^\wedge(X_{\Delta})$ of *finitely generated completed crystals of $\mathcal{O}_{\Delta}$-modules* on X_{Δ} as follows: an object is a sheaf $\mathcal{E}$ of $\mathcal{O}_{\Delta}$-modules on X_{Δ} satisfying the following:

(1) For each (A, I) in X_{Δ}, the evaluation of $\mathcal{E}$ on (A, I), i.e. $\mathcal{E}(A) := \mathcal{E}(A, I)$, is a finitely generated and (p, I)-complete A-module.

(2) For every morphism $(A, I) \to (B, IB)$ in X_{Δ}, the natural map $B\widehat{\otimes}_A\mathcal{E}(A) \to \mathcal{E}(B)$ is an isomorphism, where we set $B\widehat{\otimes}_A\mathcal{E}(A) := \lim_n(B \otimes_A \mathcal{E}(A))/(p, I)^n$.

Morphisms between two objects are morphisms of $\mathcal{O}_{\Delta}$-modules compatible with completed base change.

Remark 2.17. The category of completed crystals on X_{Δ}, denoted as $\mathrm{CR}^\wedge(X_{\Delta})$, satisfies descent for Zariski and étale topologies on X (see [DLMS24, Lemma 3.39, Remark 3.40]).

A p-adically completed O_K-algebra S is called *small* if it is p-adically completed étale over a p-adically complete Laurent polynomial O_K-algebra $O_K\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$, for some $d \in \mathbb{N}$. For such an S there exists a unique O_F-algebra R such that $O_K \otimes_{O_F} R \xrightarrow{\sim} S$ and R admits a lift of the absolute Frobenius modulo p, which we take to be the one extending the Witt vector Frobenius on O_F and such that $\varphi(X_i) = X_i^p$; denote the lift of Frobenius as $\varphi : R \to R$. Let $\mathfrak{S} = R[\![u]\!]$ equipped with a Frobenius endomorphism φ which extends the Frobenius on R by setting $\varphi(u) = u^p$. Then $(\mathfrak{S}, E(u))$ is an object of $(\mathrm{Spf}\ S)_{\Delta}$ and called the *Breuil-Kisin* prism ([DLMS24, Example 3.4]).

Definition 2.18 (Completed prismatic F-crystals, [DLMS24, Definition 3.16, Definition 3.42])**.** Let $X := \mathrm{Spf}\ S$ for S a small O_K-algebra. Define the category $\mathrm{CR}^{\wedge,\varphi}(X_{\Delta})$ of *completed F-crystals of $\mathcal{O}_{\Delta}$-modules* on X_{Δ} as follows: an object is a pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ with $\mathcal{E}$ a completed prismatic crystal on X_{Δ} such that we have an isomorphism $\varphi_{\mathcal{E}} : \varphi^*\mathcal{E}[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$. Moreover, for the Breuil-Kisin prism $(\mathfrak{S}, E)$ in X_{Δ}, assume that the $\mathfrak{S}$-module $\mathcal{E}(\mathfrak{S})$ is torsionfree, $\mathcal{E}(\mathfrak{S})[1/p]$ (resp. $\mathcal{E}(\mathfrak{S})[1/E]$) is finite projective over $\mathfrak{S}[1/p]$ (resp. $\mathfrak{S}[1/E]$) and $\mathcal{E}(\mathfrak{S}) = \mathcal{E}(\mathfrak{S})[1/p] \cap \mathcal{E}(\mathfrak{S})[1/E] \subset \mathcal{E}(\mathfrak{S})[1/p, 1/E]$. Morphisms between two objects are given as maps of $\mathcal{O}_{\Delta}$-modules compatible with $\varphi_{\mathcal{E}}$. Say that $(\mathcal{E}, \varphi_{\mathcal{E}})$ is effective if $\varphi_{\mathcal{E}}$ carries $\varphi^*\mathcal{E}$ into $\mathcal{E}$.

Let X be a smooth p-adic formal scheme over O_K. Define the category $\mathrm{CR}^{\wedge,\varphi}(X_{\Delta})$ of *completed F-crystals of $\mathcal{O}_{\Delta}$-modules* on X_{Δ} as follows: an object is a pair $(\mathcal{E}, \varphi_{\mathcal{E}})$ with $\mathcal{E}$ a completed prismatic crystal on X_{Δ} such that we have an isomorphism $\varphi_{\mathcal{E}} : \varphi^*\mathcal{E}[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$. Moreover, there exists an affine open covering $X = \cup_{\lambda\in\Lambda}U_\lambda$ by affine p-adic formal schemes, where $U_\lambda = \mathrm{Spf}\ (S_\lambda)$ is connected and small over O_K, and such that $(\mathcal{E}, \varphi_{\mathcal{E}})|_{U_{\lambda,\Delta}}$ is an object of $\mathrm{CR}^{\wedge,\varphi}(U_{\lambda,\Delta})$. Morphisms between two objects are given as maps of $\mathcal{O}_{\Delta}$-modules compatible with $\varphi_{\mathcal{E}}$. Say that $(\mathcal{E}, \varphi_{\mathcal{E}})$ is effective if $\varphi_{\mathcal{E}}$ carries $\varphi^*\mathcal{E}$ into $\mathcal{E}$.

Remark 2.19. Note that Definition 2.18 is slightly more general than loc. cit. in the sense that we do not restrict ourselves to effective completed F-crystals.

2.3.3. Analytic F-crystals. Let X be a p-adic formal scheme.

Definition 2.20 (Analytic prismatic crystals, [GR24, Definition 3.1])**.** Define the category of *analytic prismatic crystals of vector bundles* over X as

$$\mathrm{Vect}^{\mathrm{an}}(X_{\Delta}) := \lim_{(A,I)\in X_{\Delta}} \mathrm{Vect}(\mathrm{Spec}\,(A) \setminus V(p, I)).$$

Lemma 2.21 ([GR24, Lemma 3.3])**.** *Let X be a quasisyntomic p-adic formal scheme. Then, there is a natural equivalence* $\mathrm{Vect}^{\mathrm{an}}(X_{\Delta}) \xrightarrow{\sim} \mathrm{Vect}^{\mathrm{an}}(X_{\mathrm{qsyn}}) := \lim_{R\in X_{\mathrm{qrsp}}} \mathrm{Vect}(\mathrm{Spec}\,(\Delta_R) \setminus V(p, I))$.

Let (A, I) be a prism and let φ denote the Frobenius endomorphism on A. Then we have that $\varphi(I) \subset (p, I)$, since $\varphi(x) = x^p + p\delta(x)$ is in (p, I), for each x in I. In particular, the Frobenius endomorphism φ on A preserves the subscheme $\mathrm{Spec}\,(A) \setminus V(p, I)$ and we will denote the induced endomorphism on $\mathrm{Spec}\,(A) \setminus V(p, I)$ again by φ.

Definition 2.22 (Analytic prismatic F-crystals, [GR24, Definition 3.6])**.** Define the category $\mathrm{Vect}^{\mathrm{an},\varphi}(A)$ as follows: an object is a pair (M, φ_M) with M a vector bundle on $\mathrm{Spec}\,(A) \setminus V(p, I)$ equipped with an A-linear isomorphism $\varphi_M : (\varphi^*M)[1/I] \xrightarrow{\sim} M[1/I]$. Morphisms between two objects are given as maps of vector bundles compatible with φ_M. Say that (M, φ_M) is effective if φ_M carries φ^*M into M.

Define the category of analytic prismatic F-crystals on X_{Δ} as follows:

$$\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) := \lim_{(A,I)\in X_{\Delta}} \mathrm{Vect}^{\varphi}(\mathrm{Spec}\,(A) \setminus V(p, I)).$$

Say that $(\mathcal{E}, \varphi_{\mathcal{E}})$ is effective if $\varphi_{\mathcal{E}}$ carries $\varphi^*\mathcal{E}$ into $\mathcal{E}$.

Remark 2.23. From Lemma 2.11 we have an equivalence $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) \xrightarrow{\sim} \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\mathrm{qsyn}}) := \lim_{R\in X_{\mathrm{qrsp}}} \mathrm{Vect}^{\varphi}(\mathrm{Spec}\,(\Delta_R) \setminus V(p, I))$.

Remark 2.24. Let $X = \mathrm{Spf}\,(O_F)$, then from [GR24, Proposition 3.8] note that the restriction functor induces equivalence of categories $\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta})$.

2.3.4. Étale realisation functors. Now we introduce the étale realisation functors from the category of (completed/analytic) prismatic F-crystals using Lemma 2.13. Let X be a p-adic formal scheme and $(\mathcal{E}, \varphi_{\mathcal{E}})$ and object of $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta})$. Consider the open embedding $\mathrm{Spec}\,(A[1/I]) \subset \mathrm{Spec}\,(A) \setminus V(p, I)$, for any prism (A, I) in X_{Δ}, and let $\mathcal{E}[1/\mathcal{I}_{\Delta}]$ denote the pullback of $\mathcal{E}$ along this embedding, i.e. it is an object of $\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}])$. Using the equivalence $\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p) \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(X_{\eta})$ from Lemma 2.13, we have the following:

Definition 2.25. Define the étale realisation functor for completed F-crystals as,

$$\begin{aligned} T^{\wedge}_{\text{ét}} : \mathrm{CR}^{\wedge,\varphi}(X_{\Delta}) &\longrightarrow \mathrm{Loc}_{\mathbb{Z}_p}(X_{\eta}) \\ (\mathcal{E}, \varphi_{\mathcal{E}}) &\longmapsto (\mathcal{E}[1/\mathcal{I}_{\Delta}] \otimes_{\mathcal{O}[1/\mathcal{I}_{\Delta}]} \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)^{\varphi=1}, \end{aligned}$$

Similarly, define the étale realisation functor for analytic F-crystals as,

$$\begin{aligned} T^{\mathrm{an}}_{\text{ét}} : \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) &\longrightarrow \mathrm{Loc}_{\mathbb{Z}_p}(X_{\eta}) \\ (\mathcal{E}, \varphi_{\mathcal{E}}) &\longmapsto (\mathcal{E}[1/\mathcal{I}_{\Delta}] \otimes_{\mathcal{O}[1/\mathcal{I}_{\Delta}]} \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}_p)^{\varphi=1} \end{aligned}$$

Remark 2.26. The étale realisation functor from $\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta})$ in [BS23, Construction 4.8] naturally factors through the functor $T^{\wedge}_{\text{ét}}$ (see [DLMS24, Proposition 3.43]) and $T^{\mathrm{an}}_{\text{ét}}$ (see [GR24, Construction 3.9]) in Definition 2.25.

Remark 2.27. The essential images of $T^{\wedge}_{\text{ét}}$ and $T^{\mathrm{an}}_{\text{ét}}$ coincide and we have natural equivalences of categories (see [DLMS24, Theorem 1.3] and [GR24, Theorem A]):

$$\mathrm{CR}^{\wedge,\varphi}(X_{\Delta}) \xrightarrow[\sim]{T^{\wedge}_{\text{ét}}} \mathrm{Loc}^{\mathrm{cris}}_{\mathbb{Z}_p}(X_{\eta}) \xleftarrow[\sim]{T^{\mathrm{an}}_{\text{ét}}} \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}),$$

where $\mathrm{Loc}^{\mathrm{cris}}_{\mathbb{Z}_p}(X_{\eta})$ is the category of $\mathbb{Z}_p$-local systems $\mathbb{L}$ on X_{η} such that $\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is crystalline.

3. The prism $(A_R, [p]_q)$

In this section, we will define and study a prism $(A_R, [p]_q)$, which is of fundamental importance in stating and proving the results in Sections 4 and 5. We will use the setup and notations from Subsection 1.6.

3.1. Some important rings. Let $R_\infty^\flat$ and $\overline{R}^\flat$ denote the tilt of R_∞ and $\overline{R}$, respectively (see Subsection 1.6). Set $A_{\mathrm{inf}}(R_\infty) := W(R_\infty^\flat)$ and $A_{\mathrm{inf}}(\overline{R}) := W(\overline{R}^\flat)$ to be the ring of p-typical Witt vectors with coefficients in $R_\infty^\flat$ and $\overline{R}^\flat$, respectively, and equipped with the Frobenius endomorphism on Witt vectors and continuous (for the weak topology) Γ_R-action and G_R-action, respectively (see [And06, Proposition 7.2]). Moreover, from loc. cit., we have that $A_{\mathrm{inf}}(R_\infty) = A_{\mathrm{inf}}(\overline{R})^{H_R}$. We fix $\overline{\mu} := \varepsilon - 1$, where $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots)$ is in $R_\infty^\flat$ and let $q := [\varepsilon]$ be the Teichmüller lift of ε in $A_{\mathrm{inf}}(R_\infty)$. Let $\mu := q-1$ and $\xi := \mu/\varphi^{-1}(\mu)$ be some fixed elements of $A_{\mathrm{inf}}(R_\infty)$ and note that $\mu = \overline{\mu} \mod p$. Then from the description of the weak topology on $A_{\mathrm{inf}}(R_\infty)$ and $A_{\mathrm{inf}}(\overline{R})$ in loc. cit., it is easy to see that the weak topology on these rings is the same as the (p,μ)-adic topology. Next, let us note that we have $\varphi(q) = q^p$ and for any g in G_R, we have that $g(q) = q^{\chi(g)}$, where χ is the p-adic cyclotomic character, in particular, we see that $\varphi(\mu) = (1+\mu)^p - 1$ and $g(1+\mu) = (1+\mu)^{\chi(g)}$. Furthermore, we have a G_R-equivariant surjection $\theta : A_{\mathrm{inf}}(\overline{R}) \to \mathbb{C}(\overline{R})$, where $\mathbb{C}(\overline{R}) := \widehat{\overline{R}}[1/p]$ and $\mathrm{Ker}\,\theta = \xi A_{\mathrm{inf}}(\overline{R})$. The map θ further induces a Γ_R-equivariant surjection $\theta : A_{\mathrm{inf}}(R_\infty) \to \widehat{R}_\infty$, with $\mathrm{Ker}\,\theta = \xi A_{\mathrm{inf}}(R_\infty)$.

For $1 \leq i \leq d$, we fix $X_i^\flat = (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots)$ in R_∞ and we take $\{\gamma_1, \ldots, \gamma_d\}$ to be topological generators of Γ_R', such that $\gamma_j(X_i^\flat) = \varepsilon X_i^\flat$, if $i = j$ and 0 otherwise. Moreover, we may view Γ_F as a subgroup of Γ_R and for any g in Γ_F, we have $g\gamma_i g^{-1} = \gamma_i^{\chi(g)}$, for all $1 \leq i \leq d$. Let us also fix Teichmüller lifts $[X_i^\flat]$ in $A_{\mathrm{inf}}(R_\infty)$.

3.1.1. The ring A_R. Let $A_\square$ denote the (p,μ)-adic completion of the ring $O_F[\![\mu]\!][[X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]$. Note that we have a natural embedding $A_\square \subset A_{\mathrm{inf}}(R_\infty)$ and $A_\square$ is stable under the induced Frobenius endomorphism φ and the action of Γ_R (see [Abh21, Section 3]); we equip $A_\square$ with induced structures. Moreover, we have an embedding $\iota : R^\square \to A_\square$, via the map $X_i \mapsto [X_i^\flat]$, for $1 \leq i \leq d$, and it extends to an isomorphism of rings $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square$. Equip $R^\square[\![\mu]\!]$ with a Frobenius endomorphism φ extending the Frobenius on $R^\square$ by setting $\varphi(\mu) = (1+\mu)^p - 1$ and note that the map φ is finite and faithfully flat of degree p^{d+1}. Additionally, equip $R^\square[\![\mu]\!]$ with an $R^\square$-linear continuous action of Γ_F by setting $g(1+\mu) = (1+\mu)^{\chi(g)}$, for g in Γ_F. Then the embedding ι and the isomorphism $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+$ are Frobenius and Γ_F-equivariant.

Let A_R denote the (p,μ)-adic completion of the unique extension of the embedding $A_\square^+ \to A_{\mathrm{inf}}(R_\infty)$ along the p-adically completed étale map $R^\square \to R$ (see [Abh21, Subsection 3.3.2]). Then we have a natural embedding $A_R \subset A_{\mathrm{inf}}(R_\infty)$ and note that A_R is stable under the Frobenius and Γ_R-action on the latter; we equip it with induced structures. Moreover, note that the embedding $\iota : R^\square \to A_\square^+ \subset A_R$ and the isomorphism $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+ \subset A_R$ extend to a unique embedding $\iota : R \to A_R$ and an isomorphism of rings $R[\![\mu]\!] \xrightarrow{\sim} A_R$. Equip $R[\![\mu]\!]$ with a Frobenius endomorphism φ uextending the Frobenius on R by setting $\varphi(\mu) = (1+\mu)^p - 1$ and note that the map φ is finite and faithfully flat of degree p^{d+1}. Additionally, equip $R[\![\mu]\!]$ with an R-linear continuous action of Γ_F by setting $g(1+\mu) = (1+\mu)^{\chi(g)}$, for g in Γ_F. Then, it is clear that the embedding ι and the isomorphism $R[\![\mu]\!] \xrightarrow{\sim} A_R$ are Frobenius and Γ_F-equivariant. In particular, the Frobenius endomorphism φ on A_R is finite and faithfully flat of degree p^{d+1} and we have $\varphi^*(A_R) = A_R \otimes_{\varphi, A_R} A_R \xrightarrow{\sim} \oplus_\alpha \varphi(A_R) u_\alpha$, where $u_\alpha = (1+\mu)^{\alpha_0}[X_1^\flat]^{\alpha_1} \cdots [X_d^\flat]^{\alpha_d}$ and $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d)$ is a $(d+1)$-tuple with α_i taking values in $\{0, 1, \ldots, p-1\}$ for each $0 \leq i \leq d$.

Remark 3.1. For $R = O_F$, we will denote the ring A_R by A_F which is equipped with a Frobenius endomorphism φ and a continuous action of Γ_F.

Next, let us fix the following elements inside A_F:

$$\begin{aligned}
[p]_q &:= \tfrac{q^p-1}{q-1} = \tfrac{\varphi(\mu)}{\mu},\\
\mu_0 &:= \textstyle\sum_{a \in \mathbb{F}_p^\times}((1+\mu)^{[a]} - 1) = -p + \sum_{a \in \mathbb{F}_p}(1+\mu)^{[a]},\\
\tilde{p} &:= \mu_0 + p = \textstyle\sum_{a \in \mathbb{F}_p}(1+\mu)^{[a]}.
\end{aligned} \tag{3.1}$$

Remark 3.2. Note that for $p = 2$, we have $\mu_0 = \mu$ and $[p]_q = \mu + 2 = \tilde{p}$. However, such equalities do not hold for $p \geq 3$. Let us also remark that we denote the sum $-p + \sum_{a \in \mathbb{F}_p}(1+\mu)^{[a]}$ as μ_0 following

[Fon94, Subsection 5.2.5], where Fontaine denotes our μ by π and μ_0 by π_0. Moreover, the notation for $\tilde{p} = \mu_0 + p$ comes from [BL22, Subsection 3.8].

Lemma 3.3. *The element $\tilde{p}$ is the product of $[p]_q$ with a unit in A_F, in particular, both $\tilde{p}$ and $[p]_q$ generate the same ideal inside A_F.*

Proof. Note that $A_F/[p]_q = O_F[\zeta_p]$ and we have $[a] = a \bmod p\mathbb{Z}_p$. Since $\zeta_p^p = 1$ and $\tilde{p} = \sum_{a \in \mathbb{F}_p}(1+\mu)^{[a]}$, we see that $\tilde{p} = \sum_{a\in\mathbb{F}_p} \zeta_p^{[a]} = \sum_{a=0}^{p-1} \zeta_p^a = 0 \bmod [p]_q$. In particular, $\tilde{p} = [p]_q x$, for some x in A_F. Moreover, since $[p]_q = p \bmod \mu$ and $\mu_0 = 0 \bmod \mu$, we conclude that $x = 1 + \mu y$ for some y in A_F, i.e. x is a unit in A_F. This proves the claim. ∎

3.1.2. The ring $R[\![\mu_0]\!]$. From Subsection 3.1.1, recall that the ring $R[\![\mu]\!]$ is equipped with a continuous and R-linear action of Γ_F. Moreover, note that $\mu_0 = \sum_{a\in\mathbb{F}_p}(1+\mu)^{[a]}$ is an element of $R[\![\mu]\!]$ and it is invariant under the action of the subgroup $\mathbb{F}_p^\times$ of Γ_F (see the discussion after (1.6)). Furthermore, from Lemma 3.3, the element $\tilde{p}$ is the product of $[p]_q$ with a unit in $R[\![\mu]\!]$.

Now, consider an injective morphism of R-algebras $R[\![z]\!] \to R[\![\mu]\!]$ defined by sending $z \mapsto \mu_0$. Denote the image inside $R[\![\mu]\!]$ by $R[\![\mu_0]\!]$ and we will view the latter as the ring of formal power series over R in the variable μ_0. Note that the canonical injective homomorphism of R-algebras $R[\![\mu_0]\!] \to R[\![\mu]\!]$ is continuous for the (p, μ_0)-adic topology on the former and $(p, \mu_0) = (p, \mu)$-adic topology on the latter. Moreover, from the explicit description of μ_0 in (3.1), it follows that $R[\![\mu_0]\!]$ is stable under the Frobenius endomorphism and the continuous action of Γ_0 on $R[\![\mu]\!]$; we equip $R[\![\mu_0]\!]$ with the induced Frobenius endomorphism and continuous action of Γ_0.

Lemma 3.4. *Taking invariants of $R[\![\mu]\!]$ under the action of $\mathbb{F}_p^\times$ induces a (φ, Γ_0)-equivariant isomorphism of rings $R[\![\mu_0]\!] \xrightarrow{\sim} R[\![\mu]\!]^{\mathbb{F}_p^\times}$. Similarly, we have a (φ, Γ_0)-equivariant and $R[\![\mu_0]\!]$-linear isomorphism $\mu_0 R[\![\mu_0]\!] \xrightarrow{\sim} (\mu R[\![\mu]\!])^{\mathbb{F}_p^\times}$.*

Proof. Note that the map in the claim is (φ, Γ_0)-equivariant by definition. Moreover, by Remark 3.2, the claim is trivial for $p = 2$. So, assume that $p \geq 3$ and consider the following (φ, Γ_F)-equivariant diagram:

$$\begin{tikzcd}
0 \arrow[r] & R[\![\mu_0]\!] \arrow[r, "\tilde{p}"] \arrow[d] & R[\![\mu_0]\!] \arrow[r] \arrow[d] & R \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R[\![\mu]\!] \arrow[r, "\tilde{p}"] & R[\![\mu]\!] \arrow[r] & R[\zeta_p] \arrow[r] & 0,
\end{tikzcd} \tag{3.2}$$

where the vertical arrows are natural inclusions and the bottom row is exact since we have $R[\![\mu]\!]/[p]_q \xrightarrow{\sim} R[\zeta_p]$ and $[p]_q R[\![\mu]\!] = \tilde{p}R[\![\mu]\!]$ using Lemma 3.3. Now, note that the top row is $\mathbb{F}_p^\times$-invariant and for the bottom row, consider the associated long exact sequence for the cohomology of $\mathbb{F}_p^\times$-action and observe that $H^1(\mathbb{F}_p^\times, R[\![\mu]\!]) = 0$, since $p-1$ is invertible in $\mathbb{Z}_p$, and $R[\zeta_p]^{\mathbb{F}_p^\times} = R$. In particular, from the long exact sequence we obtain that $R[\![\mu_0]\!]/(\tilde{p}) \xrightarrow{\sim} R[\![\mu]\!]^{\mathbb{F}_p^\times}/(\tilde{p})$. Since, $R[\![\mu_0]\!]$ and $R[\![\mu]\!]^{\mathbb{F}_p^\times}$ are $\tilde{p}$-adically complete and $\tilde{p}$-torsion free, it follows that $R[\![\mu_0]\!] \xrightarrow{\sim} R[\![\mu]\!]^{\mathbb{F}_p^\times}$.

For the second claim, we consider the following (φ, Γ_F)-equivariant diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mu_0 R[\![\mu_0]\!] \arrow[r] \arrow[d] & R[\![\mu_0]\!] \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mu R[\![\mu]\!] \arrow[r] & R[\![\mu]\!] \arrow[r] & R \arrow[r] & 0,
\end{tikzcd}$$

where the vertical arrows are natural inclusions. Now, note that the top row is $\mathbb{F}_p^\times$-invariant and for the bottom row, consider the associated long exact sequence for the cohomology of $\mathbb{F}_p^\times$-action and observe that $H^1(\mathbb{F}_p^\times, \mu R[\![\mu]\!]) = 0$, since $p-1$ is invertible in $\mathbb{Z}_p$ and we have $R^{\mathbb{F}_p^\times} = R$ and $R[\![\mu_0]\!] \xrightarrow{\sim} R[\![\mu]\!]^{\mathbb{F}_p^\times}$ from the first part. Hence, it follows that the left vertical arrow induces a (φ, Γ_0)-equivariant isomorphism $\mu_0 R[\![\mu_0]\!] \xrightarrow{\sim} (\mu R[\![\mu]\!])^{\mathbb{F}_p^\times}$. ∎

Remark 3.5. Note that from (A.1), we have an $\mathbb{F}_p^\times$-decomposition as $R[\![\mu]\!] = \bigoplus_{i=0}^{p-2} R[\![\mu]\!]_i$. Moreover, from Lemma 3.4, it follows that we have $R[\![\mu_0]\!] \xrightarrow{\sim} R[\![\mu]\!]^{\mathbb{F}_p^\times} = R[\![\mu]\!]_0$ and the preceding decomposition is $R[\![\mu_0]\!]$-linear. In particular, for each $0 \leq i \leq p-2$, we have that $R[\![\mu]\!]_i$ is a (p, μ_0)-adically complete module over $R[\![\mu_0]\!]$, equipped with a continuous action of Γ_0.

Lemma 3.6. *Let g be any element of Γ_0, then $(g-1)\mu_0$ is an element of $\tilde{p}\mu_0 O_F[\![\mu_0]\!]$.*

Proof. Recall that $O_F[\![\mu_0]\!] \subset A_F$ is stable under the action of Γ_0. So if g is any element of Γ_0, then $(g-1)\mu_0$ is an element of $(\mu A_F)^{\mathbb{F}_p^\times} \xleftarrow{\sim} \mu_0 O_F[\![\mu_0]\!]$ (see Lemma 3.4). So let us write $(g-1)\mu_0 = \mu_0 x$. Now, let us consider the diagram in (3.2) with $R = O_F$, where the projection map $O_F[\![\mu_0]\!] \to O_F$ sends μ_0 to $-p$. In particular, we see that $(g-1)\mu_0 = -px \bmod \tilde{p}O_F[\![\mu_0]\!]$. Moreover, from the Γ_F-equivariance of the diagram, it follows that $(g-1)\mu_0 = 0 \bmod \tilde{p}O_F[\![\mu_0]\!]$. As O_F is p-torsion free, therefore, we conclude that x must be an element of $\tilde{p}O_F[\![\mu_0]\!]$. Hence, $(g-1)\mu_0$ is an element of $\tilde{p}\mu_0 O_F[\![\mu_0]\!]$. ∎

Lemma 3.7. *The element μ_0 is the product of μ^{p-1} with a unit in A_F, in particular, both μ_0 and μ^{p-1} generate the same ideal inside A_F.*

Proof. The idea of the proof is motivated from [Fon94, Proposition 5.2.6]. Let $\mu_1 = \prod_{a \in \mathbb{F}_p^\times}((1+\mu)^{[a]} - 1)$ be an element of A_F. Then, by expanding the product, it is easy to see that $\mu_1 = \mu^{p-1}u$ for some unit u in A_F. Moreover, μ_1 is invariant under the action of $\mathbb{F}_p^\times$ on A_F. Therefore, we get that μ_1 is also an element of $O_F[\![\mu]\!]^{\mathbb{F}_p^\times} \xleftarrow{\sim} O_F[\![\mu_0]\!]$ (see Lemma 3.4). So, let us write $\mu_1 = \sum_{k \in \mathbb{N}} a_k \mu_0^k$, where a_k is an element of O_F for each $k \in \mathbb{N}$. But since $\mu_1 = 0 \bmod \mu A_F$ and $\mu_0 = 0 \bmod \mu A_F$, we conclude that a_0 must be 0. In particular, in A_F, we have

$$\mu^{p-1}u = \mu_1 = \mu_0(a_1 + \textstyle\sum_{k \geq 2} a_k \mu_0^{k-1}). \tag{3.3}$$

Since A_F is μ_0-adically complete, it is enough to show that in (3.3), the element a_1 is a unit in O_F. Note that from the expression of μ_0, it is easy to see that we have $\mu_0 = \mu^{p-1} \bmod pA_F$. Therefore, reducing (3.3) modulo p, and using the preceding observation, we get that $\mu_0 u = \mu_0(a_1 + \sum_{k \geq 2} a_k \mu_0^{k-1}) \bmod pA_F$. Now note that $A_F/p = \kappa[\![\mu]\!]$ is μ_0-torsion free. Therefore, from the preceding equation, we conclude that $u = a_1 + \sum_{k \geq 2} a_k \mu^{(p-1)(k-1)} \bmod pA_F$. In particular, $a_1 = u - \sum_{k \geq 2} a_k \mu^{(p-1)(k-1)} \bmod pA_F$, is a unit in $A_F/p = \kappa[\![\mu]\!]$. Since a_1 is an element of O_F, it follows that $a_1 \bmod p$ is a unit in κ and therefore a_1 is a unit in O_F. Hence, from (3.3) we obtain that μ^{p-1} is the product of μ_0 with a unit in A_F. ∎

Remark 3.8. From Lemma 3.7 and its proof, it follows that A_F is a free $O_F[\![\mu_0]\!]$-module of rank $p-1$. In particular, the natural ring homomorphism $O_F[\![\mu_0]\!] \to A_F$ is faithfully flat and finite of degree $p-1$. Similarly, the natural ring homomorphism $R[\![\mu_0]\!] \to R[\![\mu]\!]$ is faithfully flat and finite of degree $p-1$.

Lemma 3.9. *Let us consider the ring A_F equipped with Frobenius endomorphism φ and its subring $O_F[\![\mu_0]\!]$ equipped with the induced Frobenius. Then we have the following:*

(1) *The ideal $\mu A_F \subset A_F$ is δ-stable in the sense of [BS22, Example 2.10].*

(2) *The ideal $\mu_0 O_F[\![\mu_0]\!] \subset O_F[\![\mu_0]\!]$ is δ-stable.*

(3) *We can write $\varphi(\mu_0) = u\mu_0\tilde{p}^{p-1}$, for some unit u in $O_F[\![\mu_0]\!]$.*

Proof. To prove (1), note that we have $\delta(\mu) = \frac{\varphi(\mu)-\mu^p}{p}$ belongs to μA_F. Then, using the product formula for the δ-structure from (2.1), it follows that for any x in A_F, we have $\delta(\mu x) = \mu^p\delta(x) + x^p\delta(\mu) + p\delta(\mu)\delta(x)$, i.e. $\delta(\mu x)$ belongs to μA_F. Next, by induction on $m \in \mathbb{N}$ and using the preceding observation, it easily follows that $\delta^m(\mu x)$ also belongs to μA_F.

To show (2), let us first note that since the action of Γ_F on A_F commutes with the Frobenius, therefore, the action of Γ_F also commutes with the δ-map, i.e. for any x in A_F and g in Γ_F, we have

$g(\delta(x)) = \delta(g(x))$. Now using (1), we have that for any y in $O_F[\![\mu_0]\!]$, the element $\mu_0 y$ belongs to μA_F. Therefore, for any $m \in \mathbb{N}$, we get that $\delta^m(\mu_0 y)$ belongs to μA_F. But $\delta^m(\mu_0 y)$ also belongs to $O_F[\![\mu_0]\!]$, so we get that $\delta^m(\mu_0 y)$ is an element of $O_F[\![\mu_0]\!] \cap \mu A_F = (\mu A_F)^{\mathbb{F}_p^\times} \xleftarrow{\sim} \mu_0 O_F[\![\mu_0]\!]$ (see Lemma 3.4). In particular, the ideal $\mu_0 O_F[\![\mu_0]\!] \subset O_F[\![\mu_0]\!]$ is δ-stable.

The proof of (3) is similar to that of [Fon94, Proposition 5.2.6]. Using Lemma 3.3 and Lemma 3.7, let us write $[p]_q = v\tilde{p}$ and $\mu_0 = w\mu^{p-1}$, for some units $v, w \in A_F$. Then, we have

$$\varphi(\mu_0) = \varphi(w\mu^{p-1}) = \varphi(w)\varphi(\mu)^{p-1} = \varphi(w)\mu^{p-1}[p]_q^{p-1} = \varphi(w)v^{p-1}\mu^{p-1}\tilde{p}^{p-1} = u\mu_0\tilde{p}^{p-1},$$

where $u = \varphi(w)w^{-1}v^{p-1}$ is a unit in A_F. Since, μ_0 and $\tilde{p}$ belong to $O_F[\![\mu_0]\!]$, it follows that $u = \varphi(\mu_0)/(\mu_0\tilde{p}^{p-1})$ is an element of $O_F[\![\mu]\!]^{\mathbb{F}_p^\times} \xleftarrow{\sim} O_F[\![\mu_0]\!]$ (see Lemma 3.4). ∎

3.2. The prism $(A_R, [p]_q)$. In this subsection, we will show that $(A_R, [p]_q)$ is an object of the site $(\mathrm{Spf}\ R)_{\Delta}$, covering the final object of the topos $\mathrm{Shv}((\mathrm{Spf}\ R)_{\Delta})$. Moreover, we will compute the first few terms of its prismatic Čech nerve and study the induced action of Γ_R on these terms.

3.2.1. An object of $(\mathrm{Spf}\ R)_{\Delta}$. We begin with the following important observation:

Lemma 3.10. *The pair $(A_R, [p]_q)$ is a prism and an object of $(\mathrm{Spf}\ R)_{\Delta}$.*

Proof. Note that the ring A_R is p-torsion free and equipped with a Frobenius endomorphism φ. So from Remark 2.2 we get that A_R is a δ-ring. Moreover, we have $[p]_q = \mu^{p-1} + p\mu^{p-2} + \cdots + p$, so A_R is $(p, [p]_q)$-adically $= (p, \mu)$-adically complete. Now, since $\varphi(\mu) = [p]_q\mu$, therefore, we can write $p = \varphi([p]_q) - (\mu^{p-1}[p]_q^{p-1} + p\mu^{p-2}[p]_q^{p-2} + \cdots + p(p-1)\mu[p]_q/2)$, in particular, p is an element of $[p]_q A_R + \varphi([p]_q)A_R$. So we obtain that $(A_R, [p]_q)$ is a prism in the sense of Definition 2.4. Next, we have a Γ_R-equivariant surjective map $A_\square^+ \twoheadrightarrow A_\square^+/[p]_q \xrightarrow{\sim} R^\square[\zeta_p, X_1^{1/p}, \ldots, X_d^{1/p}]$, where the isomorphism is obtained by sending $\mu \mapsto \zeta_p - 1$ and $[X_i^\flat] \mapsto X_i^{1/p}$, for $1 \leq i \leq d$. The surjective map above extends uniquely along the p-adically completed étale map $R^\square \to R$, i.e. we have a Γ_R-equivariant surjective map

$$A_R \twoheadrightarrow A_R/[p]_q \xrightarrow{\sim} R[\zeta_p, X_1^{1/p}, \ldots, X_d^{1/p}]. \tag{3.4}$$

Clearly, we have the structure map $R \to R[\zeta_p, X_1^{1/p}, \ldots, X_d^{1/p}]$ and $A_R/[p]_q$ is p-torsion free. Hence, $(A_R, [p]_q)$ satisfies the axioms of Definition 2.7 and it is an object of $(\mathrm{Spf}\ R)_{\Delta}$. ∎

From Lemma 3.10 we have that $(A_R, [p]_q)$ is regarded as a prism via $A_R \twoheadrightarrow A_R/[p]_q \leftarrow R$. Moreover, we have the following:

Lemma 3.11. *The object $(A_R, [p]_q)$ is a cover of the final object of the topos $\mathrm{Shv}((\mathrm{Spf}\ R)_{\Delta})$.*

Proof. Let us first note that $\widehat{R}_\infty$ is a perfectoid R-algebra (see [BMS19, Definition 4.18]) and the pair $(A_{\mathrm{inf}}(R_\infty), [p]_q)$ is a prism and an object of $(\mathrm{Spf}\ R)_{\Delta}$ since $A_{\mathrm{inf}}(R_\infty)/[p]_q \xrightarrow{\sim} \widehat{R}_\infty$. Furthermore, the natural map $A_R \to A_{\mathrm{inf}}(R_\infty)$ is compatible with the prism structure on both rings, in particular, we have a map $(A_R, [p]_q) \to (A_{\mathrm{inf}}(R_\infty), [p]_q)$ of prisms in $(\mathrm{Spf}\ R)_{\Delta}$. So, to show that $(A_R, [p]_q)$ is a cover of the final object of the topos $\mathrm{Shv}((\mathrm{Spf}\ R)_{\Delta})$, it is enough to show that $(A_{\mathrm{inf}}(R_\infty), [p]_q)$ is a cover of the final object of the topos $\mathrm{Shv}((\mathrm{Spf}\ R)_{\Delta})$. Now, note that the natural map $R \to \widehat{R}_\infty$ is faithfully flat and the p-complete Tor amplitude of the cotangent complex $L_{\widehat{R}_\infty/R}$ in $D(\widehat{R}_\infty)$ (see Appendix A.1), is in $[-1, 0]$ because R is a p-complete smooth $\mathbb{Z}_p$-algebra and $\widehat{R}_\infty$ is a perfectoid $\mathbb{Z}_p$-algebra, hence, quasisyntomic by [BMS19, Proposition 4.19]. Therefore, from [BS22, Proposition 7.11], we obtain that $(A_{\mathrm{inf}}(R_\infty), [p]_q)$ is a cover of the final object of the topos $\mathrm{Shv}((\mathrm{Spf}\ R)_{\Delta})$. This allows us to conclude. ∎

Lemma 3.12. *Let $g \in \Gamma_R$, then the action of g on A_R induces an automorphism of $(A_R, [p]_q)$ in $(\mathrm{Spf}\ R)_{\Delta}$. Moreover, we have $(g-1)A_R \subset \mu A_R$.*

Proof. As the action of Γ_R on A_R is continuous, it is enough to check the claim for the topological generators $\{\gamma_1, \ldots, \gamma_d\}$ of Γ'_R and each $g \in \Gamma_F$. Note that for $1 \leq i \leq d$, using the description of A_R from Subsection 3.1.1 and explicit computations, it easily follows that we have $(\gamma_i - 1)A_R \subset \mu A_R$ and $\gamma_i([p]_q A_R) \subset [p]_q A_R$. For any g in Γ_F, note that we have $(g-1)\mu = (1+\mu)^{\chi(g)} - 1 - \mu = \chi(g)\mu u - \mu$, for some unit u in A_R, therefore, it follows that $(\gamma - 1)A_R \subset \mu A_R$. Similarly, note that $g([p]_q) = g(\varphi(\mu)/\mu) = \varphi(\mu u)/(\mu u)$, which belongs to $[p]_q A_R$. This proves both the claims. ∎

3.2.2. Prismatic Čech nerve of $(A_R, [p]_q)$. In this subsubsection, we will provide a description of the simplicial object $(A_R(\bullet), I(\bullet))$ in $(\mathrm{Spf}\ R)_{\Delta}$. Such a description wowuld help in describing any prismatic F-crystal over $(\mathrm{Spf}\ R)_{\Delta}$ in terms of an A_R-module with stratification.

Construction 3.13. Let $A_R(\bullet)$ denote the cosimplicial ring obtained by taking the prismatic Čech nerve $(A_R(\bullet), I(\bullet))$ of $(A_R, [p]_q)$ in $(\mathrm{Spf}\ R)_{\Delta}$. Clearly, we have that $A_R(0) = A_R$. As products in $(\mathrm{Spf}\ R)_{\Delta}$ are computed by prismatic envelopes (which exist in our case), we can describe $A_R(n)$ a bit more explicitly. For $n \in \mathbb{N}$, let $(A_R)^{\otimes(n+1)}$ denote the $(n+1)$-fold tensor product of A_R over O_F and we consider the A_R-algebra structure on $(A_R)^{\otimes(n+1)}$ via the first component, i.e. $a \mapsto a \otimes 1^{\otimes n}$. Let $(A_R)^{\widehat{\otimes}(n+1)}$ denote the $(p, [p]_q)$-adic completion of $(A_R)^{\otimes(n+1)}$, which can be given as the $(p, [p]_q)$-adic completion of an étale algebra over a polynomial ring in finitely many variables defined over a power-series A_R-algebra in finitely many variables, in particular, we see that $(A_R)^{\widehat{\otimes}(n+1)}$ is $(p, [p]_q)$-completely flat over $(A_R, [p]_q)$. In other words, for each $m \in \mathbb{N}_{\geq 1}$, we have that the natural map $A_R/(p, [p]_q)^m \to (A_R)^{\widehat{\otimes}(n+1)}/(p, [p]_q \otimes 1)^m$ is flat. Since A_R is noetherian, from [Sta23, Tag 0912], it follows that the map $A_R \to (A_R)^{\widehat{\otimes}(n+1)}$ is flat. Now, for each $n \in \mathbb{N}$, we have a surjection

$$(A_R)^{\widehat{\otimes}(n+1)} \twoheadrightarrow R[\zeta_p, X_1^{1/p}, \ldots, X_d^{1/p}]^{\otimes(n+1)}, \tag{3.5}$$

where on the right hand term, the tensor product is taken over R. For $1 \leq i \leq n+1$, let $n_i : A_R \to (A_R)^{\widehat{\otimes}(n+1)}$ denote the map, sending $a \mapsto 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$, in the i^{th} position. Then the kernel of (3.5) is given by the ideal

$$J(n) = (n_i([p]_q), n_j([X_s^\flat]^p) - n_k([X_s^\flat]^p) \text{ for } 1 \leq s \leq d, 1 \leq i,j,k \leq d, j \neq k) \subset (A_R)^{\widehat{\otimes}(n+1)}.$$

Note that from the proof of Lemma 3.10, we have that $A_R/[p]_q \xrightarrow{\sim} R[\zeta_p, X_1^{1/p}, \ldots, X_d^{1/p}]$ (see (3.4)). Therefore, it easily follows that the sequence $\{p, n_i([p]_q), n_j([X_s^\flat]^p) - n_k([X_s^\flat]^p) \text{ for } 1 \leq s \leq d, 1 \leq i,j,k \leq d, j \neq k\}$ is regular on $(A_R)^{\widehat{\otimes}(n+1)}$. So, from [BS22, Proposition 3.13], we get that $(A_R(n), n_1([p]_q))$ is the prismatic envelope of $((A_R)^{\widehat{\otimes}(n+1)}, J(n))$ over the bounded prism $(A_R, [p]_q)$. By the universal property of prismatic envelopes, the maps n_i from above extend uniquely to $n_i : A_R \to A_R(n)$, for $1 \leq i \leq n+1$. Let us denote the cosimplical ring $A_R(\bullet)$ by the usual diagram

$$\begin{tikzcd}
A_R(\bullet): & A_R(0) \arrow[r, bend left, "p_2"] \arrow[r, bend right, "p_1"'] & A_R(1) \arrow[l, "\Delta"'] \arrow[r, bend left=40, "p_{23}"] \arrow[r, "p_{13}"] \arrow[r, bend right=40, "p_{12}"'] & A_R(2) \arrow[l, bend right=15] \arrow[l, bend left=15] \cdots,
\end{tikzcd}$$

where explicitly we write $p_1 = 1_1 : A_R(0) \to A_R(1)$ for the map $a \mapsto a \otimes 1$ and $p_2 = 1_2 : A_R(0) \to A_R(1)$ for the map $a \mapsto 1 \otimes a$. Similary, we define maps p_{12}, p_{13} and p_{23}. To avoid confusion, we will write $r_j = 2_j : A_R \to A_R(2)$ for the map sending a to the j^{th} position for $j = 1, 2, 3$. ∎

Lemma 3.14. *The pair $(A_R(n), n_1([p]_q))$ is a prism and an object of $(\mathrm{Spf}\ R)_{\Delta}$. Moreover, we have that $(A_R(n), n_1([p]_q))$ is the n-fold self product of $(A_R, [p]_q)$ over the final object of $\mathrm{Shv}((\mathrm{Spf}\ R)_{\Delta})$.*

Proof. From [BS22, Proposition 3.13] we have that $(A_R(n), n_1([p]_q))$ is a prism and moreover, it is $(p, [p]_q)$-completely flat over $(A_R, [p]_q)$, in particular, $(A_R(n), n_1([p]_q))$ is a bounded prism by [BS22, Lemma 3.7]. Furthermore, we have that $A_R(n)/(n_1([p]_q)) \xrightarrow{\sim} R[\zeta_p, X_1^{1/p}, \ldots, X_d^{1/p}]^{\otimes(n+1)} \leftarrow R$, i.e. $(A_R(n), n_1([p]_q))$ is an object of $(\mathrm{Spf}\, R)_{\Delta}$. Finally, from [BS22, Proposition 3.13], self products are computed via prismatic envelopes (when they exist), so from Construction 3.13, it follows that $(A_R(n), n_1([p]_q))$ is the n-fold self product of $(A_R, [p]_q)$ over the final object of $\mathrm{Shv}((\mathrm{Spf}\, R)_{\Delta})$. ∎

For later calculations it will be convenient to have an explicit presentation of $A_R(1)$ and $A_R(2)$. Recall that the ideal $J(1) = ([p]_q \otimes 1, 1 \otimes [p]_q, [X_i^\flat]^p \otimes 1 - 1 \otimes [X_i^\flat]^p \text{ for } 1 \leq i \leq d) \subset A_R \widehat{\otimes}_{O_F} A_R$, is the kernel of the map in (3.5) for $n = 1$. Consider a free δ-algebra over $A_R \widehat{\otimes}_{O_F} A_R$ in $d+1$ variables given as $(A_R \widehat{\otimes}_{O_F} A_R)\big\{\frac{1\otimes[p]_q}{[p]_q\otimes 1}, \frac{[X_1^\flat]^p\otimes 1 - 1\otimes[X_1^\flat]^p}{[p]_q\otimes 1}, \ldots, \frac{[X_d^\flat]^p\otimes 1 - 1\otimes[X_d^\flat]^p}{[p]_q\otimes 1}\big\}_\delta$. Then, from [BS22, Proposition 3.13], we have

$$A_R(1) = (A_R \widehat{\otimes}_{O_F} A_R)\big\{\tfrac{1\otimes[p]_q}{[p]_q\otimes 1}, \tfrac{[X_1^\flat]^p\otimes 1 - 1\otimes[X_1^\flat]^p}{[p]_q\otimes 1}, \ldots, \tfrac{[X_d^\flat]^p\otimes 1 - 1\otimes[X_d^\flat]^p}{[p]_q\otimes 1}\big\}^\wedge_{(p,[p]_q\otimes 1)}, \tag{3.6}$$

i.e. $A_R(1)$ is the $(p, [p]_q \otimes 1)$-adic completion of the free δ-algebra $(A_R \widehat{\otimes}_{O_F} A_R)\big\{\frac{J(1)}{[p]_q\otimes 1}\big\}_\delta$. Similarly, using the ideal $J(2) \subset (A_R)^{\widehat{\otimes}3}$, from [BS22, Proposition 3.13], we have

$$A_R(2) = (A_R)^{\widehat{\otimes}3}\big\{\tfrac{J(2)}{[p]_q\otimes 1}\big\}^\wedge_{(p,[p]_q\otimes 1)},$$

i.e. $A_R(2)$ is the $(p, [p]_q \otimes 1)$-adic completion of the free δ-algebra $(A_R)^{\widehat{\otimes}3}\big\{\frac{J(2)}{[p]_q\otimes 1}\big\}_\delta$.

Lemma 3.15. *The A_R-linear morphisms $n_i : A_R \to A_R(n)$, for $1 \leq i \leq n+1$, are faithfully flat. In particular, the morphisms p_i for $i = 1, 2$ and r_j for $j = 1, 2, 3$ are faithfully flat. Moreover, the element $\frac{n_i([p]_q)}{n_1([p]_q)}$ is a unit in $A_R(n)$, for $1 \leq i \leq n+1$. In particular, $\frac{1\otimes[p]_q}{[p]_q\otimes 1}$ is a unit in $A_R(1)$.*

Proof. Note that from Lemma 3.14 and [BS22, Proposition 3.13], the map $n_i : A_R \to A_R(n)$ is $(p, [p]_q)$-completely flat. In particular, $n_i : A_R/(p, [p]_q)^m \to A_R(n)/(p, [p]_q \otimes 1)^m$, is flat for each $m \in \mathbb{N}_{\geq 1}$. Since A_R is noetherian, from [Sta23, Tag 0912], it follows that n_i is flat. Moreover, as we have $\Delta \circ n_i = id$, therefore for an A_R-module M, we see that $A_R(n) \otimes_{n_i, A_R} M = 0$ if and only if $N = 0$. Hence, the morphisms n_i for $1 \leq i \leq n+1$ are faithfully flat. Finally, it is easy to see that $n_i([p]_q)$ is a distinguished element of $A_R(n)$ and writing $n_i([p]_q) = n_1([p]_q)\frac{n_i([p]_q)}{n_1([p]_q)}$, from [BS22, Lemma 2.24], we get that $\frac{n_i([p]_q)}{n_1([p]_q)}$ is a unit in $A_R(n)$. ∎

Remark 3.16. From now on we will denote the prism $(A_R(1), [p]_q \otimes 1)$ simply as $(A_R(1), [p]_q)$. In light of Lemma 3.15 this simplification should not cause any confusion to the reader.

3.2.3. Galois action on $(A_R, [p]_q)$.

Note that for $n \in \mathbb{N}$, the product $\Gamma_R^{\times(n+1)}$ of $n+1$ copies of Γ_R, naturally acts on the δ-ring $(A_R)^{\widehat{\otimes}n}$ and the ring homomorphism $(A_R)^{\widehat{\otimes}n} \twoheadrightarrow R[\zeta_p, X_1^{1/p}, \ldots, X_d^{1/p}]^{\otimes(n+1)}$ from (3.5) is easily checked to be $\Gamma_R^{\times(n+1)}$-equivariant. So, by the universal property of prismatic envelopes, the continuous action of $\Gamma_R^{\times(n+1)}$ on $(A_R)^{\widehat{\otimes}n}$ naturally extends to a continuous action on $A_R(n)$. Moreover, since the action of Γ_R is identity on A_R/μ, we claim the following:

Proposition 3.17. *The action of the i^{th}-component of $\Gamma_R^{\times(n+1)}$ is trivial on $A_R(n)/(n_i(\mu))$.*

Proof. From the definition of the action of $\Gamma_R^{\times(n+1)}$ on $(A_R)^{\widehat{\otimes}n}$, it is easy to see that the i^{th}-component acts trivially on $(A_R)^{\widehat{\otimes}n}/(n_i(\mu))$. Then, from Construction 3.13, we note that to get the claim, it is enough to show that for any x in the set of generators $\{n_j([p]_q), n_k([X_s^\flat]^p) - n_l([X_s^\flat]^p)\}$, for $1 \leq s \leq d$, $1 \leq j, k, l \leq d$ and $k \neq l$, of the ideal $J(n) \subset (A_R)^{\widehat{\otimes}n}$, we have that

$$(g-1)\delta^m\big(\tfrac{x}{n_1([p]_q)}\big) \in n_i(\mu)A_R(n), \tag{3.7}$$

for each $m \in \mathbb{N}$ and any g in the i^{th}-component of $\Gamma_R^{\times(n+1)}$. We can reduce this claim further as follows. Using Lemma 3.9 (1), let us first note that $n_i(\mu)A_R(n)$ is a δ-stable ideal of $A_R(n)$ in the sense of [BS22, Example 2.10]. Then, by using Lemma 3.19 for $A = A_R(n)$ and $\alpha = n_i(\mu)$, we see that to prove our main claim, it is enough to show (3.7) for $m = 0$, i.e. $(g-1)(\frac{x}{n_1([p]_q)})$ belongs to $n_i(\mu)A_R(n)$.

Let us first assume $1 < i \leq n+1$ and since the action of $\Gamma_R = i^{\text{th}}$-component of $\Gamma_R^{\times(n+1)}$, on $A_R(n)$ is continuous, it is enough to check the preceding claim for the topological generators $\{\gamma_1, \ldots, \gamma_d\}$ of Γ'_R and each $g \in \Gamma_F$. Now, take $x = n_k([X_s^\flat]^p) - n_l([X_s^\flat]^p)$ in $J(n)$. Then, the claim is obvious for $k \neq i$ and $l \neq i$. So, assume $k = i$ and $l \neq k$ and note that we have

$$(\gamma_s - 1)(\tfrac{x}{n_1([p]_q)}) = \tfrac{(1+n_i(\mu))^p n_i([X_s^\flat]^p) - n_i([X_s^\flat]^p)}{n_1([p]_q)} = \tfrac{n_i([p]_q) n_i(\mu) n_i([X_s^\flat]^p)}{n_1([p]_q)} \in n_i(\mu)A_R(n),$$

where we have used the fact that $\frac{n_i([p]_q)}{n_1([p]_q)}$ is a unit in $A_R(n)$ (see Lemma 3.15). Next, let $x = n_i([p]_q)$, then for any g in Γ_F, we have that

$$(g-1)(\tfrac{x}{n_1([p]_q)}) = \tfrac{n_i([p]_q) n_i(\mu) y}{n_1([p]_q)} \in n_i(\mu)A_R(n),$$

where the first equality follows from Lemma 3.18. Therefore, by using Lemma 3.19 for $A = A_R(n)$ and $\alpha = n_i(\mu)$, it follows that we have shown (3.7) for all generators of $J(n)$ and $1 < i \leq n+1$.

Let us now assume $i = 1$ and again note that since the action of $\Gamma_R = 1^{\text{st}}$-component of $\Gamma_R^{\times(n+1)}$, on $A_R(n)$ is continuous, therefore, it is enough to check the preceding claim for the topological generators $\{\gamma_1, \ldots, \gamma_d\}$ of Γ'_R and each $g \in \Gamma_F$. An argument similar to above, shows that for all x in $J(n)$ and $1 \leq s \leq d$, we have that $(\gamma_s - 1)(\frac{x}{n_1([p]_q)})$ belongs to $n_1(\mu)A_R(n)$. Now take any $g \in \Gamma_F$ and x an element of $J(n)$. Then we have that

$$(g-1)(\tfrac{x}{n_1([p]_q)}) = x(\tfrac{1}{g(n_1([p]_q))} - \tfrac{1}{n_1([p]_q)}) \in n_i(\mu)A_R(n),$$

using Lemma 3.18. Therefore, by using Lemma 3.19, it follows that we have shown (3.7) for all generators of $J(n)$ and $i = 1$. This allows us to conclude. ∎

The following observation was used above:

Lemma 3.18. *Let $g \in \Gamma_F$, then $(g-1)[p]_q$ belongs to $[p]_q\mu A_F$ and $(g-1)(\frac{1}{[p]_q})$ belongs to $\frac{\mu}{[p]_q}A_F$.*

Proof. Let us first observe that $g(\mu) = \chi(g)u_g\mu$, where χ is the p-adic cyclotomic character, and $u_g = 1 + x_g\mu$, for some $x_g \in A_F$, is a unit in A_F. Then we have

$$(g-1)[p]_q = (g-1)\tfrac{\varphi(\mu)}{\mu} = \tfrac{\varphi(g(\mu))}{g(\mu)} - \tfrac{\varphi(\mu)}{\mu} = (\tfrac{\varphi(u_g)}{u_g} - 1)[p]_q \in [p]_q\mu A_F.$$

Next, observe that

$$(g-1)(\tfrac{1}{[p]_q}) = (g-1)\tfrac{\mu}{\varphi(\mu)} = \tfrac{g(\mu)}{\varphi(g(\mu))} - \tfrac{\mu}{\varphi(\mu)} = (\tfrac{u_g}{\varphi(u_g)} - 1)\tfrac{1}{[p]_q} \in \tfrac{\mu}{[p]_q}A_F.$$

This proves the lemma. ∎

We prove a general statement for δ-rings admitting an action of Γ_R, which also be useful in Subsection 3.4.3.

Lemma 3.19. *Let A be a p-adically complete p-torsion free δ-ring such that A admits a continuous and φ-equivariant action of Γ_R and let α be an element of A such that the ideal $(\alpha) \subset A$ is δ-stable in the sense of [BS22, Example 2.10]. Assume that for some $x \in A$ and g an element of $\Gamma_R^{\times(n+1)}$, we are given that $(g-1)x$ is in (α). Then for each $m \in \mathbb{N}$, we have that $(g-1)\delta^m(x)$ belongs to (α) as well.*

Proof. Let us fix some $x \in A$ and g an element of $\Gamma_R^{\times(n+1)}$ such that $(g-1)x$ lies in (α). We will proceed by induction on $m \in \mathbb{N}$. So for some $m \in \mathbb{N}$, let us write $(g-1)\delta^m x = \alpha y$. Then, observe that

$$\begin{aligned}(g-1)(\delta^{m+1}x) &= (g-1)\tfrac{\varphi(\delta^m x)-(\delta^m x)^p}{p}\\ &= \tfrac{1}{p}(\varphi((g-1)\delta^m x) - (g-1)(\delta^m x)^p)\\ &= \tfrac{1}{p}(\varphi(\alpha y) - ((g-1)\delta^m x + \delta^m x)^p + (\delta^m x)^p)\\ &= \tfrac{1}{p}(\varphi(\alpha y) - \textstyle\sum_{k=1}^p \binom{p}{k}(\alpha y)^k(\delta^m x)^{p-k})\\ &= \delta(\alpha y) - \tfrac{1}{p}\textstyle\sum_{k=1}^{p-1}\binom{p}{k}(\alpha y)^k(\delta^m x)^{p-k}.\end{aligned} \tag{3.8}$$

Using the product formula for the delta map from (2.1), it follows that $\delta(\alpha y)$ belongs to (α). Hence, $(g-1)(\delta^{m+1}x)$ belongs to (α), thus proving the claim. ∎

Remark 3.20. For $n \in \mathbb{N}$, the δ-ring $(A_R)^{\widehat{\otimes} n}$ is equipped with a Frobenius endomorphism given as tensor product of Frobenius on each component. By compatibility of δ-structures between $(A_R)^{\widehat{\otimes} n}$ and $A_R(n)$, we deduce that the Frobenius endomorphism on $A_R(n)$ is compatible with the one on $(A_R)^{\widehat{\otimes} n}$. Then, by the description of the Galois action on $A_R(1)$, it easily follows that the natural map $p_i : A_R \to A_R(1)$ is (φ, Γ_R^2)-equivariant for $i = 1, 2$, where Γ_R^2 acts on the source via projection on to the i^{th} coordinate. Similarly, the natural map $r_j : A_R \to A_R(2)$ is (φ, Γ_R^3)-equivariant for $j = 1, 2, 3$, where Γ_R^3 acts on the source via projection on to the j^{th} coordinate.

3.3. The ring $A_R(1)/p_1(\mu)$.

From Remark 3.20 recall that we have a (φ, Γ_R^2)-equivariant maps $p_i : A_R \to A_R(1)$ for $i = 1, 2$, where A_R is equipped with a Γ_R^2-action via projection on to the i^{th} coordinate. Moreover, we note that there is an induced action of Γ_R^2 on $A_R(1)/p_1(\mu)$, where the action of the first component is identity. Additionally, from Subsection 3.1 recall that we have fixed topological generators $\{\gamma_1, \ldots, \gamma_d\}$ of Γ_R'. The goal of this subsection is to give an explicit description of the ring $A_R(1)/p_1(\mu)$ and explain some of its properties. We start with some computations.

3.3.1. Some divided power calculations.

The goal of this subsubsection is to show the following claim:

Proposition 3.21. *The following natural map is a φ-equivariant isomorphism of p-torsion free rings*

$$\beta : \Lambda_R := R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^{\wedge} \xrightarrow{\sim} R[\![\mu]\!][(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^{\wedge}. \tag{3.9}$$

Notation. For any element x, we use $x^{[k]}$ to denote $x^k/(k!)$, for all $k \in \mathbb{N}$.

Proof. Let B denote the divided power envelope of the $R[\mu]$-subalgebra $R[\mu, \mu^{p-1}/p] \subset R[\mu][1/p]$, with respect to the ideal $(\mu^{p-1}/p) \subset R[\mu, \mu^{p-1}/p]$, i.e. $B = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]$. Similarly, let B' denote the divided power envelope of the $R[\![\mu]\!]$-subalgebra $R[\![\mu]\!][\mu^{p-1}/p] \subset R[\![\mu]\!][1/p]$, with respect to the ideal $(\mu^{p-1}/p) \subset R[\![\mu]\!][\mu^{p-1}/p]$, i.e. $B' = R[\![\mu]\!][(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]$. Note that the source and target of (3.9) are the p-adic completions of B and B', respectively. Moreover, we have a natural morphism of rings $B \to B'$ induced by the inclusion $R[\mu][1/p] \subset R[\![\mu]\!][1/p]$. Evidently, the rings B and B' are p-torsion free, the natural map $B \to B'$ is injective and to show that (3.9) is an isomorphism, it is enough to show that the natural map $B \to B'$ induces an isomorphism of rings $B/p \xrightarrow{\sim} B'/p$. Since, $\mu^{p-1} = p(\mu^{p-1}/p)$ in B and B', therefore $B/p = B/(\mu^{p-1}, p) = (B/\mu^{p-1})/p$ and $B'/p = B'/(\mu^{p-1}, p) = (B'/\mu^{p-1})/p$. Now, from Lemma 3.22, using the explicit description of B and B', we have a natural commutative diagram

$$\begin{array}{ccc} ((R[\mu]/\mu^{p-1})[Y_0, Y_1, \ldots]/(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k\in\mathbb{N}})/p & \xrightarrow{\sim} & B/p \\ \downarrow & & \downarrow \\ ((R[\![\mu]\!]/\mu^{p-1})[Y_0, Y_1, \ldots]/(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k\in\mathbb{N}})/p & \xrightarrow{\sim} & B'/p, \end{array}$$

where the vertical arrows are natural maps, the top horizontal arrow is reduction modulo p of the isomorphism in (3.10) and the bottom horizontal arrow is reduction modulo p of the isomorphism in (3.11). Since $R[\mu]/\mu^{p-1} \xrightarrow{\sim} R[\![\mu]\!]/\mu^{p-1}$, it follows that the left vertical map is an isomorphism as well. So we obtain that $B/p \xrightarrow{\sim} B'/p$ and hence (3.9) is an isomorphism of p-torsion free rings. Finally, note that by definition the map β in (3.9) is φ-equivariant. This concludes our proof. ∎

The following descriptions of B and B' were used in the proof of Proposition 3.21:

Lemma 3.22. *The following natural map is an isomorphism of rings*

$$R[\mu][Y_0, Y_1, \ldots]/(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k\in\mathbb{N}} \xrightarrow{\sim} B, \tag{3.10}$$

where $Y_k \mapsto (\mu^{p-1}/p)^{[p^k]}$. Similarly, the following natural map is an isomorphism of rings

$$R[\![\mu]\!][Y_0, Y_1, \ldots]/(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k\in\mathbb{N}} \xrightarrow{\sim} B', \tag{3.11}$$

where $Y_k \mapsto (\mu^{p-1}/p)^{[p^k]}$.

Proof. We will only show that the map in (3.10) is bijective, the claim for (3.11) follows by a similar argument. Define an $R[\mu]$-linear map

$$R[\mu][Y_0, Y_1, \ldots] \longrightarrow B, \quad Y_k \longmapsto (\tfrac{\mu^{p-1}}{p})^{[p^k]}. \tag{3.12}$$

Let us first note that this map is surjective. Indeed, let $n \in \mathbb{N}$ and write $n = \sum_{i=0}^r n_i p^i$ in base p, with $0 \leq n_i \leq p-1$. Then we have,

$$(\tfrac{\mu^{p-1}}{p})^{[n]} = \tfrac{1}{n!}(\tfrac{\mu^{p-1}}{p})^{\sum_{i=0}^r n_i p^i} = \tfrac{1}{n!}\prod_{i=0}^r (p^i!)^{n_i} \prod_{i=0}^r (\tfrac{\mu^{p-1}}{p})^{[p^i]}.$$

An easy computation shows that the p-adic valuation of $\prod_{i=0}^r (p^i!)^{n_i}$ equals the p-adic valuation of $n!$. So, it follows that (3.12) is surjective and it is clear that the kernel of (3.12) is given by the ideal $(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k\in\mathbb{N}} \subset R[\mu][Y_0, Y_1, \ldots]$. This proves the claim. ∎

Remark 3.23. Let $\Lambda_F := O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge \xrightarrow{\sim} O_F[\![\mu]\!][(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ and from Proposition 3.21 note that it is a p-torsion free algebra over $A_F = O_F[\![\mu]\!]$. We equip Λ_F with a continuous (φ, Γ_F)-action by extending the (φ, Γ_F)-action on A_F and using the formulas $\varphi(\mu^{p-1}) = [p]_q^{p-1}\mu^{p-1}$ and $g(\mu^{p-1}) = \chi(g)^{p-1}\mu^{p-1}u$, for some unit u in $(A_F)^\times$. Now recall that from Subsection 3.1.1 we have a (φ, Γ_F)-equivariant isomorphism of rings $\iota : R[\![\mu]\!] \xrightarrow{\sim} A_R$. This extends to a (φ, Γ_F)-equivariant isomorphism

$$\iota : R[\![\mu]\!]\widehat{\otimes}_{O_F[\![\mu]\!]}\Lambda_F \xrightarrow{\sim} A_R\widehat{\otimes}_{A_F}\Lambda_F, \tag{3.13}$$

where we take tensor product action of φ and Γ_F and completion is with respect to the p-adic topology. Furthermore, using the completed tensor product description, we get that the isomorphism (3.9) in Proposition 3.21 is a (φ, Γ_F)-equivariant isomorphism of rings

$$\beta : \Lambda_R := R\widehat{\otimes}_{O_F}\Lambda_F \xrightarrow{\sim} R[\![\mu]\!]\widehat{\otimes}_{O_F[\![\mu]\!]}\Lambda_F. \tag{3.14}$$

For any $k \in \mathbb{N}$, let $k = (p-1)f(k) + r(k)$, with $r(k)$, $f(k)$ in $\mathbb{N}$ and $0 \leq r(k) < p-1$. Set $\mu^{\{k\}} = \frac{\mu^k}{f(k)!p^{f(k)}}$, and note that we have $\Lambda_F = O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = O_F[\mu, \mu^{\{k\}}, k \in \mathbb{N}]_p^\wedge$. In particular, for any x in Λ_F we have a unique presentation $x = \sum_{k\in\mathbb{N}} a_k\mu^{\{k\}}$ in Λ_F with a_k in O_F for all $k \in \mathbb{N}$ and p-adically $a_k \to 0$ as $k \to +\infty$. We note an important observation which will be used throughout.

Lemma 3.24. *The element $t = \log(1+\mu) = \sum_{k\in\mathbb{N}}(-1)^k\frac{\mu^{k+1}}{k+1}$ converges in $\mu\Lambda_F$. Moreover, the elements t/μ, $[p]_q/p$ and $\tilde{p}/p$ are units in Λ_F.*

Proof. Formally, write $t/\mu = \sum_{k\in\mathbb{N}}(-1)^k\frac{\mu^k}{k+1} = \sum_{k\in\mathbb{N}} a_k\mu^k$ and $\mu/t = \frac{\mu}{\log(1+\mu)} = \sum_{k\in\mathbb{N}} b_k\mu^k$, with $\upsilon_p(a_k), \upsilon_p(b_k) \geq -\frac{k}{p-1}$, for all $k \in \mathbb{N}$. From the discussion before the lemma, recall that $\mu^{\{k\}} = \frac{\mu^k}{f(k)!p^{f(k)}}$. Then we have

$$\upsilon_p\big(\tfrac{a_k}{f(k)!p^{f(k)}}\big) \geq -\tfrac{k}{p-1} + \upsilon_p(f(k)!) + f(k) > \upsilon_p(\lfloor\tfrac{k}{p-1}\rfloor!) - 1,$$

which goes to $+\infty$ as $k \to +\infty$. Similar claim holds with a_k replaced by b_k above. Hence, we conclude that t/μ and μ/t converge in Λ_F as inverse to each other. Moreover, we have $[p]_q = \varphi(\mu)/\mu = p\varphi(\mu/t)t/\mu$ in Λ_F, so $[p]_q/p$ is a unit in Λ_F. Furthermore, using Lemma 3.3, we get that $\tilde{p}/p$ is a unit in Λ_F. ∎

3.3.2. Explicit description of $A_R(1)/p_1(\mu)$. In the subsubsection, the A_R-algebra structure on $A_R(1)/p_1(\mu)$ is given by the composition $A_R \xrightarrow{p_2} A_R(1) \to A_R(1)/p_1(\mu)$ and the Γ_R-action on $A_R(1)/p_1(\mu)$ means the continuous (for the p-adic topology) action of $1 \times \Gamma_R$ on $A_R(1)/p_1(\mu)$. Let us consider the divided power A_R-algebra $A_R\big[\prod_{\mathbf{k}\in\mathbb{N}^{d+1}}(\mu^{p-1}/p)^{[k_0]}T_1^{[k_1]}\cdots T_d^{[k_d]}\big]$, where $T_1,\ldots T_d$ are variables. We equip this ring with a continuous (for the p-adic topology) action of Γ_R by extending the Γ_R-action on A_R by setting $\gamma_j(T_i) = \mu[X_i^\flat] + T_i$, for $j = i$ or T_i otherwise. Moreover, we have $\varphi(\mu^{p-1}) = [p]_q^{p-1}\mu^{p-1}$, so we can further equip this ring with a Frobenius endomorphism φ, extending the Frobenius on A_R by setting $\varphi([X_i^\flat] - T_i) = ([X_i^\flat] - T_i)^p$. Then we have the following description of $A_R(1)/p_1(\mu)$:

Proposition 3.25. *There exists a natural (φ, Γ_R)-equivariant isomorphism of p-adically complete and p-torsion free A_R-algebras*

$$\alpha : A_R(1)/p_1(\mu) \xrightarrow{\sim} A_R\big[\textstyle\prod_{\mathbf{k}\in\mathbb{N}^{d+1}}(\mu^{p-1}/p)^{[k_0]}T_1^{[k_1]}\cdots T_d^{[k_d]}\big]_p^\wedge, \tag{3.15}$$

where $p_2(\mu) \mapsto \mu$, $p_2([X_i^\flat]) \mapsto [X_i^\flat]$ and $p_1([X_i^\flat]) \mapsto [X_i^\flat] - T_i$.

Proof. The map in (3.15) is evidently (φ, Γ_R)-equivariant and we need to show that it is bijective. To show the bijectivity of (3.15), we will use the results of Appendix B. Let $A := A_F\widehat{\otimes}_{O_F}A_F$ denote the $(p, p_1(\mu))$-adic completion of $A_F \otimes_{O_F} A_F$ equipped with the tensor product Frobenius, in particular, A is a p-torsion free δ-ring. Moreover, note that $\varphi^n(p_1(\mu))$ is a nonzerodivisor on A, for each $n \in \mathbb{N}$, and $A/p_1(\mu)A \xrightarrow{\sim} A_F$ is p-torsion free. Therefore, by setting q to be $p_1(\mu) + 1$, we see that the ring A satisfies Assumption B.1. Next, let $B := A_R\widehat{\otimes}_{O_F}A_R$ denote the $(p, p_1(\mu))$-adic completion of $A_R \otimes_{O_F} A_R$ equipped with the tensor product Frobenius, in particular, B is a p-torsion free δ-algebra over A. Note that $\varphi^n(p_1(\mu))$ is a nonzerodivisor on B, for each $n \in \mathbb{N}$, and $B/p_1(\mu)B \xrightarrow{\sim} R\widehat{\otimes}_{O_F}A_R$, as the p-adic completion of $R \otimes_{O_F} A_R$, is p-torsion free. Set $Y_0 := p_2([p]_q) - p_1([p]_q)$ and $Y_i := p_2([X_i^\flat]^p) - p_1([X_i^\flat]^p)$, for each $1 \leq i \leq d$, as elements in B. Then, it is clear that the sequence $\{Y_1, \ldots, Y_d\}$ is regular on B and B/pB. Moreover, from the discussion in Construction 3.13, recall that the sequence $\{p, p_1([p]_q), p_2([p]_q), Y_1, \ldots, Y_d\}$ is regular on B and since $[p]_q = \mu^{p-1} \bmod p$, therefore, it follows that the sequence $\{p, p_1(\mu), p_2([p]_q), Y_1, \ldots, Y_d\}$ is also regular on B. In particular, we see that $\{Y_0, \ldots, Y_d\}$ is regular on $B/p_1([p]_q)B$ and on $B/(p, p_1(\mu))B$. Therefore, we obtain that the A-algebra B satisfies Assumption B.4.

Now, let I denote the set of natural numbers $\{0, \ldots, d\}$ and for each i in I, let $y_i := Y_i/p_1([p]_q)$ be an element of $B[1/p_1([p]_q)]$. Set E_0 to be the B-subalgebra of $B[1/p_1([p]_q)]$ generated by $\{y_i, i \in I\}$ (see the discussion before Lemma B.6) and E to be the δ-subalgebra of $E_0[1/p, 1/\varphi^n(p_1([p]_q)), n \in \mathbb{N}]$ generated over B by $\{y_i, i \in I\}$ (see the discussion before (B.4)). From the discussion before (3.6), note that the ring $A_R(1)$ is given as the $(p, p_1([p]_q)) = (p, p_1(\mu))$-adic completion of E, therefore, the ring $A_R(1)/p_1(\mu)$ is given as the p-adic completion of $E/p_1(\mu)E$. Next, let $\overline{A} := A/\mu A$ and $\overline{B} := B/\mu B$ equipped with the induced Frobenius, in particular, an associated δ-structure, since $\overline{A}$ and $\overline{B}$ are p-torsion free (see Remark 2.2). Let $\overline{E}_0 := E_0/\mu E_0$ and for each i in I, denote by $\overline{Y}_i$ (resp. $\overline{y}_i$), the image of Y_i (resp. y_i) in $\overline{E}_0$. Define $\overline{E}$ to be the δ-subalgebra of $\overline{E}_0[1/p]$ generated by $\{\overline{y}_i, i \in I\}$

over $\overline{B}$ (see the discussion before (B.4)). Then, from (B.4), we have a natural surjective map $E \to \overline{E}$ inducing an isomorphism of p-torsion free δ-rings $E/\mu E \xrightarrow{\sim} \overline{E}$, by Proposition B.10.

Now, for each $1 \leq i \leq d$, set $\tau_i := p_2([X_i^\flat]) - p_1([X_i^\flat])$ as an element of $\overline{E}_0$. Moreover, note that $p_2(\mu)^{p-1} = \overline{Y}_0 + pa$, for some a in $p_2(\mu)\overline{E}_0$, so we set $\tau_0 := p_2(\mu)^{p-1}/p = \overline{y}_0 + a$ as an element of $\overline{E}_0$. Let us set $\overline{E}' := \overline{E}_0[\tau_i^{[n]}, i \in I, n \in \mathbb{N}]$ as a divided power $\overline{E}_0$-subalgebra of $\overline{E}_0[1/p]$. We claim that $\overline{E} = \overline{E}'$ in $\overline{E}_0[1/p]$. Indeed, to show the inclusion $\overline{E}' \subset \overline{E}$, note that for each $1 \leq i \leq d$, we have $\tau_i^p = \overline{Y}_i - p\delta(\tau_i)$, where $\delta(\tau_i)$ is in $\overline{E}_0$ by Lemma 3.26, in particular, $\tau_i^p/p = \overline{y}_i - \delta(\tau_i)$ is in $\overline{E}_0 \subset \overline{E}$. Moreover, using the equalities above we have that $\tau_0^p = \overline{y}_0^p + pb$, for some b in $\overline{E}_0$ since $p_2(\mu)^p$ is in $p\overline{E}_0$, and by using Lemma 3.26 we have that $\overline{y}_0^p = \varphi(\overline{y}_0) - p\delta(\overline{y}_0)$ is in $p\overline{E}$. Therefore, we get that $\tau_0^p/p = b + \overline{y}_0^p/p$ is in $\overline{E}$. Hence, by using [BS22, Lemma 2.35], it follows that $\overline{E}' \subset \overline{E}$. Next, to show the inclusion $\overline{E} \subset \overline{E}'$, since we have that $\overline{y}_i$ is in $\overline{E}'$, for each $i \in I$, therefore, it is enough to show that φ on $\overline{E}[1/p]$ preserves $\overline{E}'$ and the resulting endomorphism of $\overline{E}'$ gives a δ-structure, or equivalently, that φ restricts to a lift of Frobenius on $\overline{E}'$. Note that $\varphi(\tau_0^{[n]}) = p_2([p]_q)^n \tau_0^{[n]}$ is in $p\overline{E}'$, and for each $i \in I$, we can write

$$\begin{aligned}\varphi(\tau_i^{[n]}) = \varphi(\tfrac{\tau_i^n}{n!}) = \tfrac{(\tau_i^p + p\delta(\tau_i))^n}{n!} &= \textstyle\sum_{k=0}^n \tfrac{1}{n!}\binom{n}{k}\tau_i^{pk}p^{n-k}\delta(\tau_i)^{n-k}\\ &= \textstyle\sum_{k=0}^n \tfrac{(pk)!p^{n-k}}{n!}\binom{n}{k}\tau_i^{[pk]}\delta(\tau_i)^{n-k}.\end{aligned}$$

An easy computation shows that all coefficients in the last term of the equation above lie in $p\mathbb{Z}_p$. Therefore, it follows that φ preserves $\overline{E}'$ and $\varphi(\tau_i^{[n]}) = 0 \bmod p\overline{E}'$, for each $i \in I$ and $n \in \mathbb{N}_{\geq 1}$. Moreover, note that $(\tau_i^{[n]})^p$ is also in $p\overline{E}'$, for each $n \in \mathbb{N}_{\geq 1}$. So, it follows that the endomorphism φ on $\overline{E}'$ reduces modulo p to the absolute Frobenius on all generators, hence on all elements, of $\overline{E}'$. Therefore, we obtain that $\overline{E}' = \overline{E} \subset \overline{E}_0[1/p]$.

Finally, let us consider $\overline{E}$ as an A_R-algebra via the map p_2 and define a ring homomorphism between A_R-algebras as follows:

$$\overline{E} = \overline{E}' \longrightarrow A_R\big[\textstyle\prod_{\mathbf{k}\in\mathbb{N}^{d+1}}(\mu^{p-1}/p)^{[k_0]}T_1^{[k_1]}\cdots T_d^{[k_d]}\big],$$

by sending $p_2(\mu) \mapsto \mu$, $\tau_0^n/n! \mapsto (\mu^{p-1}/p)^n/n!$, $p_2([X_i^\flat]) \mapsto [X_i^\flat]$, $p_1([X_i^\flat]) \mapsto [X_i^\flat] - T_i$ and $\tau_i^n/n! \mapsto T_i^n/n!$. Evidently, the map above is bijective and passing to its p-adic completion gives the isomorphism in (3.15). This finishes our proof. ∎

The following observation was used above:

Lemma 3.26. *For each $i \in I$, the element $\varphi(\overline{y}_i)$ is in $p\overline{E}_0$, and for each $1 \leq i \leq d$, the element $\delta(\tau_i)$ is in $\overline{E}_0$.*

Proof. For $1 \leq i \leq d$, we have that

$$\begin{aligned}\varphi(\overline{y}_i) &= \tfrac{1}{p}(\varphi(p_2([X_i^\flat])^p - p_1([X_i^\flat])^p))\\ &= \tfrac{1}{p}((p_2([X_i^\flat])^p - p_1([X_i^\flat])^p + p_1([X_i^\flat])^p)^p - p_1([X_i^\flat])^{p^2})\\ &= \tfrac{1}{p}((p_2([X_i^\flat])^p - p_1([X_i^\flat])^p)^p) + \textstyle\sum_{k=1}^{p-1}\tfrac{1}{p}\binom{p}{k}(p_2([X_i^\flat])^p - p_1([X_i^\flat])^p)^k p_1([X_i^\flat])^{p-k}\\ &= p^{p-1}\overline{y}_i^p + \textstyle\sum_{k=1}^{p-1}p^{k-1}\binom{p}{k}\overline{y}_i^k p_1([X_i^\flat])^{p-k},\end{aligned}$$

is an element of $p\overline{E}_0$. Moreover, for $i = 0$, we have that

$$\begin{aligned}\varphi(\overline{y}_0) &= \tfrac{p_2(q)^{p^2}-1}{p(p_2(q)^p-1)} - 1\\ &= \tfrac{(p_2(q)^p-1+1)^p-1}{p(p_2(q)^p-1)} - 1\\ &= \tfrac{(p_2(q)^p-1)^{p-1}}{p} + \textstyle\sum_{k=2}^{p-1}\tfrac{1}{p}\binom{p}{k}(p_2(q)^p-1)^{k-1}\\ &= p^{p-2}(p_2(q)-1)^{p-1}(\overline{y}_0+1)^{p-1} + \textstyle\sum_{k=2}^{p-1}p^{k-1}\binom{p}{k}(p_2(q)-1)^{k-1}(\overline{y}_0+1)^{k-1}.\end{aligned}$$

For $p \geq 3$, it is clear that $\varphi(\overline{y}_0)$ is in $p\overline{E}_0$, whereas for $p = 2$, note that $p_2(q) - 1 = p_2(q) + 1 - 2 = p_2([p]_q) - 2 = 2\overline{y}_0$, i.e. $\varphi(\overline{y}_0)$ is in $p\overline{E}_0$. This shows the first claim. Next, for $1 \leq i \leq d$, note that

$$\begin{aligned}\delta(\tau_i) &= \delta(p_2([X_i^\flat]) - p_1([X_i^\flat]))\\ &= \tfrac{1}{p}(p_2([X_i^\flat])^p - p_1([X_i^\flat])^p - (p_2([X_i^\flat]) - p_1([X_i^\flat]))^p)\\ &= \tfrac{-1}{p}(p_1([X_i^\flat])^p + (-p_1([X_i^\flat]))^p) - \textstyle\sum_{k=1}^{p-1} \tfrac{1}{p}\binom{p}{k} p_2([X_i^\flat])^k(-p_1([X_i^\flat]))^{p-k},\end{aligned}$$

is evidently in $\overline{E}_0$. This proves the second claim. ∎

Remark 3.27. In this remark we will use $\widehat{\otimes}$ to denote the p-adic completion of the usual tensor product. Let $\Lambda_F = O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ and note that it is a p-torsion free algebra over $A_F = O_F[\![\mu]\!]$ (see Proposition 3.21 for $R = O_F$). Note that the target ring in the isomorphism (3.15) of Proposition 3.25 is p-torsion free and by definition the ring $A_R\big[\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]$ is p-torsion free as well. Therefore, by checking modulo p, it is easy to see that we have a natural (φ, Γ_R)-equivariant isomorphism of rings,

$$A_R\big[\textstyle\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]\widehat{\otimes}_{A_F}\Lambda_F \xrightarrow{\sim} A_R\big[\prod_{\mathbf{k}\in\mathbb{N}^{d+1}} (\mu^{p-1}/p)^{[k_0]} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]_p^\wedge, \tag{3.16}$$

where we take the diagonal action of φ and Γ_R on the source. In particular, using (3.15) and (3.16) it follows that $A_R(1)/p_1(\mu)$ is the p-adic completion of a PD-polynomial algebra over $A_R\widehat{\otimes}_{A_F}\Lambda_F$.

Next, by an argument similar to (3.16), we note that the isomorphism in (3.13) readily extends to a (φ, Γ_F)-equivariant isomorphism

$$\iota : R[\![\mu]\!]\big[\textstyle\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]\widehat{\otimes}_{O_F[\![\mu]\!]}\Lambda_F \xrightarrow{\sim} A_R\big[\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]\widehat{\otimes}_{A_F}\Lambda_F.$$

Furthermore, again arguing similar to (3.16), we note that the isomorphism in (3.14) extends to a (φ, Γ_F)-equivariant isomorphism

$$\beta : R\big[\textstyle\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]\widehat{\otimes}_{O_F}\Lambda_F \xrightarrow{\sim} R[\![\mu]\!]\big[\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]\widehat{\otimes}_{O_F[\![\mu]\!]}\Lambda_F.$$

Composing these with the isomorphism in Proposition 3.25, we obtain a (φ, Γ_F)-equivariant isomorphism

$$\beta^{-1}\circ\iota^{-1}\circ\alpha : A_R(1)/p_1(\mu) \xrightarrow{\sim} R\big[\textstyle\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]\widehat{\otimes}_{O_F}\Lambda_F =: C_R. \tag{3.17}$$

Let $P_R := R\big[\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]_p^\wedge$ and note that the target ring of the isomorphism (3.17) can also be written as $C_R = P_R\widehat{\otimes}_R\Lambda_R = P_R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = \Lambda_R\big[\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]_p^\wedge$. In particular, C_R is the p-adic completion of a PD-polynomial algebra over Λ_R. Now for $k \in \mathbb{N}$, let $k = (p-1)f(k) + r(k)$, with $r(k)$, $f(k)$ in $\mathbb{N}$ and $0 \leq r(k) < p-1$. Set $\mu^{\{k\}} = \frac{\mu^k}{f(k)!p^{f(k)}}$, then we have $C_R = P_R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = P_R[\mu, \mu^{\{k\}}, k \in \mathbb{N}]_p^\wedge$. In particular, from the isomorphism (3.17), it follows that for any x in $A_R(1)/p_1(\mu)$ we have a unique presentation $(\beta^{-1}\circ\iota^{-1}\circ\alpha)(x) = \sum_{k\in\mathbb{N}} a_k\mu^{\{k\}}$ in C_R, with a_k in P_R for all $k \in \mathbb{N}$ and since $A_R(1)/p_1(\mu) \xrightarrow{\sim} C_R$ are p-adically separated, we have that $a_k \to 0$ as $k \to +\infty$.

Remark 3.28. The target ring of the isomorphism α in Proposition 3.25 admits a natural Γ_R-equivariant embedding into the crystalline period ring $\mathcal{O}A_{\mathrm{cris}}(R_\infty)$. Indeed, first of all note that the embedding $A_R \to A_{\mathrm{inf}}(R_\infty)$ from Subsection 3.1 extends to a natural (φ, Γ_R)-equivariant embedding $A_R\widehat{\otimes}_{A_F}\Lambda_F \to A_{\mathrm{inf}}(R_\infty)\widehat{\otimes}_{A_F}\Lambda_F \xrightarrow{\sim} A_{\mathrm{cris}}(R_\infty)$, where the latter isomorphism follows in a manner similar to the proof of [Bri08, Proposition 6.2.14]. Then, from the isomorphism (3.15) in Proposition 3.25 and from the description of $\mathcal{O}A_{\mathrm{cris}}(R_\infty)$ as the p-adic completion of a PD-polynomial algebra over $A_{\mathrm{cris}}(R_\infty)$ (see [Bri08, Chapitre 6] and [Abh21, Subsection 2.2]), i.e. $\mathcal{O}A_{\mathrm{cris}}(R_\infty) \xrightarrow{\sim} A_{\mathrm{cris}}(R_\infty)\big[\prod_{\mathbf{k}\in\mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}\big]_p^\wedge$ it follows that we have a (φ, Γ_R)-equivariant injective map $A_R(1)/p_1(\mu) \to \mathcal{O}A_{\mathrm{cris}}(R_\infty)$.

Remark 3.29. Recall that there are are two A_R-algebra structures on $A_R(1)$ written as $p_i : A_R \to A_R(1)$ for $i = 1, 2$ (see Construction 3.13). Composing it with the injective map $\iota : R \to A_R$ (see Subsection 3.1.1), gives morphism of rings $R \xrightarrow{\iota} A_R \xrightarrow{p_i} A_R(1)$. Reducing p_1 modulo μ induces an R-algebra structure on $A_R(1)/p_1(\mu)$ via the composition $R \xrightarrow{\sim} A_R/\mu \xrightarrow{p_1} A_R(1)/p_1(\mu)$, which we again denote by p_1 and where the first isomorphism is induced by the map ι. In particular, we have $(p_1 \circ \alpha)(X_i) = [X_i^\flat] - T_i$, for all $1 \leq i \leq d$. Moreover, we will denote the composition $R \xrightarrow{\iota} A_R \xrightarrow{p_2} A_R(1)$ just by p_2. Note that both p_1 and p_2 described above are φ-equivariant.

Lemma 3.30. *The morphisms $p_i : R \to A_R(1)/p_1(\mu)$ for $i = 1, 2$ are faithfully flat.*

Proof. From Lemma 3.15 the map $p_1 : A_R \to A_R(1)$ is faithfully flat, therefore the map $p_1 : R \xrightarrow{\sim} A_R/\mu \to A_R(1)/p_1(\mu)$ is faithfully flat. Now recall that we have $A_\square^+ \xrightarrow{\sim} R^\square[\![\mu]\!]$ (see Subsection 3.1.1) and consider the following isomorphism of rings

$$f : A_\square^+\big[\textstyle\prod_{\mathbf{k}\in\mathbb{N}^{d+1}}(\mu^{p-1}/p)^{[k_0]}T_1^{[k_1]}\cdots T_d^{[k_d]}\big]_p^\wedge \xrightarrow{\sim} A_\square^+\big[\textstyle\prod_{\mathbf{k}\in\mathbb{N}^{d+1}}(\mu^{p-1}/p)^{[k_0]}T_1^{[k_1]}\cdots T_d^{[k_d]}\big]_p^\wedge,$$

where $[X_i]^\flat \mapsto [X_i^\flat] + T_i$, $T_i \mapsto T_i$ and $\mu \mapsto \mu$. Extending this isomorphism along the p-adically completed étale map $R^\square \to R$ (see Subsection 3.1.1) gives an automorphism f of the p-adically complete PD-algebra $A_R\big[\prod_{\mathbf{k}\in\mathbb{N}^{d+1}}(\mu^{p-1}/p)^{[k_0]}T_1^{[k_1]}\cdots T_d^{[k_d]}\big]_p^\wedge$ and we consider the commutative diagram:

$$\begin{tikzcd} R \arrow[r, "p_1"] \arrow[dr, "p_2"'] & A_R(1)/p_1(\mu) \arrow[d, "\alpha^{-1}\circ f\circ\alpha", "\wr"'] \\ & A_R(1)/p_1(\mu). \end{tikzcd} \tag{3.18}$$

Since p_1 is faithfully flat, from the diagram (3.18), it follows that p_2 is faithfully flat as well. ∎

3.3.3. q-connection on $A_R(1)/p_1(\mu)$.

In this subsubection, we will interpret the action of the geometric part of Γ_R, i.e. Γ_R' as a q-connection on $A_R(1)/p_1(\mu)$. We start by recalling the definition of q-connection from [MT20, Definition 2.1] and [Abh23b, Subsection 5.1].

Let D be commutative ring and consider a D-algebra A equipped with d commuting D-algebra automorphisms $\gamma_1 \ldots, \gamma_d$, i.e. an action of $\mathbb{Z}^d$. Moreover, fix an element q in D such that $q - 1$ is a nonzerodivisor of D and $\gamma_i = 1 \bmod (q-1)A$, for all $1 \leq i \leq d$.

Definition 3.31 (q-de Rham complex)**.** Let $q\Omega^\bullet_{A/D} := \bigoplus_{k=0}^d q\Omega^k_{A/D}$ be a differential graded D-algebra defined as:

- $q\Omega^0_{A/D} = A$ and $q\Omega^1_{A/D}$ is a free left A-module on formal basis elements $d\log(U_i)$.
- The right A-module structure on $q\Omega^1_{A/D}$ is twisted by the rule $d\log(U_i) \cdot f = \gamma_i(f)\, d\log(U_i)$.
- $d\log(U_i)\, d\log(U_j) = -\, d\log(U_j)\, d\log(U_i)$ if $i \neq j$ and 0 if $i = j$.
- The following map, where $I_k = \{\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k$ such that $1 \leq i_1 < \cdots < i_k \leq d\}$, is an isomorphism of A-modules

$$\begin{aligned} \bigoplus_{\mathbf{i}\in I_k} A &\xrightarrow{\sim} q\Omega^k_{A/D} \\ (f_{\mathbf{i}}) &\longmapsto \textstyle\sum_{\mathbf{i}\in I_k} f_{\mathbf{i}}\, d\log(U_{i_1})\cdots d\log(U_{i_k}). \end{aligned}$$

- The 0^{th} differential $d_q : A \to q\Omega^1_{A/D}$ is given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{q-1}\, d\log(U_i)$.
- The elements $d\log(U_i) \in q\Omega^1_{A/D}$ are cocycles for all $1 \leq i \leq d$.

The data $d_q : A \to q\Omega^1_{A/D}$ forms a differential ring over D, i.e. $q\Omega^1_{A/D}$ is an D-bimodule and d_q is D-linear satisfying the Leibniz rule $d_q(fg) = d_q(f)g + fd_q(g)$.

Remark 3.32. In Definition 3.31, we will denote the operator $\frac{\gamma_i-1}{q-1} : A \to A$ by $\nabla_{q,i}^{\log}$ and refer to these as the q-differential operators in logarithmic coordinates. The operators $\nabla_{q,i} := U_i\nabla_{q,i}^{\log}$, will be referred to as the q-differential operators in non-logarithmic coordinates.

Remark 3.33. In most of the cases, we will fix units $U_1, \ldots, U_d \in A^\times$ such that $\gamma_i(U_j) = qU_j$ if $i = j$ or U_j if $i \neq j$.

Example 3.34. Take D to be $A_F = O_F[\![\mu]\!]$, and A to be A_R equipped with the A_F-linear action of Γ_R and let $\{\gamma_1, \ldots, \gamma_d\}$ be the topological generators of Γ_R' (see Subsection 3.1). Then setting $q = 1 + \mu$ and $U_i = [X_i^\flat]$, for $1 \leq i \leq d$, we have $\gamma_i = 1 \bmod \mu A_R$ for all $1 \leq i \leq d$. In particular, A_R satisfies the hypotheses of Definition 3.31. Moreover, in this case, $q\Omega^1_{A_R/A_F}$ identifies with $\Omega^1_{A_R/A_F}$, i.e. the (p,μ)-adic completion of the module of Kähler differentials of A_R with respect to A_F. Furthermore, note that we have an isomorphism of rings $A_R/\mu \xrightarrow{\sim} R$, so from [MT20, Remarks 2.4 and 2.10], reduction modulo $q-1$ of the differential ring $d_q : A_R \to \Omega^1_{A_R/A_F}$, is the usual continuous de Rham differential $d : R \to \Omega^1_R$.

Example 3.35. Take D to be $\Lambda_F = O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ equipped with an action of (φ, Γ_F) as in Remark 3.23 and take A to be $A^{\mathrm{PD}} := A_R[(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = A_R\widehat{\otimes}_{A_F}\Lambda_F$, equipped with the natural and continuous tensor product (φ, Γ_R)-action. From Remark 3.23, note that the structure map $\Lambda_F \to A^{\mathrm{PD}}$ is (φ, Γ_F)-equivariant and from (3.13) note that we also have a (φ, Γ_F)-equivariant isomorphism of rings $\iota : \Lambda_R \xrightarrow{\sim} A^{\mathrm{PD}}$. Moreover, the Λ_F-algebra A^{PD} is equipped with a Λ_F-linear continuous (for the p-adic topology) action of $\Gamma_R' \subset \Gamma_R$ and we have $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of Γ_R' (see Subsection 3.1). Then setting $q = 1 + \mu$ and $U_i = [X_i^\flat]$, for $1 \leq i \leq d$, it follows that $\gamma_i = 1 \bmod \mu A^{\mathrm{PD}}$, for all $1 \leq i \leq d$. In particular, A^{PD} satisfies the hypotheses of Definition 3.31. Furthermore, in this case, $q\Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$ identifies with $\Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$, i.e. the (p,μ)-adic $=$ $(p, [p]_q)$-adic $=$ p-adic (see Lemma 3.24) completion of the module of Kähler differentials of A^{PD} with respect to Λ_F. From Definition 3.31 the q-connection on A^{PD}, denoted $\nabla_q : A^{\mathrm{PD}} \to q\Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$, is given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)}\, d\log([X_i^\flat])$. Moreover, the q-connection ∇_q on A^{PD} is p-adically quasi-nilpotent because we have $\frac{\gamma_i-1}{p_2(\mu)[X_i^\flat]}([X_i^\flat]) = 1$, and it is flat because γ_i commute with each other. Furthermore, using the q-connection, A^{PD} can be equipped with a quasi-nilpotent flat connection ∇ as in Proposition 3.37, which coincides with the universal Λ_F-linear continuous de Rham differential $d : A^{\mathrm{PD}} \to \Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$. Then using Proposition 3.37, it follows that we have $(A^{\mathrm{PD}})^{\Gamma_R'} = (A^{\mathrm{PD}})^{\nabla_q=0} \xrightarrow{\sim}$ $(A^{\mathrm{PD}})^{\nabla=0} = (A^{\mathrm{PD}})^{d=0}$, where the first equality follows since the action of Γ_R on A^{PD} is continuous. Moreover, recall that from Subsection 3.1, for any g in Γ_F, we have $g\gamma_i g^{-1} = \gamma_i^{\chi(f)}$, for all $1 \leq i \leq d$. Hence, from Remark 3.38, the isomorphism $\Lambda_F \xrightarrow{\sim} (A^{\mathrm{PD}})^{\nabla_q=0} = (A^{\mathrm{PD}})^{\Gamma_R'}$, induced by the structure map, is (φ, Γ_F)-equivariant as well.

Example 3.36. Take D to be $\Lambda_R = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ equipped with an action of (φ, Γ_F) as in Remark 3.23 and take A to be $\overline{A}(1) := A_R(1)/p_1(\mu)$ as a Λ_R-algebra via the morphism of rings $\iota_\Lambda : \Lambda_R \to \overline{A}(1)$ defined by extending the R-algebra structure $p_1 : R \to \overline{A}(1)$ in Remark 3.29 via $\mu \mapsto p_2(\mu)$ and $(\mu^{p-1}/p)^{[k]} \mapsto (p_2(\mu)^{p-1}/p)^{[k]}$. Using Lemma 3.30 and the explicit description of $\overline{A}(1)$ in Proposition 3.25 and Remark 3.27, it follows that the map ι_Λ is injective and (φ, Γ_F)-equivariant. Moreover, the Λ_R-algebra $\overline{A}(1)$ is equipped with a Λ_R-linear (via ι_Λ) continuous (for the p-adic topology) action of $\Gamma_R' \subset 1 \times \Gamma_R$ and we have $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of Γ_R' (see Subsection 3.1). Then, by setting $q = 1 + \mu$ and $U_i = [X_i^\flat]$, for $1 \leq i \leq d$, from the explicit description of $\overline{A}(1)$ in Proposition 3.25, we have that $\gamma_i = 1 \bmod p_2(\mu)\overline{A}(1)$, for all $1 \leq i \leq d$. In particular, $\overline{A}(1)$ satisfies the hypotheses of Definition 3.31. Furthermore, in this case, $q\Omega^1_{\overline{A}(1)/\Lambda_R}$ identifies with $\Omega^1_{\overline{A}(1)/\Lambda_R}$, i.e. the (p,μ)-adic $=$ $(p, [p]_q)$-adic $=$ p-adic (see Lemma 3.24) completion of the module of Kähler differentials of $\overline{A}(1)$ with respect to Λ_R. From Definition 3.31 the q-connection on $\overline{A}(1)$, denoted $\nabla_q : \overline{A}(1) \to q\Omega^1_{\overline{A}(1)/\Lambda_R}$, is given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)}\, d\log([X_i^\flat])$. Moreover, the q-connection ∇_q on $\overline{A}(1)$ is p-adically quasi-nilpotent because we have $\frac{\gamma_i-1}{p_2(\mu)[X_i^\flat]}([X_i^\flat]) = 1$, and it is flat because γ_i commute with each other. Furthermore, using the q-connection, $\overline{A}(1)$ can be equipped with a p-adically quasi-nilpotent flat connection ∇ as in Proposition 3.37 which coincides with the universal Λ_R-linear continuous de Rham differential $d : \overline{A}(1) \to \Omega^1_{\overline{A}(1)/\Lambda_R}$. Then using Proposition 3.37, it follows that we have $\overline{A}(1)^{\nabla_q=0} = \overline{A}(1)^{\Gamma'_R} \xrightarrow{\sim} \overline{A}(1)^{\nabla=0} = \overline{A}(1)^{d=0}$, where the isomorphism follows since the action of Γ_R on $\overline{A}(1)$ is continuous. Moreover, from Subsection 3.1, recall that for any g in Γ_F, we have $g\gamma_i g^{-1} = \gamma_i^{\chi(f)}$, for all $1 \leq i \leq d$. Hence, the isomorphism $\Lambda_R \xrightarrow{\sim} \overline{A}(1)^{\Gamma'_R} = \overline{A}(1)^{\nabla_q=0} \xrightarrow{\sim} \overline{A}(1)^{\nabla=0}$, induced by ι_Λ in Remark 3.38, is moreover (φ, Γ_F)-equivariant. In particular, it follows that $R \xrightarrow{\sim} \Lambda_R^{\Gamma_F} \xrightarrow{\sim} (\overline{A}(1)^{\nabla_q=0})^{\Gamma_F} \xrightarrow{\sim} \overline{A}(1)^{\Gamma_R}$, induced by $p_1 : R \to \overline{A}(1)$.

From Lemma 3.24, we have that $t = \log(1+\mu)$ converges in $\mu\Lambda_F \subset \mu\Lambda_R$ and t/μ is a unit.

Proposition 3.37. *Let $D = \Lambda_R$ (resp. Λ_F) and $A = A_R(1)/p_1(\mu)$ (resp. A^{PD}). Then for $1 \leq i \leq d$, the series of operators $\nabla_i^{\log} = \frac{\log \gamma_i}{t} = \frac{1}{t}\sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_i-1)^k}{k+1}$ converge p-adically on A. This defines a D-linear p-adically quasi-nilpotent flat connection on A, denoted $\nabla : A \to \Omega^1_{A/D}$ and given as $f \mapsto \sum_{i=1}^d \nabla_i^{\log}(f)\, d\log([X_i^\flat])$. The connection ∇ coincides with the universal D-linear continuous de Rham differential operator $d : A \to \Omega^1_{A/D}$. Moreover, the data of the connection ∇ on A is equivalent to the data of the q-connection ∇_q described in Example 3.36 (resp. Example 3.35), i.e. one may be recovered from the other. Furthermore, the q-de Rham complex $q\Omega^\bullet_{A/D}$ is naturally quasi-isomorphic to the de Rham complex $\Omega^\bullet_{A/D}$. In particular, we have $A^{\nabla_q=0} \xrightarrow{\sim} A^{\nabla=0}$.*

Proof. The first two claims, i.e. the convergence of the operators $\nabla_i^{\log}$ on A and the fact that the operator ∇ defines a D-linear p-adically quasi-nilpotent flat connection on A and the claim of equivalence between the connection ∇ and quasi-isomorphism of complexes, will be shown in Proposition 4.13 for finite A-modules adimitting a continuous action of Γ_R (trivial modulo μ), in particular, A itself. Moreover, in the proof of Proposition 4.13, we will also show that the connection ∇ coincides with the universal D-linear continuous de Rham differential operator $d : A \to \Omega^1_{A/D}$. Note that the proof of Proposition 4.13 is independent of the subsequent claims proved in this section. This allows us to conclude. ∎

Remark 3.38. Let $\overline{A}(1) = A_R(1)/p_1(\mu)$ as in Proposition 3.37. From the isomorphism (3.17) in Remark 3.27 recall that we have a ring $C_R \xleftarrow{\sim} \overline{A}(1)$. Moreover, from Remark 3.27, we have that C_R is the p-adic completion of a PD-polynomial algebra over Λ_R and the structure map $\Lambda_R \to C_R$ coincides with the composition $\Lambda_R \xrightarrow{\iota_\Lambda} \overline{A}(1) \xrightarrow{(3.17)} C_R$. Now note that C_R is equipped with the universal Λ_R-linear continuous de Rham differential $d : C_R \to \Omega^1_{C_R/\Lambda_R}$, which can be given as $f \mapsto \sum_{i=1}^d \partial_i^{\log}(f)\, d\log(X_i)$, where $\partial_i^{\log} = X_i\partial_i$ and $\partial_i : C_R \to C_R$ is the unique Λ_R-linear continuous differential operator such that $\partial_i(X_j) = 1$ if $i = j$ and 0 otherwise. So, the isomorphism in (3.17) is Λ_R-linear, and it induces an isomorphism between complexes of $\overline{A}(1)$-modules $\Omega^\bullet_{\overline{A}(1)/\Lambda_R} \xrightarrow{\sim} \Omega^\bullet_{C_R/\Lambda_R} = C_R \otimes_{P_R} \Omega^1_{P_R/R}$, where the right hand side is an $\overline{A}(1)$-module via the isomorphism (3.17). Now, let f be any element of C_R, then from Remark 3.27 we have a unique presentation $f = \sum_{k\in\mathbb{N}} a_k\mu^{\{k\}}$, with a_k in P_R for all $k \in \mathbb{N}$ and p-adically $a_k \to 0$ as $k \to +\infty$. Then, it follows that $f \in C_R^{d=0}$ if and only if a_k in R, i.e. f is in Λ_R. In particular, we get that $\Lambda_R \xrightarrow{\sim} C_R^{d=0}$, in particular, $\Lambda_R \xrightarrow{\sim} \overline{A}(1)^{d=0}$ via ι_Λ. Using the same argument as above by replacing Λ_R by Λ_F, $\overline{A}(1)$ by A^{PD}, C_R by Λ_R and P_R by O_F, we obtain that $\Lambda_F \xrightarrow{\sim} (A^{\mathrm{PD}})^{d=0}$.

Remark 3.39. Let us consider the ring A^{PD} from Example 3.35, equipped with a continuous action of (φ, Γ_R). From (3.13) in Remark 3.23, note that we have a (φ, Γ_F)-equivariant isomorphism of rings $\iota : \Lambda_R \xrightarrow{\sim} A^{\mathrm{PD}}$. Moreover, we have a (φ, Γ_R)-equivariant embedding $A^{\mathrm{PD}} \to A_{\mathrm{cris}}(R_\infty)$ from Remark 3.28. Furthermore, the injective structure map $p_2 : A_R \to A_R(1)/p_1(\mu) = \overline{A}(1)$ from Remark 3.29 extends to an injective ring homomorphism $p_2 : A^{\mathrm{PD}} \to \overline{A}(1)$ and the target is the p-adic completion of a PD-polynymial algebra over the source (see Remark 3.27). Now, consider the universal A^{PD}-linear continuous de Rham differential $d : \overline{A}(1) \to \Omega^1_{\overline{A}(1)/A^{\mathrm{PD}}}$, which can be given as $f \mapsto \sum_{i=1}^d \partial_i^{\log}(f) dY_i$, where $Y_i = p_1(X_i)$ (see Remark 3.29), $\partial_i^{\log} = Y_i\partial_i$ and $\partial_i : \overline{A}(1) \to \overline{A}(1)$ is the unique A^{PD}-linear continuous differential operator such that $\partial_i(Y_j) = 1$ if $i = j$ and 0 otherwise. Then, it easily follows that the (φ, Γ_R)-equivariant injective map $\overline{A}(1) \to \mathcal{O}A_{\mathrm{cris}}(R_\infty)$ from Remark 3.28 is A^{PD}-linear and compatible with the aforementioned A^{PD}-linear differential operator on $\overline{A}(1)$ and the unique $A_{\mathrm{cris}}(R_\infty)$-linear connection on $\mathcal{O}A_{\mathrm{cris}}(R_\infty)$ from [Bri08, Section 6.2].

Remark 3.40. Let $\overline{A}(1) = A_R(1)/p_1(\mu)$ and A^{PD} as in Example 3.35. Note that we have a natural injective (φ, Γ_R)-equivariant homomorphism of rings $p_2 : A^{\mathrm{PD}} \to \overline{A}(1)$ (see Remark 3.39). Then using the Leibniz rule for the connection on $\overline{A}(1)$ (see the proof of Proposition 4.13) it follows that the respective connections on $\overline{A}(1)$ and A^{PD} are compatible and we have $\Omega^1_{\overline{A}(1)/\Lambda_R} = \overline{A}(1) \otimes_{A^{\mathrm{PD}}} \Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$.

3.4. Galois action on $A_R(1)$.

Note that we have the φ-equivariant multiplication map $\Delta : A_R(1) \to A_R$. Moreover, recall that in Remark 3.20, we described the action of Γ_R^2 on $A_R(1)$. In this subsection, our goal is to prove the following claim:

Proposition 3.41. *The φ-equivariant homomorphism $\Delta : A_R(1) \to A_R$ restricts to a φ-equivariant isomorphism $A_R(1)^{1\times\Gamma_R} \xrightarrow{\sim} A_R$. Moreover, the preceding isomorphism is compatible with the action of Γ_R on each side.*

In order to prove Proposition 3.41, we will study the action of $1 \times \Gamma_R$ on $A_R(1)$ in three steps, namely, the geometric part, i.e. Γ'_R below, the torsion part, i.e. $\mathbb{F}_p^\times$ below, and the (free) arithmetic part, i.e. Γ_0 below. From Subsection 1.6, recall that Γ_R fits into the following exact sequence:

$$1 \longrightarrow \Gamma'_R \longrightarrow \Gamma_R \longrightarrow \Gamma_F \longrightarrow 1.$$

Furthermore, from (1.6), recall that $\Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$, via the p-adic cyclotomic character, and it fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1,$$

where, for $p \geq 3$, we have $\Gamma_0 \xrightarrow{\sim} 1+p\mathbb{Z}_p$ and for $p = 2$, we have $\Gamma_0 \xrightarrow{\sim} 1+4\mathbb{Z}_2$. Moreover, for $p \geq 3$, we have that $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$ and the projection map $\Gamma_F \to \Gamma_{\mathrm{tor}}$, admits a section $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \to \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$, where the second map is given as $a \mapsto [a]$, the Teichmüller lift of a. Finally, for $p = 2$, we have $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \{\pm 1\}$, as groups.

Let us note that the results for the action of the geometric part of Γ_R in Subsection 3.4.1, are applicable for all primes p. However, for $p = 2$, since $\mathbb{F}_p^\times$ is the trivial group, in Subsections 3.4.2 and 3.4.3 we assume that $p \geq 3$. For $p = 2$, the arithmetic action of Γ_R will be handled in Subsection 3.4.4.

Remark 3.42. Note that it is possible to prove Proposition 3.41, by studying the action of $1 \times \Gamma_R$ at once and using Galois cohomology arguments, instead of the "3-step" argument presented here. However, the methods used in the "3-step" proof are applicable to the case of Wach modules as well (see Subsection 5.2), whereas the Galois cohomology arguments do not seem to generalise.

3.4.1. The action of Γ'_R.

In this subsubsection, our first goal is to show the following claim:

Lemma 3.43. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant sequence is exact:*

$$0 \longrightarrow (A_R(1)/p_1(\mu))^{1\times\Gamma'_R} \xrightarrow{p_1(\mu)^n} (A_R(1)/p_1(\mu)^{n+1})^{1\times\Gamma'_R} \longrightarrow (A_R(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \longrightarrow 0. \tag{3.19}$$

Before proving the Lemma 3.43, we provide a more explicit description of the ring $A_R(1)^{1\times\Gamma'_R}$ equipped with the induced $(\varphi, \Gamma_R \times \Gamma_F)$-action (introduced in the proof of Lemma 3.43). We start with the following:

Construction 3.44. From Construction 3.13, let us consider the (φ, Γ_R^2)-equivariant surjective map in (3.5), for $n = 1$. Taking invariants of this map, under the action of $1 \times \Gamma_R'$, we obtain a surjective map

$$A_R \widehat{\otimes}_{O_F} A_F \twoheadrightarrow R[\zeta_p, X_1^{1/p}, \ldots, X_d^{1/p}] \otimes_{O_F} O_F[\zeta_p]. \tag{3.20}$$

Using the description of the kernel of the map in (3.5), it follows that the kernel of the surjective map (3.20), is generated by the ideal $J = (1 \otimes [p]_q, [p]_q \otimes 1) \subset A_R \widehat{\otimes}_{O_F} A_F$. Clearly, the sequence $\{p, 1 \otimes [p]_q, [p]_q \otimes 1\}$ is regular on $A_R \widehat{\otimes}_{O_F} A_F$, so from [BS22, Proposition 3.13], we see that $(\tilde{\Lambda}_R, [p]_q \otimes 1)$ is the prismatic envelope of $(A_R \widehat{\otimes}_{O_F} A_F, J)$ over the bounded prism $(A_R, [p]_q)$. More explicitly, similar to the description of $A_R(1)$ before Lemma 3.15, let us consider a free δ-algebra over $A_R \widehat{\otimes}_{O_F} A_F$ in one variable given as $(A_R \widehat{\otimes}_{O_F} A_F)\big\{\frac{1 \otimes [p]_q}{[p]_q \otimes 1}\big\}_\delta$. Then, from [BS22, Proposition 3.13], we have that

$$\tilde{\Lambda}_R = (A_R \widehat{\otimes}_{O_F} A_F)\Big\{\frac{1 \otimes [p]_q}{[p]_q \otimes 1}\Big\}^{\wedge}_{(p, [p]_q \otimes 1)},$$

i.e. $\tilde{\Lambda}_R$ is the $(p, [p]_q \otimes 1)$-adic completion of the free δ-algebra $(A_R \widehat{\otimes}_{O_F} A_F)\big\{\frac{J}{[p]_q \otimes 1}\big\}_\delta$. Moreover, from Lemma 3.3, recall that $\tilde{p} = \mu_0 + p$ is the product of $[p]_q$ with a unit in A_F. Therefore, the ring $\tilde{\Lambda}_R$ has another presentation given as

$$\tilde{\Lambda}_R = (A_R \widehat{\otimes}_{O_F} A_F)\Big\{\frac{1 \otimes \tilde{p}}{\tilde{p} \otimes 1}\Big\}^{\wedge}_{(p, \tilde{p} \otimes 1)}. \tag{3.21}$$

Using notations similar to Lemma 3.15, we will denote the two projection maps as $p_1 : A_R \to \tilde{\Lambda}_R$ and $p_2 : A_F \to \tilde{\Lambda}_R$. Then, similar to Lemma 3.15, it is easy to see that both the preceding maps are faithfully flat, in particular, $\tilde{\Lambda}_R$ is $p_1(\mu)$-torsion free.

Next, note that we have a natural (φ, Γ_R)-equivariant injective map $A_F \to A_R$, where the action of Γ_R factors through $\Gamma_F \subset \Gamma_R$. So, by the universal property of prismatic envelopes the continuous action of $\Gamma_R \times \Gamma_F$ on $A_R \widehat{\otimes}_{O_F} A_F$ naturally extends to a continuous action on $\tilde{\Lambda}_R$ and since the morphism in (3.20) is compatible with the morphism in (3.5), therefore, we obtain the following natural $(\varphi, \Gamma_R \times \Gamma_R)$-equivariant map

$$\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R \longrightarrow A_R(1), \tag{3.22}$$

where the action of $\Gamma_R \times \Gamma_R$ on the source of $\iota_{\tilde{\Lambda}}$ factors through $\Gamma_R \times \Gamma_F \subset \Gamma_R \times \Gamma_R$. ■

Remark 3.45. Using an argument similar to the proof of Proposition 3.25, it can be shown that we have a natural (φ, Γ_F)-equivariant isomorphism of rings

$$\tilde{\Lambda}_R / p_1(\mu) \xrightarrow{\sim} \Lambda_R = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]^{\wedge}_p, \tag{3.23}$$

where $p_2(\mu) \mapsto \mu$ and $p_1([X_i]^\flat) \mapsto X_i$ for $1 \leq i \leq d$ (see Proposition 3.21 for equivalent descriptions of Λ_R).

Remark 3.46. In Construction 3.44, by switching the two components in (3.20) and carrying out essentially the same steps, we can define another ring $E := (A_F \widehat{\otimes}_{O_F} A_R)\big\{\frac{1 \otimes [p]_q}{[p]_q \otimes 1}\big\}^{\wedge}_{(p, [p]_q \otimes 1)}$, as the prismatic envelope of $(A_F \widehat{\otimes}_{O_F} A_R, J)$ over the bounded prism $(A_F, [p]_q)$, where $J = (1 \otimes [p]_q, [p]_q \otimes 1) \subset A_F \widehat{\otimes}_{O_F} A_R$, i.e. E is the $(p, [p]_q \otimes 1)$-adic completion of the free δ-algebra $(A_R \widehat{\otimes}_{O_F} A_F)\big\{\frac{J}{[p]_q \otimes 1}\big\}_\delta$. Now, if we use notations similar to Lemma 3.15 to denote the two projection maps as $p_1 : A_F \to E$ and $p_2 : A_R \to E$. Then, similar to Lemma 3.15, it is easy to see that both the preceding maps are faithfully flat. Moreover, by the universal property of prismatic envelopes, we see that the continuous action of $\Gamma_F \times \Gamma_R$ on $A_F \widehat{\otimes}_{O_F} A_R$ naturally extends to a continuous action on E. Furthermore, using an argument similar to the proof of Proposition 3.25, it can be shown that we have a natural (φ, Γ_R)-equivariant isomorphism of rings $E/p_1(\mu) \xrightarrow{\sim} A^{\mathrm{PD}} = A_R[(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]^{\wedge}_p$, where $p_2(\mu) \mapsto \mu$ and $p_2([X_i]^\flat) \mapsto [X_i^\flat]$ for $1 \leq i \leq d$ (see Example 3.35 for the definition of A^{PD}).

Lemma 3.47. *For each $n \geq 1$, the natural map in (3.22) induces a $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism*

$$\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R/p_1(\mu)^n \xrightarrow{\sim} (A_R(1)/p_1(\mu)^n)^{1\times\Gamma'_R}. \tag{3.24}$$

In particular, (3.22) induces a $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism $\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R \xrightarrow{\sim} A_R(1)^{1\times\Gamma'_R}$.

Proof. As the action of $\Gamma_R \times \Gamma_R$ on $\tilde{\Lambda}_R$ factors through $\Gamma_R \times \Gamma_F$, we see that reducing (3.22) modulo $p_1(\mu)^n$ and taking $(1 \times \Gamma'_R)$-invariants, we obtain the $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant map in (3.24). It remains to show that it is an isomorphism. For $n = 1$, we consider the following (φ, Γ_F)-equivariant diagram,

$$\begin{tikzcd}
\tilde{\Lambda}_R/p_1(\mu) \arrow[r, "(3.24)"] \arrow[d, "\wr"', "(3.23)"] & (A_R(1)/p_1(\mu))^{1\times\Gamma'_R} \\
\Lambda_R, \arrow[ur, "\iota_\Lambda", "\sim"'] &
\end{tikzcd}$$

where the isomorphism ι_Λ is from Example 3.36. It is easy to see that the diagram above commutes. Therefore, the top horizontal map of the diagram is an isomorphism, i.e. $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} (A_R(1)/p_1(\mu))^{1\times\Gamma'_R}$. Now as both $A_R(1)$ and $\tilde{\Lambda}_R$ are $p_1(\mu)$-torsion free (see Lemma 3.15 and Construction 3.44), we consider the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\Lambda}_R/p_1(\mu) \arrow[r, "p_1(\mu)^n"] \arrow[d, "\wr"', "(3.24)"] & \tilde{\Lambda}_R/p_1(\mu)^{n+1} \arrow[r] \arrow[d, "(3.24)"] & \tilde{\Lambda}_R/p_1(\mu)^n \arrow[r] \arrow[d, "(3.24)"] & 0 \\
0 \arrow[r] & (A_R(1)/p_1(\mu))^{1\times\Gamma'_R} \arrow[r, "p_1(\mu)^n"] & (A_R(1)/p_1(\mu)^{n+1})^{1\times\Gamma'_R} \arrow[r] & (A_R(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \arrow[r] & 0,
\end{tikzcd}$$

where the second exact sequence is from (3.19). Using the diagram, an easy induction on $n \geq 1$, gives the natural $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism induced from the map (3.22) as $\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R/p_1(\mu)^n \xrightarrow{\sim} (A_R(1)/p_1(\mu)^n)^{1\times\Gamma'_R}$, i.e. the isomorphism in (3.24). Finally, since both $\tilde{\Lambda}_R$ and $A_R(1)$ are $p_1(\mu)$-adically complete, therefore, by taking the limit over $n \geq 1$ and noting that limit commutes with right adjoint functors, in particular, with taking $(1 \times \Gamma'_R)$-invariants, we obtain the $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism $\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R \xrightarrow{\sim} A_R(1)^{1\times\Gamma'_R}$. ■

Let us note an important observation for the action of $\Gamma_R \times \Gamma_F$ on $\tilde{\Lambda}_R$.

Lemma 3.48. *The action of $1 \times \Gamma_F$ is trivial on $\tilde{\Lambda}_R/p_2(\mu)$ and the action of $\Gamma_R \times 1$ is trivial on $\tilde{\Lambda}_R/p_1(\mu)$.*

Proof. The proof of the claim is similar to that of Proposition 3.17. For the first part, let us note that the action of $1 \times \Gamma_F$ is trivial modulo $p_2(\mu)$ on $A_R \widehat{\otimes}_{O_F} O_F[\![\mu]\!]$. Then, from the explicit description of $\tilde{\Lambda}_R$ in Construction 3.44, it is enough to show that for any $m \in \mathbb{N}$ and g in $1 \times \Gamma_F$, we have that

$$(g-1)\delta^m\big(\tfrac{1\otimes[p]_q}{[p]_q\otimes 1}\big) \in p_2(\mu)\tilde{\Lambda}_R.$$

Now, using Lemma 3.9 (2), let us note that $p_2(\mu)\tilde{\Lambda}_R$ is a δ-stable ideal of $\tilde{\Lambda}_R$, in the sense of [BS22, Example 2.10]. Then, using Lemma 3.19 for $A = \tilde{\Lambda}_R$ and $\alpha = p_2(\mu)$, we see that to prove our claim, it is enough to show that $(g-1)(\frac{1\otimes[p]_q}{[p]_q\otimes 1})$ belongs to $p_2(\mu)\tilde{\Lambda}_R$, which follows from Lemma 3.18. In particular, we have shown that the action of $1 \times \Gamma_F$ is trivial on $\tilde{\Lambda}_R/p_2(\mu)$. For the second claim, note that the action of $\Gamma_R \times 1$ is trivial modulo $p_1(\mu)$ on $A_R \widehat{\otimes}_{O_F} O_F[\![\mu]\!]$. Now, from the explicit description of $\tilde{\Lambda}_R$ in Construction 3.44, it is enough to show that for any $m \in \mathbb{N}$ and g in $\Gamma_R \times 1$, we have that

$$(g-1)\delta^m\big(\tfrac{1\otimes[p]_q}{[p]_q\otimes 1}\big) \in p_1(\mu)\tilde{\Lambda}_R.$$

Similar to the first part, using Lemma 3.9 (1), we note that $p_1(\mu)\tilde{\Lambda}_R$ is a δ-stable ideal of $\tilde{\Lambda}_R$. Then, using Lemma 3.19 for $A = \tilde{\Lambda}_R$ and $\alpha = p_1(\mu)$, we see that to prove our claim, it is enough to show that $(g-1)(\frac{1\otimes[p]_q}{[p]_q\otimes 1})$ belongs to $p_1(\mu)\tilde{\Lambda}_R$, which again follows from Lemma 3.18. Hence, the action of $\Gamma_R \times 1$ is trivial on $\tilde{\Lambda}_R/p_1(\mu)$. ∎

Remark 3.49. From Lemma 3.48, note that the action of $1 \times \Gamma_F$ is trivial on $\tilde{\Lambda}_R/p_2(\mu)$ and the element $p_1(\mu)$ is invariant under this action. Therefore, it follows that for any g in $1 \times \Gamma_F$ and any x in $\tilde{\Lambda}_R/p_1(\mu)^n$ we have that $(g-1)x$ is an element of $p_2(\mu)\tilde{\Lambda}_{R,0}/p_1(\mu)^n$. In particular, for $n = 1$, using the isomorphism $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$ from (3.23), we get that for any g in Γ_F and any x in Λ_R the element $(g-1)x$ belongs to $\mu\Lambda_R$.

We are now ready to prove the main claim:

Proof of Lemma 3.43. To lighten notations, let us set $A(1) := A_R(1)$ and $\overline{A}(1) := A_R(1)/p_1(\mu)$. Instead of working with the action of $1\times\Gamma'_R$, we will work with the q-connection arising from this action. More precisely, in the notation of Definition 3.31, take D to be $\tilde{\Lambda}_R \xrightarrow{\sim} A(1)^{1\times\Gamma'_R}$ (see Lemma 3.47), and A to be $A(1)$ equipped with a $\tilde{\Lambda}_R$-linear action of $1 \times \Gamma'_R$ and let $\{\gamma_1, \ldots, \gamma_d\}$ be the topological generators of Γ'_R (see Subsection 3.1). Then setting $q = 1 + p_2(\mu)$ and $U_i = p_2([X_i^\flat])$ for $1 \leq i \leq d$, we have that $\gamma_i = 1 \bmod p_2(\mu)A_R(1)$ for all $1 \leq i \leq d$ (see Proposition 3.17). In particular, $A(1)$ satisfies the hypotheses of Definition 3.31. Moreover, in this case, $q\Omega^1_{A(1)/\tilde{\Lambda}_R}$ identifies with $\Omega^1_{A(1)/\tilde{\Lambda}_R}$, and given as the (p,μ)-adic completion of the module of Kähler differentials of $A(1)$ with respect to $\tilde{\Lambda}_R$. From Definition 3.31, the q-connection on $A(1)$, denoted $\nabla_q : A(1) \to q\Omega^1_{A(1)/\tilde{\Lambda}_R}$, is given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)}\, d\log(p_2([X_i^\flat]))$. Moreover, the q-connection ∇_q on $A(1)$ is $(p, p_1(\mu))$-adically quasi-nilpotent because we have $\frac{\gamma_i - 1}{p_2(\mu)p_2([X_i^\flat])}(p_2([X_i^\flat])) = 1$, and it is flat because γ_i commute with each other. Let us also note that $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R \xrightarrow{\sim} (A(1)/p_1(\mu))^{1\times\Gamma'_R}$ (see (3.23), Example 3.36 and the proof of Lemma 3.47). Now consider the following exact sequence of q-de Rham complexes:

$$0 \longrightarrow q\Omega^\bullet_{\overline{A}(1)/\Lambda_R} \xrightarrow{p_1(\mu)^n} A(1)/p_1(\mu)^{n+1} \otimes_{A(1)} q\Omega^\bullet_{A(1)/\tilde{\Lambda}_R} \longrightarrow A(1)/p_1(\mu)^n \otimes_{A(1)} q\Omega^\bullet_{A(1)/\tilde{\Lambda}_R} \longrightarrow 0.$$

Since the action of $1\times\Gamma'_R$ is continuous for the $(p, p_1(\mu))$-adic topology on $A(1)$, therefore, we get that $(A(1)/p_1(\mu)^n)^{1\times\Gamma'_R} = (A(1)/p_1(\mu)^n)^{\nabla_q=0}$. In particular, showing that (3.19) is exact, is equivalent to showing that $H^1(q\Omega^\bullet_{\overline{A}(1)/\Lambda_R}) = 0$. Now, from Proposition 3.37, recall that the q-de Rham complex $q\Omega^\bullet_{\overline{A}(1)/\Lambda_R}$ is naturally quasi-isomorphic to the de Rham complex $\Omega^\bullet_{\overline{A}(1)/\Lambda_R}$. Then, from the explicit description of $\overline{A}(1)$ in Proposition 3.25, it is clear that the de Rham complex $\Omega^\bullet_{\overline{A}(1)/\Lambda_R}$ is acyclic in positive degree, in particular, $H^1(q\Omega^\bullet_{\overline{A}(1)/\Lambda_R}) = H^1(\Omega^\bullet_{\overline{A}(1)/\Lambda_R}) = 0$. Hence, it follows that (3.19) is exact. ∎

For computations to be carried out in Subsection 5.2, we will consider a "diagonal map for the geometric variables" from $A_R(1)$ to $\tilde{\Lambda}_R$. More precisely, consider the following surjective A_R-linear (via p_1) and $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant map

$$\Delta' : A_R(1) \longrightarrow \tilde{\Lambda}_R,$$

where $p_2(\mu) \mapsto p_2(\mu)$ and $p_2([X_i^\flat]) \mapsto p_1([X_i^\flat])$ for $1 \leq i \leq d$. Then it is easy to verify that the composition $\Delta' \circ \iota_{\tilde{\Lambda}}$ (see (3.22)), is the identity on $\tilde{\Lambda}_R$. As $p_1(\mu)$ is invariant under the action of $1\times\Gamma'_R$, we see that the reduction of Δ' modulo $p_1(\mu)^n$ is $(\varphi, \Gamma_R\times\Gamma_F)$-equivariant, for each $n \geq 1$. Now taking invariants of the source under the action of $1 \times \Gamma'_R$, we obtain the following (φ, Γ_F)-equivariant morphism:

$$\Delta' : (A_R(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \longrightarrow \tilde{\Lambda}_R/p_1(\mu)^n. \tag{3.25}$$

Then, we claim the following:

Lemma 3.50. *The (φ, Γ_F)-equivariant map in (3.25) is an isomorphism. Moreover , (3.25) induces a (φ, Γ_F)-equivariant isomorphism $\Delta' : A_R(1)^{1\times\Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R$*

Proof. Let us first consider the following (φ, Γ_F)-equivariant diagram

$$\Lambda_R \xrightarrow[\sim]{\iota_\Lambda} (A_R(1)/p_1(\mu))^{1\times\Gamma'_R} \xrightarrow{(3.25)} \tilde{\Lambda}_R/p_1(\mu) \xrightarrow[\sim]{(3.23)} \Lambda_R.$$

Using the isomorphism (3.24) in Lemma 3.47, it is easy to verify that the composition above is the identity on Λ_R. Therefore, it follows that (3.25) is an isomorphism for $n = 1$. Now as both $A_R(1)$ and $\tilde{\Lambda}_R$ are $p_1(\mu)$-torsion free (see Lemma 3.15 and Construction 3.44), we consider the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (A_R(1)/p_1(\mu))^{1\times\Gamma'_R} \arrow[r, "p_1(\mu)^n"] \arrow[d, "(3.25)", "\wr"'] & (A_R(1)/p_1(\mu)^{n+1})^{1\times\Gamma'_R} \arrow[r] \arrow[d, "(3.25)"] & (A_R(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \arrow[r] \arrow[d, "(3.25)"] & 0 \\
0 \arrow[r] & \tilde{\Lambda}_R/p_1(\mu) \arrow[r, "p_1(\mu)^n"] & \tilde{\Lambda}_R/p_1(\mu)^{n+1} \arrow[r] & \tilde{\Lambda}_R/p_1(\mu)^n \arrow[r] & 0,
\end{tikzcd}$$

where the first exact sequence is from (3.19). Using the diagram, an easy induction on $n \geq 1$, gives the natural $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism $\Delta' : (A_R(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R/p_1(\mu)^n$ in (3.25). Finally, since both $A_R(1)$ and $\tilde{\Lambda}_R$ are $p_1(\mu)$-adically complete, therefore, by taking the limit over $n \geq 1$ and noting that limit commutes right adjoint functors, in particulat, with taking $(1 \times \Gamma'_R)$-invariants, we obtain the (φ, Γ_F)-equivariant isomorphism $\Delta' : A_R(1)^{1\times\Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R$. ∎

3.4.2. The action of $\mathbb{F}_p^\times$. In this subsubsection, we will assume that $p \geq 3$ and consider the invariants of the exact sequence (3.19) (more precisely, its image under the inverse of the map $\iota_{\tilde{\Lambda}}$ of (3.24)), for the action of $1 \times \mathbb{F}_p^\times$.

Lemma 3.51. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant sequence is exact:*

$$0 \longrightarrow (\tilde{\Lambda}_R/p_1(\mu))^{1\times\mathbb{F}_p^\times} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1\times\mathbb{F}_p^\times} \longrightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\mathbb{F}_p^\times} \longrightarrow 0. \tag{3.26}$$

Proof. Using the discussion in Construction 3.44 and the isomorphism (3.24) in Lemma 3.47, the exact sequence (3.19) in Lemma 3.43 can be written as the following $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant exact sequence:

$$0 \longrightarrow \tilde{\Lambda}_R/p_1(\mu) \xrightarrow{p_1(\mu)^n} \tilde{\Lambda}_R/p_1(\mu)^{n+1} \longrightarrow \tilde{\Lambda}_R/p_1(\mu)^n \longrightarrow 0.$$

By considering the associated long exact sequence for the cohomology of $(1 \times \mathbb{F}_p^\times)$-action and noting that $H^1(1 \times \mathbb{F}_p^\times, \Lambda_R) = 0$, since $p - 1$ is invertible in $\mathbb{Z}_p$, we obtain that the sequence in (3.26) is exact. ∎

Next, let us describe the rings $(\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\mathbb{F}_p^\times}$, more explicitly.

Construction 3.52. Note that from (A.1), the action of $1 \times \mathbb{F}_p^\times$ induces a decomposition $\tilde{\Lambda}_R = \bigoplus_{i=0}^{p-2} (\tilde{\Lambda}_R)_i$. Let us set $\tilde{\Lambda}_{R,0} := (\tilde{\Lambda}_R)_0 = (\tilde{\Lambda}_R)^{1\times\mathbb{F}_p^\times}$, complete for the p-adic topology and equipped with induced Frobenius and a continuous action of $\Gamma_R \times \Gamma_0$, from the corresponding structures on $\tilde{\Lambda}_R$. Using the explicit presentation of $\tilde{\Lambda}_R$ from Construction 3.44 (see (3.21)), we obtain an explicit presentation of $\tilde{\Lambda}_{R,0}$ as follows:

$$\tilde{\Lambda}_{R,0} = (A_R \widehat{\otimes}_{O_F} O_F[\![\mu_0]\!])\Big\{\tfrac{1\otimes\tilde{p}}{\tilde{p}\otimes 1}\Big\}^\wedge_{(p,\tilde{p}\otimes 1)}. \tag{3.27}$$

Using notations from Construction 3.44, we will denote the two projection maps as $p_1 : A_R \to \tilde{\Lambda}_{R,0}$ and $p_2 : O_F[\![\mu_0]\!] \to \tilde{\Lambda}_{R,0}$. Then, similar to Lemma 3.15, it is easy to see that both the preceding maps are faithfully flat, in particular, $\tilde{\Lambda}_{R,0}$ is $p_1(\mu)$-torsion free.

Furthermore, note that from (3.23), we have a natural (φ, Γ_F)-equivariant isomorphism of rings $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$. Now, using (A.1), the action of $1 \times \mathbb{F}_p^\times$ induces a decomposition $\Lambda_R = \bigoplus_{i=0}^{p-2} (\Lambda_R)_i$. Note that $p_1(\mu)$ in $\tilde{\Lambda}_R$ is invariant under the action of $1 \times \mathbb{F}_p^\times$ and $p-1$ is invertible in $\mathbb{Z}_p$, in particular,

$H^1(1 \times \mathbb{F}_p^\times, \tilde{\Lambda}_R) = 0$. Therefore, it follows that for each $n \geq 1$, we have a $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism of rings

$$\tilde{\Lambda}_{R,0}/p_1(\mu)^n \xrightarrow{\sim} (\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\mathbb{F}_p^\times}. \tag{3.28}$$

In particular, for $n = 1$, we have a $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism of rings $\tilde{\Lambda}_{R,0}/p_1(\mu) \xrightarrow{\sim} (\tilde{\Lambda}_R/p_1(\mu))^{1\times\mathbb{F}_p^\times}$. Additionally, using (3.23) induces a (φ, Γ_0)-equivariant isomorphism of rings

$$\tilde{\Lambda}_{R,0}/p_1(\mu) \xrightarrow{\sim} (\tilde{\Lambda}_R/p_1(\mu))^{1\times\mathbb{F}_p^\times} \xrightarrow{\sim} \Lambda_R^{\mathbb{F}_p^\times} =: \Lambda_{R,0}. \tag{3.29}$$

Let us describe the ring $\Lambda_{R,0}$ more explicitly. From Proposition 3.21, recall that we have $\Lambda_R = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$. Moreover, recall that $t = \log(1+\mu)$ is an element of Λ_F and since t/μ is a unit in Λ_F (see Lemma 3.24), we may write $\Lambda_R = R[t, (t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$. Now, since the action of $\mathbb{F}_p^\times$ is trivial on t^{p-1}, it follows that $\Lambda_{R,0} = \Lambda_R^{\mathbb{F}_p^\times} = R[(t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$. Furthermore, recall that μ_0 is the product of μ^{p-1} with a unit in A_F (see Lemma 3.7) and since t/μ is a unit in Λ_F, therefore, we can write $\mu_0 = \upsilon t^{p-1}$, where υ is a unit in Λ_F. As the action of $\mathbb{F}_p^\times$ is trivial on μ_0 and t^{p-1}, it follows that υ belongs to $\Lambda_{F,0} = \Lambda_F^{\mathbb{F}_p^\times}$. In particular, μ_0 is a product of t^{p-1} with a unit in $\Lambda_{F,0}$ and we have (φ, Γ_0)-equivariant identifications

$$\Lambda_{R,0} = \Lambda_R^{\mathbb{F}_p^\times} = R[(t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = R[(\mu_0/p)^{[k]}, k \in \mathbb{N}]_p^\wedge. \tag{3.30}$$

Note that μ^{p-1} is not an element of $\Lambda_{R,0}$ as the action of $\mathbb{F}_p^\times$ is non-trivial on μ^{p-1}. ∎

3.4.3. The action of $1 + p\mathbb{Z}_p$. In this subsubsection, we will assume that $p \geq 3$ and consider the invariants of the exact sequence (3.26), for the action of $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + p\mathbb{Z}_p)$, and show the following:

Lemma 3.53. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times 1)$-equivariant sequence is exact:*

$$0 \longrightarrow (\tilde{\Lambda}_R/p_1(\mu))^{1\times\Gamma_F} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1\times\Gamma_F} \longrightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\Gamma_F} \longrightarrow 0. \tag{3.31}$$

Remark 3.54. Via the natural $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism $\iota_{\tilde{\Lambda}}$ in Lemma 3.47 (see (3.24)), we see that the exact sequence in (3.26) is the invariants, under the natural action of $1 \times \mathbb{F}_p^\times$, of the exact sequence (3.19) in Lemma 3.43. Then it follows that the exact sequence in (3.31) is the invariants, under the natural action of $1 \times \Gamma_F$, of the exact sequence (3.19) in Lemma 3.43.

Note that using the $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism in (3.28) and (3.29), the sequence in (3.31) can be rewritten as the following $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant sequence:

$$0 \longrightarrow \Lambda_{R,0}^{\Gamma_0} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1})^{1\times\Gamma_0} \longrightarrow (\tilde{\Lambda}_{R,0}/p_1(\mu)^n)^{1\times\Gamma_0} \longrightarrow 0. \tag{3.32}$$

In order to prove that (3.32) is exact, we will now look at the action of $\Gamma_R \times \Gamma_0$ on the rings introduced in Construction 3.52. We start with the following observation:

Lemma 3.55. *The action of $1 \times \Gamma_0$ is trivial on $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$ and the action of $\Gamma_R \times 1$ is trivial on $\tilde{\Lambda}_{R,0}/p_1(\mu)$.*

Proof. The proof of the claim is similar to that of Lemma 3.48. Let us first note that if g is any element of Γ_0, then $(g-1)\mu_0$ is an element of $\mu_0 O_F[\![\mu_0]\!]$ (see Lemma 3.6). Now, for the first part, let us note that the action of $1 \times \Gamma_0$ is trivial modulo $p_2(\mu_0)$ on $A_R \widehat{\otimes}_{O_F} O_F[\![\mu_0]\!]$. Then, from the explicit description of $\tilde{\Lambda}_{R,0}$ in (3.27), it is enough to show that for any $m \in \mathbb{N}$ and g in $1 \times \Gamma_0$, we have that

$$(g-1)\delta^m\left(\tfrac{1\otimes\tilde{p}}{\tilde{p}\otimes 1}\right) \in p_2(\mu_0)\tilde{\Lambda}_{R,0}.$$

We can reduce this claim further, as follows. Using Lemma 3.9 (2), we first note that $p_2(\mu_0)\tilde{\Lambda}_{R,0}$ is a δ-stable ideal of $\tilde{\Lambda}_{R,0}$, in the sense of [BS22, Example 2.10]. Then, using Lemma 3.19 for $A = \tilde{\Lambda}_{R,0}$

and $\alpha = p_2(\mu_0)$, we see that to prove our claim, it is enough to show that $(g-1)(\frac{1\otimes\tilde{p}}{\tilde{p}\otimes 1})$ belongs to $p_2(\mu_0)\tilde{\Lambda}_{R,0}$. The assertion now follows from the earlier observation that $(g-1)\mu_0$ is an element of $\mu_0 O_F[\![\mu_0]\!]$. In particular, we have shown that the action of $1\times\Gamma_0$ is trivial on $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$. The second claim easily follows from Lemma 3.48. ∎

Remark 3.56. From Lemma 3.55, note that the action of $1\times\Gamma_0$ is trivial on $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$ and the element $p_1(\mu)$ is invariant under this action. Therefore, it follows that for any g in $1\times\Gamma_0$ and any x in $\tilde{\Lambda}_{R,0}/p_1(\mu)^n$ we have that $(g-1)x$ is an element of $p_2(\mu_0)\tilde{\Lambda}_{R,0}/p_1(\mu)^n$. In particular, for $n=1$, using the isomorphism $\tilde{\Lambda}_{R,0}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,0}$ from (3.29), we get that for any g in Γ_0 and any x in $\Lambda_{R,0}$ the element $(g-1)x$ belongs to $\mu_0\Lambda_{R,0}$. We can show this more explicitly. Indeed, let γ_0 be the topological generator of Γ_0 such that $\chi(\gamma_0) = 1+pa$, for a unit a in $\mathbb{Z}_p$, and we will write $(1+pa)^{p-1} = 1+pb$, where b is also a unit in $\mathbb{Z}_p$. Now, from Construction 3.52, recall that μ_0 is the product of t^{p-1} with a unit in $\Lambda_{F,0}$ (see (3.30) and the discussion preceding it). Therefore, it is enough to show that for any x in $\Lambda_{R,0}$, we have that $(\gamma_0-1)x$ is an element of $t^{p-1}\Lambda_{R,0}$. Recall that $\Lambda_{R,0} = R[(t^{p-1}/p)^{[k]}, k\in\mathbb{N}]_p^\wedge$, so we can write $x = \sum_{k\in\mathbb{N}} x_k(t^{p-1}/p)^{[k]}$, where x_k is an element of R for each $k\in\mathbb{N}$. Then we have,

$$\begin{aligned}(\gamma_0-1)x &= (\gamma_0-1)\big(\textstyle\sum_{k\in\mathbb{N}} x_k(\frac{t^{p-1}}{p})^{[k]}\big)\\ &= \textstyle\sum_{k\in\mathbb{N}}(\chi(\gamma_0)^{(p-1)k}-1)x_k(\frac{t^{p-1}}{p})^{[k]} = t^{p-1}\sum_{k\in\mathbb{N}}\frac{((1+pb)^k-1)}{pk}x_k(\frac{t^{p-1}}{p})^{[k-1]} = t^{p-1}y,\end{aligned}$$

where y converges in $\Lambda_{F,0}$ because for any $k\in\mathbb{N}$, an easy computation shows that we have $(1+pb)^k-1 = pku_k$, for some unit u_k in $\mathbb{Z}_p$. Therefore, it follows that $(\gamma_0-1)x$ is an element of $t^{p-1}\Lambda_{R,0} = \mu_0\Lambda_{R,0}$.

Using the action of $1\times\Gamma_0$ on $\tilde{\Lambda}_{R,0}$, we will define a q-de Rham complex (see Definition 3.31). For such a definition we will treat the following element in $\tilde{\Lambda}_{R,0}$ as a parameter:

$$\tilde{s} := \tfrac{1\otimes\tilde{p}-\tilde{p}\otimes 1}{\tilde{p}\otimes 1} = \tfrac{p_2(\tilde{p})-p_1(\tilde{p})}{p_1(\tilde{p})} \tag{3.33}$$

Lemma 3.57. *Let γ_0 be any element of $1\times\Gamma_0$. Then we have that $(\gamma_0-1)\tilde{s} = u\,p_2(\mu_0)$, for some unit u in $\tilde{\Lambda}_{R,0}$.*

Proof. Note that it is enough to show the claim for a topological generator γ_0 of Γ_0 such that $\chi(\gamma_0) = 1+pa$, for a unit a in $\mathbb{Z}_p$. Now, recall that we have $\tilde{p} = \mu_0+p$. Moreover, from Lemma 3.6, recall that we can write $(\gamma_0-1)\mu_0 = \tilde{p}\mu_0 x$, for some x in $O_F[\![\mu_0]\!]$. So from (3.33) we can write

$$(\gamma_0-1)\tilde{s} = \tfrac{(\gamma_0-1)p_2(\tilde{p})}{p_1(\tilde{p})} = \tfrac{(\gamma_0-1)p_2(\mu_0)}{p_1(\tilde{p})} = \tfrac{p_2(\tilde{p}\mu_0 x)}{p_1(\tilde{p})} = \tfrac{p_2(\tilde{p})}{p_1(\tilde{p})}p_2(x)p_2(\mu_0). \tag{3.34}$$

From Lemma 3.3 and Lemma 3.15, it follows that $\frac{1\otimes\tilde{p}}{\tilde{p}\otimes 1}$ is a unit in $A_R(1)$. Moreover, from the description of $\tilde{\Lambda}_{R,0}$ as the $\mathbb{F}_p^\times$-invariants of $\tilde{\Lambda}_R$, in Construction 3.52, it follows that $\frac{1\otimes\tilde{p}}{\tilde{p}\otimes 1}$ is also a unit in $\tilde{\Lambda}_{R,0}$. Therefore, to show the claim, it is enough to show that $p_2(x)$ is a unit in $\tilde{\Lambda}_{R,0}$. Again, note that from Construction 3.52, the ring $\tilde{\Lambda}_{R,0}$ is $p_1(\mu)$-adically complete and we have a (φ,Γ_0)-equivariant isomorphism $\tilde{\Lambda}_{R,0}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,0}$ (see (3.29)), therefore, we are reduced to showing that $p_2(\overline{x})$, the image of $p_2(x)$ under the preceding isomorphism, is a unit. Also note that, under the preceding isomorphism, $p_2(\mu_0)$ and $p_2(\tilde{p})$ are the respective images of μ_0 and $\tilde{p}$ in $\Lambda_{R,0}$. Now, reducing the equalities in (3.34) modulo $p_1(\mu)$, we obtain the following expression in $\Lambda_{R,0}$:

$$\tfrac{(\gamma_0-1)\mu_0}{p} = p_2(\overline{x})\mu_0\tfrac{\tilde{p}}{p}.$$

Note that $\tilde{p}/p$ is a unit in $\Lambda_{R,0}$, because it is the image of the unit $p_2(\tilde{p})/p_1(\tilde{p})$ in $\tilde{\Lambda}_{R,0}$ (also see Lemma 3.24). Now, from Lemma 3.58, it follows that $p_2(\overline{x})$ is a unit in $\Lambda_{R,0}$. Therefore, $p_2(x)$ is a unit in $\tilde{\Lambda}_{R,0}$. Hence, we have shown that $(\gamma_0-1)\tilde{s} = u\,p_2(\mu_0)$, for $u = (p_2(\tilde{p})/p_1(\tilde{p}))x$ a unit in $\tilde{\Lambda}_{R,0}$. ∎

The following observation was used above:

Lemma 3.58. *Let γ_0 be a topological generator of Γ_0 such that $\chi(\gamma_0) = 1 + pa$, for a unit a in $\mathbb{Z}_p$. Then we have $(\gamma_0 - 1)\frac{\mu_0}{p} = v\mu_0$, for some unit v in $\Lambda_{F,0} = \Lambda_F^{\mathbb{F}_p^\times}$.*

Proof. From Construction 3.52, recall that μ_0/p is the product of t^{p-1}/p with a unit in $\Lambda_{F,0}$ (see (3.30) and the discussion preceding it). So let us write $\mu_0/p = ut^{p-1}/p$, for some unit u in $\Lambda_{F,0}$. Note that, from Remark 3.56, we have that $(\gamma_0 - 1)u = \mu_0 x$, for some x in $\Lambda_{F,0}$. Moreover, we write $(1+pa)^{p-1} - 1 = pb$, where b is a unit in $\mathbb{Z}_p$. Therefore, we can write

$$\begin{aligned}(\gamma_0 - 1)\tfrac{\mu_0}{p} = (\gamma_0 - 1)\tfrac{ut^{p-1}}{p} &= \tfrac{t^{p-1}}{p}(\gamma_0 - 1)u + \gamma_0(u)(\gamma_0 - 1)\tfrac{t^{p-1}}{p}\\ &= \tfrac{t^{p-1}}{p}\mu_0 x + \gamma_0(u)(\chi(\gamma_0)^{p-1} - 1)\tfrac{t^{p-1}}{p}\\ &= \mu_0(\tfrac{t^{p-1}}{p}x + \gamma_0(u)u^{-1}b) = \mu_0 v,\end{aligned}$$

where $v = (\frac{t^{p-1}}{p}x + \gamma_0(u)u^{-1}b)$ is a unit in $\Lambda_{F,0}$ because $\gamma_0(u)u^{-1}b$ is a unit and t^{p-1}/p is p-adically nilpotent in $\Lambda_{F,0}$. Hence, the lemma is proved. ∎

In the rest of this subsubsection, we will fix the choice of a topological generator γ_0 of $1 \times \Gamma_0$ such that $\chi(\gamma_0) = 1 + pa$, for a unit a in $\mathbb{Z}_p$. Let us now consider the following operator on $\tilde{\Lambda}_{R,0}$:

$$\begin{aligned}\nabla_{q,\tilde{s}} : \tilde{\Lambda}_{R,0} &\longrightarrow \tilde{\Lambda}_{R,0}\\ x &\mapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)\tilde{s}}.\end{aligned} \tag{3.35}$$

From the triviality of the action of $1 \times \Gamma_0$ on $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$ (see Lemma 3.55) and from Lemma 3.57, it follows that the operator $\nabla_{q,\tilde{s}}$ is well-defined. For each $n \in \mathbb{N}$, using Remark 3.56, the operator in (3.35), induces well-defined operators $\nabla_{q,\tilde{s}} : \tilde{\Lambda}_{R,0}/p_1(\mu)^n \longrightarrow \tilde{\Lambda}_{R,0}/p_1(\mu)^n$. In particular, for $n = 1$, set $s := \mu_0/p$ in $\Lambda_{R,0}$, then using Remark 3.56 and Lemma 3.58, we have a well-defined operator

$$\begin{aligned}\nabla_{q,s} : \Lambda_{R,0} &\longrightarrow \Lambda_{R,0}\\ x &\mapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)s}.\end{aligned} \tag{3.36}$$

Remark 3.59. Considering $\tilde{s}$ as a parameter, the operator $\nabla_{q,\tilde{s}}$ in (3.35), may be considered as a q-differential operator in non-logarithmic coordinates, in the sense of Definition 3.31 and Remark 3.32, where $q\Omega^1_{A/D}$ identifies with the $(p, p_1(\mu))$-adic completion of the module of Kähler differentials of $\tilde{\Lambda}_{R,0}$ with respect to $p_1 : A_R \to \tilde{\Lambda}_{R,0}$. Similarly, considering s as a parameter, the operator $\nabla_{q,s}$ in (3.36), may be also considered as a non-logarithmic q-differential operator in the sense of Definition 3.31 and Remark 3.32, where the $q\Omega^1_{A/D}$ identifies with the p-adic completion of the module of Kähler differentials of $\Lambda_{R,0}$ with respect to R.

For each $n \in \mathbb{N}$, the operator $\nabla_{q,\tilde{s}}$ is an endomorphism of $\tilde{\Lambda}_{R,0}/p_1(\mu)^n$, so we can define the following two term Koszul complex:

$$K_{\tilde{\Lambda}_{R,0}/p_1(\mu)^n}(\nabla_{q,\tilde{s}}) := [\tilde{\Lambda}_{R,0}/p_1(\mu)^n \xrightarrow{\nabla_{q,\tilde{s}}} \tilde{\Lambda}_{R,0}/p_1(\mu)^n]. \tag{3.37}$$

For $n = 1$, we have the following claim:

Lemma 3.60. *The cohomology of the Koszul complex $K_{\Lambda_{R,0}}(\nabla_{q,s})$ vanishes in degree 1, i.e. we have $H^1(K_{\Lambda_{R,0}}(\nabla_{q,s})) = 0$.*

Proof. The proof of this claim will be carried out in Proposition 4.24, where the claim will be shown, more generally, for finite $\Lambda_{R,0}$-modules admitting a continuous action of Γ_0 (trivial modulo μ_0), in particular, $\Lambda_{R,0}$ itself. We give a brief sketch of the main steps below.

Recall that, $s = \mu_0/p$ is the product of t^{p-1}/p with a unit in $\Lambda_{F,0}$ (see Construction 3.52, in particular, (3.30) and the discussion preceding it). Moreover, from Lemma 3.58 we have that $(\gamma_0 - 1)s$ is the product of μ_0 with a unit in $\Lambda_{F,0}$. Now let $z := t^{p-1}/p$, then in the proof of Proposition 4.16, it will be shown that $(\gamma_0 - 1)z$ is the product of μ_0 with a unit in $\Lambda_{F,0}$. Therefore, it follows that the complex $K_{\Lambda_{R,0}}(\nabla_{q,s})$ is quasi-isomorphic to the following complex

$$K_{\Lambda_{R,0}}(\nabla_{q,z}) := [\Lambda_{R,0} \xrightarrow{\nabla_{q,z}} \Lambda_{R,0}].$$

To show the vanishing of $H^1(K_{\Lambda_{R,0}}(\nabla_{q,z}))$, we switch from the q-differential operator to a differential operator. So, in Proposition 4.24, we show that $\nabla_0^{\log} := \frac{\log(\gamma_0)}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))}\sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_0-1)^{k+1}}{k+1}$, converge as a series of operators on $\Lambda_{R,0}$ and that $\nabla_0(-) \otimes dz : \Lambda_{R,0} \to \Omega^1_{\Lambda_{R,0}/R}$, coincides with the universal R-linear continuous de Rham differntial operator $d : \Lambda_{R,0} \to \Omega^1_{\Lambda_{R,0}/R}$. In particular, we have the following identification of complexes:

$$\Omega^\bullet_{\Lambda_{R,0}/R} = K_{\Lambda_{R,0}}(\nabla_0) := [\Lambda_{R,0} \xrightarrow{\nabla_0} \Lambda_{R,0}].$$

Recall that $\Lambda_{R,0} = R[z^{[k]}, k \in \mathbb{N}]^\wedge_p$ (see (3.30)), therefore it follows that the de Rham complex $\Omega^\bullet_{\Lambda_{R,0}/R} = K_{\Lambda_{R,0}}(\nabla_0)$ is acyclic in positive degrees and we have that $R \xrightarrow{\sim} \Lambda_{R,0}^{\nabla_0=0}$. Furthermore, in Proposition 4.24, we show a natural quasi-isomorphism of complexes

$$K_{\Lambda_{R,0}}(\nabla_{q,z}) \xrightarrow{\sim} K_{\Lambda_{R,0}}(\nabla_0).$$

Hence, we conclude that $H^1(K_{\Lambda_{R,0}}(\nabla_{q,s})) \xrightarrow{\sim} H^1(K_{\Lambda_{R,0}}(\nabla_0)) = 0$. ∎

Proof of Lemma 3.53. Using the $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism of rings from (3.28), the exact sequence in (3.26) can be rewritten as follows:

$$0 \longrightarrow \Lambda_{R,0} \xrightarrow{p_1(\mu)^n} \tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1} \longrightarrow \tilde{\Lambda}_{R,0}/p_1(\mu)^n \longrightarrow 0.$$

Then, using the operator $\nabla_{q,\tilde{s}}$ in (3.35) and the Koszul complex defined in (3.37), we obtain an exact sequence of Koszul complexes:

$$0 \longrightarrow K_{\Lambda_{R,0}}(\nabla_{q,s}) \xrightarrow{p_1(\mu)^n} K_{\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1}}(\nabla_{q,\tilde{s}}) \longrightarrow K_{\tilde{\Lambda}_{R,0}/p_1(\mu)^n}(\nabla_{q,\tilde{s}}) \longrightarrow 0.$$

Considering the associated long exact sequence, and noting that $H^1(K_{\Lambda_{R,0}}(\nabla_{q,s})) = 0$ from Lemma 3.60, we obtain the following exact sequence:

$$0 \longrightarrow \Lambda_{R,0}^{\nabla_{q,s}=0} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{s}}=0} \longrightarrow (\tilde{\Lambda}_{R,0}/p_1(\mu)^n)^{\nabla_{q,\tilde{s}}=0} \longrightarrow 0.$$

Since the action of $1 \times \Gamma_0$ on $\tilde{\Lambda}_{R,0}$ is continuous for the $(p, p_1(\mu))$-adic topology, therefore, we that $(\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{s}}=0} = (\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1})^{1\times\Gamma_0}$, for each $n \in \mathbb{N}$. Hence, from the preceding exact sequence we obtain that the sequence in (3.32) is exact, therefore, the sequence (3.31) is also exact. ∎

3.4.4. The case $p = 2$. In this subsubsection, our goal is to prove a statement similar to Lemma 3.53, for $p = 2$. From (1.6), recall that Γ_F fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1,$$

where, we have $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$ and $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \{\pm 1\}$, as groups.

We will first look at the action of Γ_{tor} on $\tilde{\Lambda}_R$. Let σ denote a generator of Γ_{tor}. Then, from (A.2), recall that by setting $\tilde{\Lambda}_{R,+} := \{x \in \tilde{\Lambda}_R \text{ such that } \sigma(x) = x\}$ and $\tilde{\Lambda}_{R,-} := \{x \in \tilde{\Lambda}_R \text{ such that } \sigma(x) = -x\}$, we have a natural injective map of $\tilde{\Lambda}_{R,+}$-modules

$$\tilde{\Lambda}_{R,+} \oplus \tilde{\Lambda}_{R,-} \longrightarrow \tilde{\Lambda}_R, \tag{3.38}$$

given as $(x, y) \mapsto x + y$. Note that the action of $1 \times \Gamma_F$ on $\tilde{\Lambda}_R$, is continuous for the $(p, p_1(\mu))$-adic topology, so it follows that $\tilde{\Lambda}_{R,+}$ is a $(p, p_1(\mu))$-adically complete subring of $\tilde{\Lambda}_R$ stable under the action of $(\varphi, \Gamma_R \times \Gamma_F)$ on $\tilde{\Lambda}_R$ and similarly, $\tilde{\Lambda}_{R,-}$ is a complete $\tilde{\Lambda}_{R,+}$-submodule stable under the action of $(\varphi, \Gamma_R \times \Gamma_F)$. Equipping $\tilde{\Lambda}_{R,+}$ and $\tilde{\Lambda}_{R,-}$ with induced structures, we see that (3.38) is $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant.

Now, from (3.23), recall that $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$. Similar to above, by setting $\Lambda_{R,+} := \{x \in \Lambda_R \text{ such that } \sigma(x) = x\}$ and $\Lambda_{R,-} := \{x \in \Lambda_R \text{ such that } \sigma(x) = -x\}$, from (A.2), we have a natural (φ, Γ_F)-equivariant injective map of $\Lambda_{R,+}$-modules

$$\Lambda_{R,+} \oplus \Lambda_{R,-} \longrightarrow \Lambda_R. \tag{3.39}$$

Lemma 3.61. *The natural map in (3.39) is bijective.*

Proof. Recall that $t = \log(1+\mu)$ is an element of Λ_F and t/μ is a unit in Λ_F (see Lemma 3.24), so we may write, $\Lambda_R = R[t, (t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = R[(t/2)^{[k]}, k \in \mathbb{N}]_p^\wedge$. Since $\sigma(t) = -t$, therefore it easily follows that $\Lambda_{R,+} = R[(t/2)^{[k]}, k \in 2\mathbb{N}]_p^\wedge$ and $\Lambda_{R,-} = R[(t/2)^{[k]}, k \in 2\mathbb{N}+1]_p^\wedge$. Then it is immediate that $\Lambda_R = \Lambda_{R,+} \oplus \Lambda_{R,-}$. Hence, (3.39) is bijective. ∎

Next, we will give another description of $\Lambda_{R,+}$ and $\Lambda_{R,-}$. Let us consider the following element in A_F from [Fon94, Subsection 5.2.5]:

$$\nu := q - 1 + \sigma(q-1) = q + q^{-1} - 2 = \tfrac{(q-1)^2}{q} = \tfrac{\mu^2}{1+\mu}. \tag{3.40}$$

Using Lemma 3.24, note that the element ν is the product of t^2 with a unit in Λ_F. Let $\tau := \nu/8$ and we claim the following:

Lemma 3.62. *The element τ is the product of $t^2/8$ with a unit in $\Lambda_{F,+}$, i.e. we have $\tau = ut^2/8$, for some unit u in $\Lambda_{F,+}$. In particular, we have*

$$\begin{aligned} \Lambda_{R,+} &= R[(t^2/8)^{[k]}, k \in \mathbb{N}]_p^\wedge = R[\tau^{[k]}, k \in \mathbb{N}]_p^\wedge, \\ \Lambda_{R,-} &= R[(t/2)(t^2/8)^{[k]}, k \in \mathbb{N}]_p^\wedge = R[(t/2)\tau^{[k]}, k \in \mathbb{N}]_p^\wedge = (t/2)\Lambda_{R,+}, \end{aligned} \tag{3.41}$$

Proof. Recall that $\sigma(t) = -t$, so $t^2/8$ is an element of $\Lambda_{F,+}$. Moreover, $\nu = ut^2$ for some unit u in Λ_F, so τ is an element of Λ_F. Since, $\sigma(\nu) = \nu$, it follows that $\tau = \nu/8$ belongs to $\Lambda_{F,+}$, and therefore, u is a unit in $\Lambda_{F,+}$. Next, from the proof of Lemma 3.61, note that $\Lambda_{R,+} = R[(t/2)^{[n]}, n \in 2\mathbb{N}]_p^\wedge$ and $\Lambda_{R,-} = R[(t/2)(t/2)^{[n]}, n \in 2\mathbb{N}]_p^\wedge$. Now, let $n = 2k$ for $k \in \mathbb{N}$, and note that

$$(\tfrac{t}{2})^{[n]} = \tfrac{t^{2k}}{(2k)!4^k} = \tfrac{k!2^k}{(2k)!}(\tfrac{t^2}{8})^{[k]} = \tfrac{k!2^k}{(2k)!u^k}\tau^{[k]}.$$

The equalities in (3.41) now follow since an easy computation shows that we have $v_2(2^k) + v_2(k!) = v_2((2k)!)$. ∎

Let us now consider a lifting of the element $\tau = \nu/8$ to $\tilde{\Lambda}_R$, under the surjective map $\tilde{\Lambda}_R \to \tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$, where the isomorphism is the (φ, Γ_F)-equivariant isomorphism in (3.23). We have the following element in $\tilde{\Lambda}_R$:

$$\tilde{\tau} := \tfrac{1}{p_2(q)}\delta\big(\tfrac{p_2([p]_q)}{p_1([p]_q)}\big) = \tfrac{1}{1\otimes q}\delta\big(\tfrac{1\otimes[p]_q}{[p]_q\otimes 1}\big) = \tfrac{1}{1\otimes q}\delta\big(\tfrac{1\otimes\tilde{p}}{\tilde{p}\otimes 1}\big), \tag{3.42}$$

where the last equality follows from Remark 3.2.

Lemma 3.63. *The element $\tilde{\tau}$ belongs to $\tilde{\Lambda}_{R,+}$ and we have $\tilde{\tau} = \tau$ mod $p_1(\mu)\tilde{\Lambda}_R$.*

Proof. Let σ be a generator of $1 \times \Gamma_{\mathrm{tor}}$ and note that $\sigma(1 \otimes q) = 1 \otimes q^{-1}$. Moreover, since the action of $1 \times \Gamma_R$ commutes with the δ-structure on $\tilde{\Lambda}_R$, we have

$$\sigma(\tilde{\tau}) = \tfrac{1}{\sigma(1\otimes q)}\delta\big(\tfrac{\sigma(1\otimes[p]_q)}{[p]_q\otimes 1}\big) = (1\otimes q)\delta\big(\tfrac{1}{1\otimes q}\tfrac{1\otimes[p]_q}{[p]_q\otimes 1}\big) = \tfrac{1}{1\otimes q}\delta\big(\tfrac{1\otimes[p]_q}{[p]_q\otimes 1}\big) = \tilde{\tau},$$

where the third equality follows from using the product formula for δ-structure from (2.1) and the fact that $\delta(1 \otimes q) = 0$. Therefore, $\tilde{\tau}$ is an element of $\tilde{\Lambda}_{R,+}$. Next, since the isomorphism $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$

in (3.23) is compatible with the respective Frobenii, in particular, with respective δ-structures, we have the following:

$$\tilde{\tau} \text{ mod } p_1(\mu) = \tfrac{1}{p_2(q)}\delta\big(\tfrac{p_2([p]_q)}{p_1([p]_q)}\big) \text{ mod } p_1(q-1) = \tfrac{1}{q}\delta\big(\tfrac{[p]_q}{2}\big) = \tfrac{(q-1)^2}{8q} = \tfrac{\nu}{8} = \tau.$$

This proves the second claim. ∎

Lemma 3.64. *For each $n \geq 1$, reduction modulo $p_1(\mu)^n$ of (3.38), induces a natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism*

$$\tilde{\Lambda}_{R,+}/p_1(\mu)^n \xrightarrow{\sim} (\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}}. \tag{3.43}$$

Moreover, for $n = 1$, the (φ, Γ_F)-equivariant isomorphism $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$ from (3.23), induces a natural (φ, Γ_F)-equivariant isomorphism

$$\tilde{\Lambda}_{R,+}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,+}. \tag{3.44}$$

Proof. Let us consider the following natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \tilde{\Lambda}_{R,+}/p_1(\mu) \arrow[r, "p_1(\mu)^n"] \arrow[d, "(3.38)"] & \tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1} \arrow[r] \arrow[d, "(3.38)"] & \tilde{\Lambda}_{R,+}/p_1(\mu)^n \arrow[r] \arrow[d, "(3.38)"] & 0 \\
0 \arrow[r] & (\tilde{\Lambda}_R/p_1(\mu))^{1\times\Gamma_{\mathrm{tor}}} \arrow[r, "p_1(\mu)^n"] & (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1\times\Gamma_{\mathrm{tor}}} \arrow[r] & (\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}},
\end{tikzcd} \tag{3.45}$$

where the vertical maps are injective because we have $p_1(\mu)^n\tilde{\Lambda}_R \cap \tilde{\Lambda}_{R,+} = p_1(\mu)^n\tilde{\Lambda}_{R,+}$, as $p_1(\mu)$ is invariant under the action of $1\times\Gamma_F$. Composing the left vertical arrow in (3.45) with the (φ, Γ_F)-equivariant isomorphism $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$ from (3.23), we obtain a natural (φ, Γ_F)-equivariant injective map $\tilde{\Lambda}_{R,+}/p_1(\mu) \to \Lambda_{R,+}$, and we claim that it is surjective as well. Indeed, note that $\tilde{\Lambda}_{R,+}/p_1(\mu)$ is a p-torsion free ring equipped with an induced Frobenius, in particular, a δ-structure compatible and the left vertical map in (3.45) is compatible withe the respective δ-structures. Now, from (3.41) in Lemma 3.62, recall that we have $\Lambda_{R,+} = R[\tau^{[k]}, k \in \mathbb{N}]^{\wedge}_p$. If we again denote by τ, its preimage under the isomorphism (3.23), then from Lemma 3.63 we have that τ is an element of $\tilde{\Lambda}_{R,+}/p_1(\mu)$ and we need to show that $\tau^{[k]}$ belongs to $\tilde{\Lambda}_{R,+}/p_1(\mu)$, for each $k \in \mathbb{N}$. Moreover, using [BS22, Lemma 2.35], we see that it is enough to show that $\frac{\tau^2}{2}$ is an element of $\tilde{\Lambda}_{R,+}/p_1(\mu)$. Since, $\tilde{\Lambda}_{R,+}/p_1(\mu)$ is a δ-ring, we have the following:

$$\delta(\tau) = \delta\big(\tfrac{(q-1)^2}{8q}\big) = \tfrac{1}{2q^2}\big(\tfrac{(q^2-1)^2}{8} - \tfrac{(q-1)^4}{64}\big) = \tfrac{1}{2q}\big(\tau(q+1)^2 - \tfrac{\tau^2}{q}\big).$$

As $[2]_q = q + 1$ can be written as the product of 2 with a unit in Λ_F (see Lemma 3.24), therefore, it follows that $\frac{\tau^2}{2} = \frac{q\tau(q+1)^2}{2} - q^2\delta(\tau)$ is an element of $\tilde{\Lambda}_{R,+}/p_1(\mu)$. Hence, we conclude that $\tilde{\Lambda}_{R,+}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,+}$, in particular, the composition in (3.44) and the left vertical arrow in (3.45) are bijective. Now, using the diagram (3.45), an easy induction on $n \geq 1$, gives that for each $n \geq 1$, the right vertical arrow is bijective and the bottom right horizontal arrow is surjective. Hence, it follows that the natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant map $\tilde{\Lambda}_{R,+}/p_1(\mu)^n \to (\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}}$, induced by (3.38), is bijective for each $n \geq 1$. ∎

Remark 3.65. Lemma 3.64 can be proved using an alternative method as in the proof of Lemma 5.23, where a crucial input is Lemma A.11. However, the proof given above is conceptually more satisfying.

From Lemma 3.64, we obtain the following:

Lemma 3.66. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant sequence is exact:*

$$0 \longrightarrow (\tilde{\Lambda}_R/p_1(\mu))^{1\times\Gamma_{\mathrm{tor}}} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1\times\Gamma_{\mathrm{tor}}} \longrightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}} \longrightarrow 0. \tag{3.46}$$

Proof. The sequence (3.46) is the same as the second row of the diagram (3.43), which was shown to be exact in the proof of Lemma 3.64. ■

Next, we will look at the action of $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + 4\mathbb{Z}_2)$ on $\tilde{\Lambda}_{R,+}$ and prove a result similar to Lemma 3.53, for $p = 2$. In particular, we will show the following:

Lemma 3.67. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times 1)$-equivariant sequence is exact:*

$$0 \longrightarrow (\tilde{\Lambda}_R/p_1(\mu))^{1\times\Gamma_F} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1\times\Gamma_F} \longrightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1\times\Gamma_F} \longrightarrow 0. \tag{3.47}$$

Remark 3.68. Via the natural $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism $\iota_{\tilde{\Lambda}}$ in Lemma 3.47 (see (3.24)), we see that the exact sequence in (3.47) is the invariants, under the natural action of $1 \times \Gamma_F$, of the exact sequence in (3.19) in Lemma 3.43, for $p = 2$.

Note that using the $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism in (3.43) and (3.44), the sequence in (3.47) can be rewritten as the following $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant sequence:

$$0 \longrightarrow \Lambda_{R,+}^{\Gamma_0} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1})^{1\times\Gamma_0} \longrightarrow (\tilde{\Lambda}_{R,+}/p_1(\mu)^n)^{1\times\Gamma_0} \longrightarrow 0. \tag{3.48}$$

In order to show that the squence (3.48) is exact, we start with the following observation:

Lemma 3.69. *The action of $1 \times \Gamma_0$ is trivial on $\tilde{\Lambda}_{R,+}/p_2(\nu)$ and the action of $\Gamma_R \times 1$ is trivial on $\tilde{\Lambda}_{R,+}/p_1(\mu)$.*

Proof. For the first claim, note that we have $p_2(\nu)\tilde{\Lambda}_R \cap \tilde{\Lambda}_{R,+} = p_2(\nu)\tilde{\Lambda}_{R,+}$. So, if x is an element of $\tilde{\Lambda}_{R,+}$ and g any element of $1 \times \Gamma_0$, then it is enough to show that $(g-1)x$ is an element of $p_2(\nu)\tilde{\Lambda}_R$. Moreover, recall that ν is the product of μ^2 with a unit in A_F. Therefore, we are reduced to showing that $(g-1)x$ is an element of $p_2(\mu)^2\tilde{\Lambda}_R$. Now, using Lemma 3.48, we can write $(g-1)x = p_2(\mu)y$, for some y in $\tilde{\Lambda}_R$. Let σ be a generator of $1 \times \Gamma_{\text{tor}}$ and note that $\sigma(x) = x$. Then, we have $\sigma(p_2(\mu))\sigma(y) = p_2(\mu)y$, in particular, $(\sigma-1)y = -(2+p_2(\mu))y$. Again, using Lemma 3.48, we can write $-p_2([p]_q)y = (\sigma-1)y = p_2(\mu)z$, for some z in $\tilde{\Lambda}_R$. So we get that $-py = 0 \bmod p_2(\mu)\tilde{\Lambda}_R$. Note that $(p_2(\mu), p)$ is a regular sequence on $\tilde{\Lambda}_R$ since $p_2 : A_F \to \tilde{\Lambda}_R$ is flat (see Construction 3.44). Therefore, we conclude that $y = 0 \bmod p_2(\mu)\tilde{\Lambda}_R$, i.e. y is an element of $p_2(\mu)\tilde{\Lambda}_R$ and $(g-1)x = p_2(\mu)y$ is an element of $p_2(\mu)^2\tilde{\Lambda}_R$, as desired. The second claim easily follows from Lemma 3.48. ■

Remark 3.70. From Lemma 3.69, note that the action of $1 \times \Gamma_0$ is trivial on $\tilde{\Lambda}_{R,+}/p_2(\nu)$ and the element $p_1(\mu)$ is invariant under this action. Therefore, it follows that for any g in $1 \times \Gamma_0$ and any x in $\tilde{\Lambda}_{R,+}/p_1(\mu)^n$ we have that $(g-1)x$ is an element of $p_2(\nu)\tilde{\Lambda}_{R,+}/p_1(\mu)^n$. In particular, for $n = 1$, using the isomorphism $\tilde{\Lambda}_{R,+}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,+}$ from (3.44), we get that for any g in Γ_0 and any x in $\Lambda_{R,+}$ the element $(g-1)x$ belongs to $\nu\Lambda_{R,+}$.

Using the action of $1 \times \Gamma_0$ on $\tilde{\Lambda}_{R,+}$, we will define a q-de Rham complex (see Definition 3.31). For such a definition we will treat the element $\tilde{\tau}$ in $\tilde{\Lambda}_{R,+}$ as a parameter. We start with the following observation:

Lemma 3.71. *Let γ_0 be any element of $1 \times \Gamma_0$. Then we have $(\gamma_0 - 1)\tilde{\tau} = u\,p_2(\nu)$, for some unit u in $\tilde{\Lambda}_{R,+}$.*

Proof. Note that it is enough to show the claim for a topological generator γ_0 of Γ_0 such that $\chi(\gamma_0) = 1 + 4a$, for a unit a in $\mathbb{Z}_2$. Let us set $v := \frac{p_2([p]_q)}{p_1([p]_q)}$, then we have $\tilde{\tau} = \frac{\delta(v)}{p_2(q)}$. Now observe that

$$\begin{aligned}(\gamma_0 - 1)\tilde{\tau} = (\gamma_0 - 1)\big(\tfrac{\delta(v)}{p_2(q)}\big) &= (\gamma_0 - 1)\big(\tfrac{1}{p_2(q)}\big)\delta(v) + \tfrac{(\gamma_0-1)\delta(v)}{\gamma_0(p_2(q))} \\ &= \tfrac{1}{p_2(q^5)}\big((\gamma_0 - 1)\delta(v) - (p_2(q^4) - 1)\delta(v)\big)\end{aligned}$$

Since δ and γ_0 commute with each other, an easy computation shows that

$$(\gamma_0 - 1)\delta(v) = \delta((\gamma_0 - 1)v) - v(\gamma_0 - 1)v$$
$$(p_2(q^4) - 1)\delta(v) = \delta((p_2(q^4) - 1)v) - v^2(p_2(q^2) - 1).$$

Then we note that

$$\begin{aligned} v(\gamma_0 - 1)v - v^2(p_2(q^2) - 1) &= v(p_2(q^2) - 1)(\tfrac{p_2(q^3)+p_2(q)}{p_1(q)+1} - v) = -v^2(p_2(q) - 1)(p_2(q^3) - 1) \\ &= -v^2 p_2(q)(p_2(q^2) + p_2(q) + 1)p_2(\nu) = p_2(\nu)x_1, \end{aligned}$$

for some x_1 in $\tilde{\Lambda}_R$ and $(\gamma_0 - 1)v = p_2(q)(p_2(q) - 1)(p_2(q^2) + 1)v$. Now, let $a = p_2(q)(p_2(q^2) + 1)$, $b = p_2(q) + 1$ and $c = (p_2(q) - 1)v$, and note that $a - b$, $\delta(a - b)$ and $\delta(c)$ are elements of $(p_2(q) - 1)\tilde{\Lambda}_R$, since the latter is a δ-stable ideal of $\tilde{\Lambda}_R$, in the sense of [BS22, Example 2.10]. So we obtain that

$$\begin{aligned} \delta((\gamma_0 - 1)v) - \delta((p_2(q^4) - 1)v) &= \delta(ac) - \delta(bc) = \delta(ac - bc) + b^2c^2 + abc^2 \\ &= \delta(a - b)c^2 + (a - b)^2\delta(c) + 2\delta(a - b)\delta(c) + b^2c^2 + abc^2 = p_2(\nu)x_2, \end{aligned}$$

for some x_2 in $\tilde{\Lambda}_R$, and in the third equality we have used that $\nu = \frac{(q-1)^2}{q}$ from (3.40). Set $u := \frac{x_1+x_2}{p_2(q^5)}$ in $\tilde{\Lambda}_R$, and by putting everything together, we have that

$$(\gamma_0 - 1)\tilde{\tau} = up_2(\nu). \tag{3.49}$$

Since $\sigma(\tilde{\tau}) = \tilde{\tau}$ (see Lemma 3.63), $\sigma(\nu) = \nu$, the group $1 \times \Gamma_F$ is commutative and $\tilde{\Lambda}_{R,+}$ is $p_2(\nu)$-torsion free, therefore, we get that u is an element of $\tilde{\Lambda}_{R,+}$. So to show the claim, it is enough to show that u is a unit in $\tilde{\Lambda}_R$. Now, note that from the discussion after (3.38), the ring $\tilde{\Lambda}_{R,+}$ is $p_1(\mu)$-adically complete and we have a (φ, Γ_0)-equivariant isomorphism $\tilde{\Lambda}_{R,+}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,+}$ (see (3.44) in Lemma 3.64), therefore, we are reduced to showing that $\overline{u}$, the image of u under the preceding isomorphism, is a unit. By reducing the equalities in (3.49), modulo $p_1(\mu)$, we obtain the following expression in $\Lambda_{R,+}$:

$$(\gamma_0 - 1)\tau = (\gamma_0 - 1)\tfrac{\nu}{8} = \overline{u}\nu.$$

But from Lemma 3.58, we see that $\overline{u}$ must be a unit in $\Lambda_{R,+}$. Hence, u is a unit in $\tilde{\Lambda}_{R,+}$, proving the claim. ∎

The following observation was used above:

Lemma 3.72. *Let γ_0 be a topological generator of Γ_0 such that $\chi(\gamma_0) = 1 + 4a$, for a unit a in $\mathbb{Z}_2$. Then we have $(\gamma_0 - 1)\tau = (\gamma_0 - 1)\frac{\nu}{8} = u\nu$, for some unit u in $\Lambda_{F,+} = \Lambda_F^{\Gamma_{\mathrm{tor}}}$.*

Proof. From Lemma 3.62, recall that $\nu/8$ is the product of $t^2/8$ with a unit in $\Lambda_{F,+}$. So let us write $\nu/8 = et^2/8$, for some unit e in $\Lambda_{F,+}$. Note that, from Remark 3.70, we have that $(\gamma_0 - 1)e = \nu x$, for some x in $\Lambda_{F,+}$. Therefore, we can write

$$\begin{aligned} (\gamma_0 - 1)\tfrac{\nu}{8} = (\gamma_0 - 1)\tfrac{et^2}{8} &= \tfrac{t^2}{8}(\gamma_0 - 1)e + \gamma_0(e)(\gamma_0 - 1)\tfrac{t^2}{8} \\ &= \tfrac{t^2}{8}\nu x + \gamma_0(e)(\chi(\gamma_0)^2 - 1)\tfrac{t^2}{8} = \nu(\tfrac{t^2}{8}x + \gamma_0(e)e^{-1}a(2a + 1)) = \nu u, \end{aligned}$$

where $u = (\frac{t^2}{8}x + \gamma_0(e)e^{-1}a(2a + 1))$ is a unit in $\Lambda_{F,+}$ because $\gamma_0(e)e^{-1}a(2a + 1)$ is a unit and $t^2/8$ is p-adically nilpotent in $\Lambda_{F,+}$. Hence, the lemma is proved. ∎

In the rest of this subsubsection, we will fix a topological generator γ_0 of $1 \times \Gamma_0$ such that $\chi(\gamma_0) = 1 + 4a$, for a unit a in $\mathbb{Z}_2$. Similar to (3.35), let us now consider the following operator on $\tilde{\Lambda}_{R,+}$:

$$\begin{aligned} \nabla_{q,\tilde{\tau}} : \tilde{\Lambda}_{R,+} &\longrightarrow \tilde{\Lambda}_{R,+} \\ x &\mapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)\tilde{\tau}}. \end{aligned} \tag{3.50}$$

From the triviality of the action of $1 \times \Gamma_0$ on $\tilde{\Lambda}_{R,+}/p_2(\nu)$ (see Lemma 3.69) and from Lemma 3.71, it follows that the operator $\nabla_{q,\tilde{\tau}}$ is well-defined. For each $n \in \mathbb{N}$, using Remark 3.70, the operator in (3.50), induces well-defined operators $\nabla_{q,\tilde{\tau}} : \tilde{\Lambda}_{R,+}/p_1(\mu)^n \longrightarrow \tilde{\Lambda}_{R,+}/p_1(\mu)^n$. In particular, for $n = 1$, we have $\tau = \nu/8$ in $\Lambda_{R,0}$, and using Remark 3.70 and Lemma 3.72, we have a well-defined operator

$$\begin{aligned} \nabla_{q,\tau} : \Lambda_{R,+} &\longrightarrow \Lambda_{R,+} \\ x &\mapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)\tau}. \end{aligned} \tag{3.51}$$

Remark 3.73. Considering $\tilde{\tau}$ as a variable, the operator $\nabla_{q,\tilde{\tau}}$ in (3.35), may be considered as a q-differential operator in non-logarithmic coordinates, in the sense of Definition 3.31 and Remark 3.32, where $q\Omega^1_{A/D}$ identifies with the $(p, p_1(\mu))$-adic completion of the module of Kähler differentials of $\tilde{\Lambda}_{R,+}$ with respect to $p_1 : A_R \to \tilde{\Lambda}_{R,+}$. Similarly, considering τ as a variable, the operator $\nabla_{q,\tau}$ in (3.51), may be also considered as a non-logarithmic q-differential operator in the sense of Definition 3.31 and Remark 3.32, where the $q\Omega^1_{A/D}$ identifies with the p-adic completion of the module of Kähler differentials of $\Lambda_{R,+}$ with respect to R.

For each $n \in \mathbb{N}$, the operator $\nabla_{q,\tilde{\tau}}$ is an endomorphism of $\tilde{\Lambda}_{R,+}/p_1(\mu)^n$, so we can define the following two term Koszul complex:

$$K_{\tilde{\Lambda}_{R,+}/p_1(\mu)^n}(\nabla_{q,\tilde{\tau}}) : [\tilde{\Lambda}_{R,+}/p_1(\mu)^n \xrightarrow{\nabla_{q,\tilde{\tau}}} \tilde{\Lambda}_{R,+}/p_1(\mu)^n]. \tag{3.52}$$

For $n = 1$, we have the following claim:

Lemma 3.74. *The cohomology of the Koszul complex $K_{\Lambda_{R,+}}(\nabla_{q,\tau})$ vanishes in degree 1, i.e. we have $H^1(K_{\Lambda_{R,+}}(\nabla_{q,\tau})) = 0$.*

Proof. The idea of the proof is similar to the proof of Lemma 3.60, with slightly different computations. Main arguments for the proof of the claim will be given in Proposition 4.24, where the claim will be shown, more generally, for finite $\Lambda_{R,0}$-modules admitting a continuous action of Γ_0 (trivial modulo μ_0), in particular, $\Lambda_{R,0}$ itself. We give a brief sketch of the main steps below.

Recall that $\tau = \nu/8$ is the product of $t^2/8$ with a unit in $\Lambda_{F,+}$ (see Lemma 3.62). Moreover, from Lemma 3.72 we have that $(\gamma_0 - 1)\tau$ is the product of ν with a unit in $\Lambda_{F,+}$. Now let $w := t^2/8$, then from the proof of Proposition 4.30, we note that $(\gamma_0 - 1)w$ is the product of ν with a unit in $\Lambda_{F,+}$. Therefore, it follows that the complex $K_{\Lambda_{R,+}}(\nabla_{q,\tau})$ is quasi-isomorphic to the following complex

$$K_{\Lambda_{R,+}}(\nabla_{q,w}) : [\Lambda_{R,+} \xrightarrow{\nabla_{q,w}} \Lambda_{R,+}].$$

To show the vanishing of $H^1(K_{\Lambda_{R,+}}(\nabla_{q,w}))$, we switch from the q-differential operator to a differential operator. So, in Proposition 4.30 we show that $\nabla_0^{\log} := \frac{\log(\gamma_0)}{\log(\chi(\gamma_0))}$, converge as a series of operators on $\Lambda_{R,+}$ and $\nabla_0(-) \otimes dz : \Lambda_{R,+} \to \Omega^1_{\Lambda_{R,+}/R}$, coincides with the universal R-linear continuous de Rham differntial operator $d : \Lambda_{R,+} \to \Omega^1_{\Lambda_{R,+}/R}$. In particular, we obtain the following identification of complexes:

$$\Omega^\bullet_{\Lambda_{R,+}/R} = K_{\Lambda_{R,+}}(\nabla_0) : [\Lambda_{R,+} \xrightarrow{\nabla_0} \Lambda_{R,+}].$$

Recall that $\Lambda_{R,+} = R[w^{[k]}, k \in \mathbb{N}]^\wedge_p$ (see Lemma 3.62), therefore it follows that the de Rham complex $\Omega^\bullet_{\Lambda_{R,+}/R} = K_{\Lambda_{R,+}}(\nabla_0)$ is acyclic in positive degrees and we have that $R \xrightarrow{\sim} \Lambda_{R,+}^{\nabla_0=0}$. Furthermore, in Proposition 4.30, we show a natural quasi-isomorphism of complexes

$$K_{\Lambda_{R,+}}(\nabla_{q,w}) \xrightarrow{\sim} K_{\Lambda_{R,+}}(\nabla_0).$$

Hence, we conclude that $H^1(K_{\Lambda_{R,+}}(\nabla_{q,\tau})) \xrightarrow{\sim} H^1(K_{\Lambda_{R,+}}(\nabla_0)) = 0$. ∎

Proof of Lemma 3.67. The proof is similar to the proof of Lemma 3.53. Using the $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism in (3.43), the exact sequence in (3.46) can be rewritten as follows:

$$0 \longrightarrow \Lambda_{R,+} \xrightarrow{p_1(\mu)^n} \tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1} \longrightarrow \tilde{\Lambda}_{R,+}/p_1(\mu)^n \longrightarrow 0.$$

Then, using the operator $\nabla_{q,\tilde{\tau}}$ in (3.50) and the Koszul complex defined in (3.52), we obtain an exact sequence of Koszul complexes:

$$0 \longrightarrow K_{\Lambda_{R,+}}(\nabla_{q,\tau}) \xrightarrow{p_1(\mu)^n} K_{\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1}}(\nabla_{q,\tilde{\tau}}) \longrightarrow K_{\tilde{\Lambda}_{R,+}/p_1(\mu)^n}(\nabla_{q,\tilde{\tau}}) \longrightarrow 0.$$

Considering the associated long exact sequence, and noting that $H^1(K_{\Lambda_{R,+}}(\nabla_{q,\tau})) = 0$ from Lemma 3.74, we obtain the following exact sequence:

$$0 \longrightarrow \Lambda_{R,+}^{\nabla_{q,\tau}=0} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{\tau}}=0} \longrightarrow (\tilde{\Lambda}_{R,+}/p_1(\mu)^n)^{\nabla_{q,\tilde{\tau}}=0} \longrightarrow 0.$$

Since the action of $1 \times \Gamma_0$ is continuous on $\tilde{\Lambda}_{R,+}$ for the $(p, p_1(\mu))$-adic topology, therefore, we have $(\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{\tau}}=0} = (\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1})^{1\times\Gamma_0}$ for each $n \in \mathbb{N}$. Hence, from the preceding exact sequence we obtain that the sequence in (3.48) is exact, therefore, the sequence (3.47) is also exact. ∎

3.4.5. Proof of Proposition 3.41. Note that from the explicit description of $A_R(1)/p_1(\mu)$ in Proposition 3.25 it is easy to see that reduction modulo μ of $p_1 : A_R \to A_R(1)$ induces an isomorphism $p_1 : R \xrightarrow{\sim} (A_R(1)/p_1(\mu))^{1\times\Gamma_R}$. More generally, we have the following:

Lemma 3.75. *For $n \in \mathbb{N}_{\geq 1}$, reduction modulo μ^n of the (φ, Γ_R^2)-equivariant map $p_1 : A_R \to A_R(1)$ induces a $(\varphi, \Gamma_R \times 1)$-equivariant isomorphism $p_1 : A_R/\mu^n \xrightarrow{\sim} (A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R}$.*

Proof. Note that p_1 is (φ, Γ_R^2)-equivariant, so it is enough to show the bijectivity of the map modulo μ^n. By considering the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & A_R/\mu \arrow[r, "\mu^n"] \arrow[d, "p_1", "\wr"'] & A_R/\mu^{n+1} \arrow[r] \arrow[d, "p_1"] & A_R/\mu^n \arrow[r] \arrow[d, "p_1"] & 0 \\
0 \arrow[r] & (A_R(1)/p_1(\mu))^{1\times\Gamma_R} \arrow[r, "p_1(\mu)^n"] & (A_R/p_1(\mu)^{n+1})^{1\times\Gamma_R} \arrow[r] & (A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R}, &
\end{tikzcd} \tag{3.53}$$

an easy induction on $n \geq 1$ gives the claimed $(\varphi, \Gamma_R \times 1)$-equivariant isomorphism $p_1 : A_R/\mu^{n+1} \xrightarrow{\sim} (A_R(1)/p_1(\mu)^{n+1})^{1\times\Gamma_R}$. ∎

Remark 3.76. From the description of the action of Γ_R^3 on $A_R(2)$ in Remark 3.20, we note that there is an induced action of Γ_R^3 on $A_R(2)/p_1(\mu)$, where the action of the first component is identity. Moreover, we have a (φ, Γ_R^3)-equivariant map $r_1 : A_R \to A_R(2)$, where A_R is equipped with an action of Γ_R^3 via projection on to the first coordinate. Then similar to above it can easily be shown that reduction modulo μ of $r_1 : A_R \to A_R(2)$ induces an isomorphism $r_1 : R \xrightarrow{\sim} (A_R(2)/p_1(\mu))^{1\times\Gamma_R\times\Gamma_R}$.

Now, recall that we have the φ-equivariant multiplication map $\Delta : A_R(1) \to A_R$. The map Δ induces an A_R/μ^n-linear (via p_1) and φ-equivariant maps $\Delta : A_R(1)/p_1(\mu)^n \to A_R/\mu^n$ for $n \in \mathbb{N}_{\geq 1}$. For $n = 1$, using Lemma 3.75, the map Δ restricts to a φ-equivariant isomorphism $(A_R(1)/p_1(\mu))^{1\times\Gamma_R} \xrightarrow{\sim} R$. More generally, we have the following:

Lemma 3.77. *For $n \in \mathbb{N}_{\geq 1}$, reduction modulo $p_1(\mu)^n$ of the φ-equivariant homomorphism $\Delta : A_R(1) \to A_R$ restricts to a φ-equivariant isomorphism $(A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R} \xrightarrow{\sim} A_R/\mu^n$.*

Proof. Let us first note that using Lemma 3.43, together with Lemma 3.51, Lemma 3.53 and Remark 3.54 for $p \geq 3$ and Lemma 3.66, Lemma 3.67 and Remark 3.68 for $p = 2$, we obtain that for each $n \geq 1$, the following $(\varphi, \Gamma_R \times 1)$-equivariant sequence is exact:

$$0 \longrightarrow (A_R(1)/p_1(\mu))^{1\times\Gamma_R} \xrightarrow{p_1(\mu)^n} (A_R(1)/p_1(\mu)^{n+1})^{1\times\Gamma_R} \longrightarrow (A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R} \longrightarrow 0. \tag{3.54}$$

Next, we note that Δ is φ-equivariant, so it is enough to show that the map modulo $p_1(\mu)^n$ is bijective. Now, consider the following natural diagram:

$$\begin{tikzcd} 0 \arrow[r] & (A_R(1)/p_1(\mu))^{1\times\Gamma_R} \arrow[r, "p_1(\mu)^n"] \arrow[d, "\Delta", "\wr"'] & (A_R/p_1(\mu)^{n+1})^{1\times\Gamma_R} \arrow[r] \arrow[d, "\Delta"] & (A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R} \arrow[r] \arrow[d, "\Delta"] & 0 \\ 0 \arrow[r] & A_R/\mu \arrow[r, "\mu^n"] & A_R/\mu^{n+1} \arrow[r] & A_R/\mu^n \arrow[r] & 0, \end{tikzcd}$$

where the top row is the exact sequence in (3.54). Using the diagram, an easy induction on $n \geq 1$, gives the φ-equivariant isomorphism $\Delta : (A_R(1)/p_1(\mu)^{n+1})^{1\times\Gamma_R} \xrightarrow{\sim} A_R/\mu^{n+1}$. ■

Finally, recall that the ring $A_R(1)$ is equipped with an action of (φ, Γ_R^2), and the rings $A_R(1)^{1\times\Gamma_R}$ and $(A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R}$ are equipped with a residual action of $\Gamma_R = \Gamma_R \times 1 \subset \Gamma_R^2$. Then the following observation proves Proposition 3.41:

Lemma 3.78. *For $n \in \mathbb{N}_{\geq 1}$ the isomorphism $(A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R} \xrightarrow{\sim} A_R/\mu^n$ of Proposition 3.41, is compatible with (φ, Γ_R)-action. Passing to the limit over n gives a (φ, Γ_R)-equivariant isomorphism $A_R(1)^{1\times\Gamma_R} \xrightarrow{\sim} A_R$.*

Proof. The isomorphism in Proposition 3.41 is φ-equivariant. To check Γ_R-equivariance note that if g is in Γ_R and a in $A_R(1)/p_1(\mu)^n$, then we have that $\Delta((g, g)a) = g(a)$. So if a is $(1 \times \Gamma_R)$-invariant, then for g_1, g_2 in Γ_R, we have that $\Delta((g_1, g_2)f) = \Delta((g_1, g_1)f) = g_1(\Delta(f))$. This proves the first claim. Next, as inverse limit commutes with right adjoint functors, in particular, with taking $(1 \times \Gamma_R)$-invariants, therefore, it follows that we have a (φ, Γ_R)-equivariant isomorphism $A_R(1)^{1\times\Gamma_R} = (\lim_n A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R} = \lim_n (A_R(1)/p_1(\mu)^n)^{1\times\Gamma_R} \xrightarrow{\sim} \lim_n A_R/\mu^n = A_R$. This proves the second claim. ■

4. An integral comparison isomorphism

In this section, we will prove an integral comparison isomorphism for Wach modules, which will be the most important input in building a stratification on Wach modules in Subsection 5.2.5. We will use the setup and notations of Subsection 1.6.

Definition 4.1 (Wach modules, [Abh23b, Definition 1.3, Lemma 3.10])**.** A *Wach module* over A_R is a finitely generated A_R-module N equipped with a semilinear action of Γ_R satisfying the following:

(1) The sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N.

(2) The action of Γ_R is trivial on $N/\mu N$.

(3) N is equipped with a Frobenius of finite $[p]_q$-height, i.e. an A_R-linear and Γ_R-equivariant isomorphism $\varphi_N : (\varphi^* N)[1/[p]_q] = (A_R \otimes_{\varphi, A_R} N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$.

Say that N is *effective* if φ_N carries $\varphi^* N$ into N. Denote the category of Wach modules over A_R as $(\varphi, \Gamma)\text{-Mod}_{A_R}^{[p]_q}$ with morphisms between objects being A_R-linear, Γ_R-equivariant and φ_N-equivariant (after inverting $[p]_q$) morphisms.

Remark 4.2. Note that the action of Γ_R is automatically continuous on N for the (p, μ)-adic topology (see [Abh23b, Lemma 3.7]). Moreover, from [Abh23b, Proposition 3.11], note that for a Wach module N over A_R, the $A_R[1/p]$-module $N[1/p]$ is finite projective, the $A_R[1/\mu]$-module $N[1/\mu]$ is finite projective and by [Abh23b, Remark 3.12] the $A_R[1/[p]_q]$-module $N[1/[p]_q]$ is finite projective. Furthermore, from loc. cit., the sequences $\{p, [p]_q\}$ and $\{[p]_q, p\}$ are regular on N and equivalent to condition (1) in Definition 4.1.

Remark 4.3. For $R = O_F$, a Wach module over A_F is necessarily finite free (see [Abh23b, Remark 1.6]).

From Remark 3.20 recall that we have a (φ, Γ_R^2)-equivariant maps $p_i : A_R \to A_R(1)$ for $i = 1, 2$, where A_R is equipped with a Γ_R^2-action via projection onto the i^{th} coordinate. Moreover, we note that there is an induced action of Γ_R^2 on $A_R(1)/p_1(\mu)$, where the action of the first component is identity. In this section, we will identify Γ_R with $1 \times \Gamma_R$ and say that $A_R(1)/p_1(\mu)$ is equipped with a natural continuous action of Γ_R.

Remark 4.4. Let N a be Wach module over A_R, then extending the isomorphism in Definition 4.1 (3) along the $(\varphi, 1 \times \Gamma_R)$-equivariant map $p_2 : A_R \to A_R(1)/p_1(\mu)$, we obtain an isomorphism $(A_R(1)/p_1(\mu) \otimes_{\varphi, A_R} N)[1/[p]_q] \xrightarrow{\sim} (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)[1/[p]_q]$. Now note that $p_1([p]_q) = p \mod p_1(\mu)A_R(1)$ and $p_1([p]_q)/p_2([p]_q)$ is a unit in $A_R(1)$ (see Lemma 3.15). So, by setting $M := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\Gamma_R}$, and taking Γ_R-invariants in the preceding isomorphism gives an $R[1/p]$-linear isomorphism $(\varphi^* M)[1/p] = (R \otimes_{\varphi, R} M)[1/p] \xrightarrow{\sim} M[1/p]$.

Notation. In this section, by the φ-equivariance of a morphism we always mean φ-equivariance after inverting p. However, we will not always mention this explicitly.

The goal of this section is to prove Theorem 4.5 below, which is an important ingredient for the proof of Theorem 5.12.

Theorem 4.5. *Let N be Wach module over A_R and set $M := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\Gamma_R}$ as an R-module equipped with tensor product Frobenius. Then we have a natural (φ, Γ_R)-equivariant isomorphism*

$$\begin{aligned} A_R(1)/p_1(\mu) \otimes_{p_1, R} M &\xrightarrow{\sim} A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N \\ a \otimes b \otimes x &\longmapsto ab \otimes x. \end{aligned} \tag{4.1}$$

Moreover, M is a finitely generated p-torsion free R-module and we have a φ-equivariant isomorphism of R-modules $M \xrightarrow{\sim} N/\mu N$.

Remark 4.6. For $R = O_F$, the R-module M in Theorem 4.5 is finite free of rank $= \mathrm{rk}_{A_F} N$.

In Subsections 4.2 and 4.3, we will prove the comparison isomorphism (4.1) claimed in Theorem 4.5. Our proof is broadly divided into three main steps: a geometric descent for the action of Γ_R' (see Proposition 4.17) and a "two-step" arithmetic descent for the action of Γ_F (see Propositions 4.22 and 4.25, for $p \geq 3$, and Propositions 4.26 and 4.31, for $p = 2$). Then in Subsection 4.4, we will put everything together to complete the proof of Theorem 4.5. We begin by interpreting the action of Γ_R on a Wach module as a q-connection.

4.1. Wach modules and q-connections.

In this subsection, we will interpret Wach modules and its scalar extension to $A_R(1)/p_1(\mu)$ as modules with q-connections similar to [Abh23b, Section 5]. We will work with the notation described in Subsection 3.3.3.

Definition 4.7 (q-connection, [MT20, Definition 2.2])**.** A module with *q-connection* over D is a right D-module N equipped with an A-linear map $\nabla_q : N \to N \otimes q\Omega^1_{D/A}$ satisfying the Leibniz rule $\nabla_q(xf) = \nabla_q(x)f + x \otimes d_q(f)$ for all f in D and x in N. The q-connection ∇_q extends uniquely to a map of graded A-modules $\nabla_q : N \otimes q\Omega^\bullet_{D/A} \to N \otimes q\Omega^{\bullet+1}_{D/A}$ satisfying $\nabla_q((n \otimes \omega) \cdot \omega') = \nabla_q(n \otimes \omega) \cdot \omega' + (-1)^{\deg \omega}(n \otimes \omega) \cdot d_q(\omega')$. The q-connection ∇_q is said to be *flat* or *integrable* if $\nabla_q \circ \nabla_q = 0$.

Example 4.8. From Example 3.34, take $D = A_F = O_F[\![\mu]\!]$, $A = A_R$ equipped with the action of Γ_R and $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of Γ_R' (see Subsection 3.1). Taking $q = 1 + \mu$ and $U_i = [X_i^\flat]$ for $1 \leq i \leq d$, we know that A_R satisfies the hypotheses of Definition 3.31. In particular, A_R is equipped with an A_F-linear q-connection $\nabla_q : A_R \to q\Omega^1_{A_R/A_F}$, given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f) - f}{p_2(\mu)} \, d\log(p_2([X_i^\flat]))$. Now let N be a Wach module A_R. Then from [Abh23b, Proposition 5.3], the geometric q-connection $\nabla_q : N \to N \otimes_{A_R} \Omega^1_{A_R/A_F}$ given as $x \mapsto \sum_{i=1}^d \frac{\gamma_i(x) - x}{\mu} \, d\log([X_i^\flat])$ describes (N, ∇_q) as a φ-module equipped with $(p, [p]_q)$-adically quasi-nilpotent flat q-connection over A_R (see [Abh23b, Subsection 5.2] for details).

Example 4.9. From Example 3.35, take $D = \Lambda_F$ and $A = A^{\mathrm{PD}}$ as a Λ_F-algebra. Note that A^{PD} is equipped with a Λ_F-linear action of Γ'_R and we have $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of Γ'_R (see Subsection 3.1). Then setting $q = 1 + \mu$ and $U_i = [X_i^\flat]$ for $1 \leq i \leq d$, we know that A^{PD} satisfies the hypotheses of Definition 3.31. In particular, A^{PD} is equipped with a Λ_F-linear q-connection $\nabla_q : A^{\mathrm{PD}}(1) \to q\Omega^1_{A^{\mathrm{PD}}(1)/\Lambda_F}$, given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)} \, d\log(p_2([X_i^\flat]))$. Now let N be a Wach module over A_R and set $N^{\mathrm{PD}} := A^{\mathrm{PD}} \otimes_{A_R} N$, equipped with tensor product Frobenius and tensor product action of Γ_R. Then note that for any $f \otimes y$ in N^{PD} and g in Γ_R, we have $(g-1)(f \otimes y) = (g-1)f \otimes y + g(f) \otimes (g-1)y$ in μN^{PD}. Therefore, the operator $\nabla_q : N^{\mathrm{PD}} \to N^{\mathrm{PD}} \otimes_{A^{\mathrm{PD}}} \Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$ given as $x \mapsto \sum_{i=1}^d \frac{\gamma_i(x)-x}{\mu} \, d\log([X_i^\flat])$ satisfies the assumptions of Definition 4.7. Moreover, the q-connection ∇_q on N^{PD} is p-adically quasi-nilpotent using Example 3.35 and Example 4.8, and it is flat because γ_i commute with each other.

Example 4.10. From Example 3.36, take D to be Λ_R and A to be $\overline{A}(1) = A_R(1)/p_1(\mu)$ as a Λ_R-algebra via the morphism of rings $\iota_\Lambda : \Lambda_R \to \overline{A}(1)$. Then, the Λ_R-algebra $\overline{A}(1)$ is equipped with a Λ_R-linear (via ι_Λ) action of Γ'_R and we take $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of Γ'_R (see Subsection 3.1). Then setting $q = 1 + \mu$ and $U_i = [X_i^\flat]$ for $1 \leq i \leq d$, we know that $\overline{A}(1)$ satisfies the hypotheses of Definition 3.31. In particular, $\overline{A}(1)$ is equipped with a Λ_R-linear q-connection $\nabla_q : \overline{A}(1) \to q\Omega^1_{\overline{A}(1)/\Lambda_R}$, given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)} \, d\log(p_2([X_i^\flat]))$. Now let N be a Wach module over A_R and set $N(1) := A_R(1) \otimes_{p_2, A_R} N$, equipped with tensor product Frobenius and tensor product action of Γ_R^2, where Γ_R^2 acts on N via projection onto the second coordinate. Let $\overline{N}(1) := N(1)/p_1(\mu) = \overline{A}(1) \otimes_{p_2, A_R} N$ equipped with induced action of Frobenius and $\Gamma_R = 1 \times \Gamma_R$. Note that for any $f \otimes y$ in $\overline{N}(1)$ and g in Γ_R, we have that $(g-1)(f \otimes y) = (g-1)y \otimes y + g(f) \otimes (g-1)y$ is in $p_2(\mu)\overline{N}(1)$. Therefore, the operator $\nabla_q : \overline{N}(1) \to \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^1_{\overline{A}(1)/\Lambda_R}$, given as $x \mapsto \sum_{i=1}^d \frac{\gamma_i(x)-x}{\mu} \, d\log([X_i^\flat])$ satisfies the assumptions of Definition 4.7. Moreover, the q-connection ∇_q on $\overline{N}(1)$ is p-adically quasi-nilpotent using Example 3.36 and Example 4.8, and it is flat because γ_i commute with each other.

Lemma 4.11. *Let N be a Wach module over A_R and $\overline{A}(1) = A_R(1)/p_1(\mu)$. Then the $\overline{A}(1)$-module $\overline{A}(1) \otimes_{p_2, A_R} N$ is p-adically complete and p-torsion free.*

Proof. Note that the morphism $p_2 : A_R \to A_R(1)$ is faithfully flat from Lemma 3.15. Moreover, from Definition 4.1 recall that $\{p, \mu\}$ and $\{\mu, p\}$ are regular sequences on N. Then from the flatness of p_2, it follows that $\{p, p_2(\mu)\}$ and $\{p_2(\mu), p\}$ are regular sequences on $A_R(1) \otimes_{p_2, A_R} N$. Using the fact that $p_1([p]_q)/p_2([p]_q)$ is a unit in $A_R(1)$ (see Lemma 3.15), we get that $p_1(\mu)^{p-1}$ is an element of $(p, p_2(\mu))A_R(1)$. Now, since both $\{p, p_2(\mu)\}$ and $\{p_2(\mu), p\}$ are regular sequences on $A_R(1) \otimes_{p_2, A_R} N$, therefore, it follows that both $\{p, p_1(\mu)^{p-1}\}$ and $\{p_1(\mu)^{p-1}, p\}$ are regular sequences on $A_R(1) \otimes_{p_2, A_R} N$. In particular, using [Sta23, Tag 07DV] we get that both $\{p, p_1(\mu)\}$ and $\{p_1(\mu), p\}$ are regular sequences on $A_R(1) \otimes_{p_2, A_R} N$, and we conclude that $\overline{A}(1) \otimes_{p_1, A_R} N$ is p-torsion free. Next, note that N is a finitely generated and (p, μ)-adically complete A_R-module. So, it follows that $A_R(1) \otimes_{p_2, A_R} N$ is a finitely generated and $(p, p_2(\mu))$-adically $= (p, p_1(\mu))$-adically complete $A_R(1)$-module, in particular, it is $(p, p_1(\mu))$-adically separated. Consequently, we get that $\overline{A}(1) \otimes_{p_2, A_R} N$ is p-adically separated and it is clearly finitely generated over $\overline{A}(1)$-module, therefore, p-adically complete. ∎

Remark 4.12. Let N be a Wach module over A_R and let A^{PD} be the ring defined in Example 3.35. Moreover, recall that we defined a ring E in Remark 3.46, which admits faithfully flat maps $p_1 : A_F \to E$ and $p_2 : A_R \to E$, and we have that $E/p_1(\mu) \xrightarrow{\sim} A^{\mathrm{PD}}$. Then, by an argument similar to the proof of Lemma 4.11, it follows that the A^{PD}-module $A^{\mathrm{PD}} \otimes_{A_R} N$ is p-adically complete and p-torsion free.

From Lemma 3.24, we have that $t = \log(1 + \mu)$ converges in $\mu\Lambda_F \subset \mu\Lambda_R$ and t/μ is a unit.

Proposition 4.13. *Let N be a Wach module over A_R and $\overline{A}(1) = A_R(1)/p_1(\mu)$. Then, for $1 \leq i \leq d$, the series of operators $\nabla_i^{\log} = \frac{\log \gamma_i}{t} = \frac{1}{t} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1}$ converge p-adically on $\overline{N}(1) = \overline{A}(1) \otimes_{p_2, A_R} N$. This defines a Λ_R-linear p-adically quasi-nilpotent flat connection on $\overline{N}(1)$, denoted* *as* $\nabla : \overline{N}(1) \to \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^1_{\overline{A}(1)/\Lambda_R}$ *and given as* $x \mapsto \sum_{i=1}^d \nabla_i^{\log}(x)\, d\log([X_i^\flat])$. *The data of the connection* ∇ *on* $\overline{N}(1)$ *is equivalent to the data of the* q*-connection* ∇_q *described in Example 4.10, i.e. either may be recovered from the other. Moreover, the* q*-de Rham complex* $\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega^\bullet_{\overline{A}(1)/\Lambda_R}$ *is naturally quasi-isomorphic to the de Rham complex* $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega^\bullet_{\overline{A}(1)/\Lambda_R}$. *In particular, we have* $\overline{N}(1)^{\nabla_q = 0} = \overline{N}(1)^{\nabla=0}$.

Proof. Our proof employs ideas similar to that of [Abh21, Lemmas 4.36 & 4.38], [MT20, Theorem 4.2] and [BMS18, Corollary 12.5]. Note that the $\overline{A}(1)$-module $\overline{N}(1)$ is p-adically complete and p-torsion free by Lemma 4.11. Let us first show that for $1 \leq i \leq d$ the series of operators $\nabla_i^{\log} = \frac{1}{t}\sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_i-1)^{k+1}}{k+1}$ converge on $\overline{N}(1)$. Indeed, note that we have

$$\tfrac{(\gamma_i-1)^{k+1}}{k+1} = \mu k! \tfrac{\mu^k}{(k+1)!}\tfrac{(\gamma_i-1)^{k+1}}{\mu^{k+1}},$$

where $\mu^k/(k+1)!$ converges p-adically to 0 in Λ_R as k goes to $+\infty$, and t/μ is a unit in Λ_R.

Next, let us check that the operator $\nabla_i^{\log}$ satisfies the Leibniz rule. To show this, we claim that for x in $\overline{N}(1)$, we have

$$\lim_{m\to+\infty} \tfrac{\gamma_i^{p^m}-1}{p^m}(x) = t\nabla_i^{\log}(x). \tag{4.2}$$

Indeed, note that since $t^k/k!$ converges p-adically to 0 in $\overline{A}(1)$ as k goes to $+\infty$, so we can write $\gamma_i^n = \exp(nt\nabla_i^{\log})$, for $n \in \mathbb{Z}$. Expanding the preceding exponential, we see that

$$\tfrac{\gamma_i^n-1}{n} = t\nabla_i^{\log} + n\textstyle\sum_{k\geq 2} n^{k-2}\tfrac{t^k}{k!}(\nabla_i^{\log})^k : \overline{N}(1) \to \overline{N}(1),$$

is well-defined. Taking $n = p^m$ and letting $m \to +\infty$, we get the formula in (4.2). Now, for any f in $\overline{A}(1)$ and x in $\overline{N}(1)$, we have that

$$(\gamma_i^{p^m}-1)(fx) = (\gamma_i^{p^m}-1)(f)\cdot x + \gamma_i^{p^m}(f)(\gamma_i^{p^m}-1)(x).$$

Dividing out the preceding equality by tp^m, letting $m \to +\infty$ and using (4.2) we get that $\nabla_i^{\log}(fx) = \nabla_i^{\log}(f)x + f\nabla_i^{\log}(x)$, where the first operator on the right is $\nabla_i^{\log} := \log(\gamma_i)/t : \overline{A}(1) \to \overline{A}(1)$, whose well-definedness and the equality (4.2) can be checked similar to above. In particular, we have shown that the operators $\nabla_i^{\log}$ are well defined and satisfy a Leibniz rule.

To show that $\nabla : \overline{N}(1) \to \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^1_{\overline{A}(1)/\Lambda_R}$, given as $x \mapsto \sum_{i=1}^d \nabla_i^{\log}(x)\, d\log([X_i^\flat])$, is a well defined connection, we need to show that $\nabla : \overline{A}(1) \to \Omega^1_{\overline{A}(1)/\Lambda_R}$ is the usual de Rham differential $d : \overline{A}(1) \to \Omega^1_{\overline{A}(1)/\Lambda_R}$. As each $\nabla_i^{\log}$ is a continuous Λ_R-linear derivation, we can write $\nabla_i^{\log} = h_i \circ d$ for some unique continuous Λ_R-linear map $h_i : \Omega^1_{\overline{A}(1)/\Lambda_R} \to \Lambda_R$. Then it is easy to see that $h_i(d[X_i^\flat]) = \nabla_i^{\log}([X_i^\flat]) = [X_i^\flat]$ for $i = j$ and 0 otherwise. Therefore, we have $d = \sum_i [X_i^\flat]^{-1}\nabla_i^{\log}(-) \otimes d[X_i^\flat] = \sum_i \nabla_i^{\log}(-) \otimes d\log([X_i^\flat])$, as desired.

Next, let us show that the operators $\nabla_i = \nabla_i^{\log}/[X_i^\flat] = (\log\gamma_i)/(t[X_i^\flat])$ are p-adically quasi-nilpotent. Indeed, we first note that from the commutativity of φ and γ_i, it follows that $\log\gamma_i \circ \varphi = \varphi \circ \log\gamma_i$. Therefore, it is easy to see that $\nabla_i \circ \varphi = p[X_i^\flat]^{p-1}\varphi \circ \nabla_i$. Recall that N is equipped with an A_R-linear isomorphism $\varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$, and $[p]_q/p$ is a unit in Λ_F (see Lemma 3.24), therefore, $\overline{N}(1)$ is equipped with an $\overline{A}(1)$-linear isomorphism $\varphi^*(\overline{N}(1))[1/p] \xrightarrow{\sim} \overline{N}(1)[1/p]$. In particular, for any x in $\overline{N}(1)$, there exists $r \in \mathbb{N}$ large enough, such that $p^r x$ belongs to $\varphi^*(\overline{N}(1))$. Then, from the relation $\nabla_i \circ \varphi = p\varphi \circ \nabla_i$, we see that $\nabla_i^k(p^r x)$ converges p-adically to 0 as $k \to +\infty$. Hence, it follows that $\nabla_i^k(x) = p^{-r}\nabla_i^k(p^r x)$ converges p-adically to 0 as $k \to +\infty$, in particular, ∇_i are p-adically quasi-nilpotent.

So far, we have shown that $\nabla : \overline{N}(1) \to \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^1_{\overline{A}(1)/\Lambda_R}$ is a p-adically quasi-nilpotent connection and it is flat since γ_i, and therefore $\nabla_i^{\log}$, commute with each other. Moreover, we defined the connection ∇ using the action of Γ'_R and conversely, we have shown that the action of Γ'_R can be recovered by the formula $\gamma_i := \exp(t\nabla_i^{\log})$ and it remains to check that, the action of γ_i thus obtained, is semi-linear. Note that the Leibniz rule implies that $\frac{1}{k!}(t\nabla_i^{\log})^k(xf) = \sum_{a+b=k}\frac{1}{a!}(t\nabla_i^{\log})^a(x)\frac{1}{b!}(t\nabla_i^{\log})^b(f)$ for x in $\overline{N}(1)$ and f in $\overline{A}(1)$. Now taking the sum over all $k > 0$, we get that $\exp(t\nabla_i^{\log}(xf)) = \exp(t\nabla_i^{\log})(x)\exp(t\nabla_i^{\log})(f) = \exp(t\nabla_i^{\log})(x)\gamma_i(f)$, as required.

From the discussion above, it is clear that we have a q-de Rham complex $\overline{N}(1)\otimes_{\overline{A}(1)} q\Omega^\bullet_{\overline{A}(1)/\Lambda_R}$ and a de Rham complex $\overline{N}(1)\otimes_{\overline{A}(1)}\Omega^\bullet_{\overline{A}(1)/\Lambda_R}$. We claim that these complexes are naturally quasi-isomorphic. Indeed, let us first note that the endomorphisms $\nabla_{q,i}^{\log} = \frac{\gamma_i-1}{\mu} : \overline{N}(1) \to \overline{N}(1)$ are commuting. So, let $K_{\overline{N}(1)}(\nabla_{q,1}^{\log}, \ldots, \nabla_{q,d}^{\log})$ denote the corresponding Koszul complex (see Definition A.8). Then we have a natural identification of complexes

$$K_{\overline{N}(1)}(\nabla_{q,1}^{\log}, \ldots, \nabla_{q,d}^{\log}) = \overline{N}(1) \otimes_{\overline{A}(1)} q\Omega^\bullet_{\overline{A}(1)/\Lambda_R}.$$

Next, let $\nabla : \overline{A}(1) \to \Omega^1_{\overline{A}(1)/\Lambda_R}$ denote the Λ_R-linear p-adically quasi-nilpotent flat connection on $\overline{A}(1)$ (arising from the q-connection, see Proposition 3.37), given as $f \mapsto \sum_{i=1}^d \nabla_i^{\log}(f)\, d\log([X_i^\flat])$. Next, let $t = \log(1+\mu)$, and again note that the endomorphisms $\nabla_i^{\log} = \frac{\log\gamma_i}{t} : \overline{N}(1) \to \overline{N}(1)$ are commuting. So, let $K_{\overline{N}(1)}(\nabla_1^{\log}, \ldots, \nabla_d^{\log})$ denote the corresponding Koszul complex, and again we have a natural identification of complexes

$$K_{\overline{N}(1)}(\nabla_1^{\log}, \ldots, \nabla_d^{\log}) = \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^\bullet_{\overline{A}(1)/\Lambda_R}.$$

Now, from the discussion above, recall that we have the relation $\gamma_i = \exp(t\nabla_i^{\log})$ on $\overline{N}(1)$. Therefore, we can write

$$\nabla_{q,i} = \nabla_i^{\log}\big(1 + \textstyle\sum_{k\geq 1}\frac{t^k}{k!\mu}(\nabla_i^{\log})^{k-1}\big), \tag{4.3}$$

where the term inside the parentheses is invertible, because t/μ is a unit in $\overline{A}(1)$ (see Lemma 3.24) and $\mu^{k-1}/(k!)$ is topologically nilpotent because $(\mu^{p-1}/p)^k$ is topologically nilpotent in $\overline{A}(1)$ (see Proposition 3.25). Now, in the notation of Lemma A.9, let us set $M = \overline{N}(1)$, $f_i = \nabla_i^{\log}$ and take h_i to be the formula in the parentheses in (4.3), in particular, $f_ih_i = \nabla_{q,i}^{\log}$. Then, from Lemma A.9, we obtain a natural quasi-isomorphism of complexes

$$\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega^\bullet_{\overline{A}(1)/\Lambda_R} = K_{\overline{N}(1)}(\nabla_{q,1}^{\log}, \ldots, \nabla_{q,d}^{\log}) \xrightarrow{\sim} K_{\overline{N}(1)}(\nabla_1^{\log}, \ldots, \nabla_d^{\log}) = \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^\bullet_{\overline{A}(1)/\Lambda_R}.$$

Finally, from the quasi-isomorphism above, it follows that we have $\overline{N}(1)^{\nabla_q=0} = \overline{N}(1)^{\Gamma'_R} \xrightarrow{\sim} \overline{N}(1)^{\nabla=0}$. This conlcudes our proof. ∎

Remark 4.14. From remark 4.12, recall that the A^{PD}-module $N^{\mathrm{PD}} = A^{\mathrm{PD}} \otimes_{A_R} N$ is p-adically complete and p-torsion free. Then, in Proposition 4.13, replacing Λ_R by Λ_F, $\overline{A}(1)$ by A^{PD} and $\overline{N}(1)$ by $N^{\mathrm{PD}} = A^{\mathrm{PD}} \otimes_{A_R} N$, and using essentially the same arguments, we see that for $1 \leq i \leq d$, the series of operators $\nabla_i^{\log} = \frac{\log\gamma_i}{t} = \frac{1}{t}\sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_i-1)^k}{k+1}$ converge p-adically on N^{PD}. This defines a Λ_F-linear p-adically quasi-nilpotent flat connection on N^{PD}, denoted as $\nabla : N^{\mathrm{PD}} \to N^{\mathrm{PD}} \otimes_{A^{\mathrm{PD}}} \Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$ and given as $x \mapsto \sum_{i=1}^d \nabla_i^{\log}(x)\, d\log([X_i^\flat])$. The data of the connection ∇ on N^{PD} is equivalent to the data of the q-connection ∇_q described in Example 4.9, i.e. either may be recovered from the other. Moreover, the q-de Rham complex $N^{\mathrm{PD}} \otimes_{A^{\mathrm{PD}}} q\Omega^\bullet_{A^{\mathrm{PD}}/\Lambda_F}$ is naturally quasi-isomorphic to the de Rham complex $N^{\mathrm{PD}} \otimes_{A^{\mathrm{PD}}} \Omega^\bullet_{A^{\mathrm{PD}}/\Lambda_F}$. In particular, we have $(N^{\mathrm{PD}})^{\nabla_q=0} \xrightarrow{\sim} (N^{\mathrm{PD}})^{\nabla=0}$.

Remark 4.15. Let $\overline{A}(1) = A_R(1)/p_1(\mu)$ and A^{PD} as in Example 3.35. Recall that we have a natural injective (φ, Γ_R)-equivariant homomorphism of rings $p_2 : A^{\mathrm{PD}} \to \overline{A}(1)$ (see Remark 3.39). Now let N be a Wach module over A_R, then we have that $\overline{A}(1)\otimes_{p_2,A^{\mathrm{PD}}} N^{\mathrm{PD}} = \overline{A}(1)\otimes_{p_2,A_R} N = \overline{N}(1)$, compatible with the action of (φ, Γ_R). Then, using the compatibility of the corresponding connections on $\overline{A}(1)$ and A^{PD} (see Remark 3.40) and the Leibniz rule for the connection on $\overline{N}(1)$, proven in Proposition 4.13, it follows that the respective connections on $\overline{N}(1)$ and N^{PD} are compatible, in partiular, the connection on $\overline{N}(1)$ is given as the tensor product of respective connections on $\overline{A}(1)$ and N^{PD}.

4.2. Geometric descent. From Example 4.8, recall that a Wach module N can be seen as a φ-module over A_R equipped with a $(p, [p]_q)$-adically quasi-nilpotent flat q-connection. Then from Proposition 4.13 we have that $\overline{N}(1) = A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$ is a finite module over $\overline{A}(1) := A_R(1)/p_1(\mu)$, equipped with a p-adically quasi-nilpotent flat connection and a Frobenius-semilinear endomorphism φ (after inverting p). Similarly, from Remark 4.14 we have that $N^{\mathrm{PD}} = A^{\mathrm{PD}} \otimes_{A_R} N$ is a finite module over A^{PD}, equipped with a p-adically quasi-nilpotent flat connection and a Frobenius-semilinear endomorphism φ (after inverting p). In order to prove Theorem 4.5, we need to first interpret the preceding φ-modules with connection, as F-crystals over the crystalline site $\mathrm{CRIS}(A^{\mathrm{PD}}/\Lambda_F)$. So we begin this subsection by recalling some standard facts about relevant crystalline sites and crystals. Using the formalism of crystalline sites, we will also be able to obtain the geometric descent step in the proof of Theorem 4.5 mentioned above.

4.2.1. Crystalline site and F-crystals. Let A be a p-adically complete PD-ring and let D be the p-adic completion of a divided power A-algebra. For $m \geq 1$, we set $\Sigma_m = \mathrm{Spec}\,(A/p^m)$ and $X_m = \mathrm{Spec}\,(D/p^m)$. Let $\mathrm{CRIS}(X_m/\Sigma_m)$ denote the big crystalline site of X_m over Σ_m with the PD-structure given by $p(A/p^m) + JA/p^m$, where J denotes the PD-ideal of A, and let $\mathcal{O}_{X_m/\Sigma_m}$ denote the structure sheaf of rings. Let $\mathrm{CR}(X_m/\Sigma_m)$ denote the category of finitely generated crystals of $\mathcal{O}_{X_m/\Sigma_m}$-modules over $\mathrm{CRIS}(X_m/\Sigma_m)$. Note that the homomorphisms $\Sigma_m \to \Sigma_{m+1}$ and $X_m \to X_{m+1}$ induce the pullback functor $i^*_{m,m+1} : \mathrm{CR}(X_{m+1}/\Sigma_{m+1}) \to \mathrm{CR}(X_m/\Sigma_m)$. One can define $\mathrm{CRIS}(X_1/\Sigma_m)$ and $\mathrm{CR}(X_1/\Sigma_m)$ similarly and the pullback functor $i^*_m : \mathrm{CR}(X_m/\Sigma_m) \xrightarrow{\sim} \mathrm{CR}(X_1/\Sigma_m)$ is an equivalence of categories by [Ber74, Chapitre IV, Théorème 1.4.1]. We define a finitely generated crystal of $\mathcal{O}_{X/\Sigma}$-modules $\mathcal{E}$ on X/Σ to be a system $(\mathcal{E}_m)_{m\geq 1}$, where $\mathcal{E}_m$ is an object of $\mathrm{CR}(X_m/\Sigma_m)_{\mathrm{cris}}$ for each $m \geq 1$ and we have isomorphisms $i^*_{m,m+1}\mathcal{E}_{m+1} \xrightarrow{\sim} \mathcal{E}_m$. A locally free crystal on X_1/Σ can be defined similarly, using $\mathrm{CRIS}(X_1/\Sigma_m)$. Write $\mathrm{CR}(X/\Sigma)$ and $\mathrm{CR}(X_1/\Sigma)$ for the category of finitely generated crystals on X/Σ and X_1/Σ respectively, then the obvious pullback functor $i^* : \mathrm{CR}(X/\Sigma) \to \mathrm{CR}(X_1/\Sigma)$ is an equivalence of categories. Let $\mathrm{MIC}_{\mathrm{conv}}(D)$ denote the category of finitely generated D-modules equipped with an A-linear p-adically quasi-nilpotent flat connection. Then by [Ber74, Chapitre IV, Théorème 1.6.5], we have an equivalence of categories

$$\mathrm{CR}(X/\Sigma) \xrightarrow{\sim} \mathrm{MIC}_{\mathrm{conv}}(D), \tag{4.4}$$

obtained by sending $(\mathcal{E}_m)_{m\geq 1}$ to the inverse limit of the evaluation of $\mathcal{E}_m$ on the object $\mathrm{Spec}\,(X_m) \xrightarrow{id} \mathrm{Spec}\,(X_m)$ of the site $\mathrm{CRIS}(X_m/\Sigma_m)$, equipped with the natural A-linear p-adically quasi-nilpotent flat connection.

Let us now describe F-crystals on $\mathrm{CRIS}(X/\Sigma)$. Note that the absolute Frobenius on X_1 and the natural Frobenius on Σ induce Frobenius pullbacks $\varphi^* : \mathrm{CR}(X_1/\Sigma_m) \to \mathrm{CR}(X_1/\Sigma_m)$ and $\varphi^* : \mathrm{CR}(X_1/\Sigma) \to \mathrm{CR}(X_1/\Sigma)$. A finitely generated F-crystal $\mathcal{E}$ on $\mathrm{CRIS}(X/\Sigma)$ is an object of $\mathrm{CR}(X_1/\Sigma)$ equipped with an isomorphism $\varphi_{\mathcal{E}} : (\varphi^* i^* \mathcal{F})_{\mathbb{Q}} \xrightarrow{\sim} (i^*\mathcal{F})_{\mathbb{Q}}$ in the isogeny category $\mathrm{CR}(X/\Sigma) \otimes \mathbb{Q}$. We will denote the category of finitely generated F-crystals on X/Σ as $\mathrm{CR}^{\varphi}(X/\Sigma)$. Let $\mathrm{MIC}^{\varphi}(D)$ denote the following category : an object M is a finitely generated D-module, equipped with an A-linear p-adically quasi-nilpotent flat connection and an isomorphism $\varphi_M : \varphi^* M[1/p] \xrightarrow{\sim} M[1/p]$; morphisms between two objects are D-linear maps compatible with respective Frobenii and connections. In fact, in the presence of Frobenius structures, similar to [MT20, Lemma 2.24], it can be shown that the p-adic quasi-nilpotence of the connection is automatic. Therefore, the equivalence in (4.4) refines to an equivalence,

$$\mathrm{CR}^{\varphi}(X/\Sigma) \xrightarrow{\sim} \mathrm{MIC}^{\varphi}(D), \tag{4.5}$$

Remark 4.16. In the discussion above, we can take A to be Λ_F with the PD-structure on it given by $p(\Lambda_F/p^m) + J\Lambda_F/p^m$, where $J = ((\mu^{p-1}/p)^{[k]}, k \geq 1) \subset \Lambda_F$, and D to be A^{PD}, i.e. we set $\Sigma_m = \mathrm{Spec}\,(\Lambda_F/p^m)$ and $X_m = \mathrm{Spec}\,(A^{\mathrm{PD}}/p^m)$. Now, recall that we have an isomorphism of rings $\iota : \Lambda_R \xrightarrow{\sim} A^{\mathrm{PD}}$ from (3.13). Moreover, we have an inclusion $\mathrm{Spec}\,(\Lambda_R/p^m) \hookrightarrow \mathrm{Spec}\,(\overline{A}(1)/p^m)$ induced by the surjection $\overline{A}(1) \twoheadrightarrow \Lambda_R$, which is defined by sending $[X_i^\flat] \mapsto X_i$ and $T_i^{[k_i]} \mapsto 0$, for all $1 \leq i \leq d$, and note that the composition $\Lambda_R \xrightarrow{\iota_\Lambda} \overline{A}(1) \twoheadrightarrow \Lambda_R$ is the identity. In particular,

$\mathrm{Spec}\,(\Lambda_R/p^m) \hookrightarrow \mathrm{Spec}\,(\overline{A}(1)/p^m)$ is an object of $\mathrm{CRIS}(X_m/\Sigma_m)$. Furthermore, using the isomorphism $\iota : \Lambda_R \xrightarrow{\sim} A^{\mathrm{PD}}$, it is easy to see that $\overline{A}(1)$ is the self product of A^{PD} in $\mathrm{CRIS}(X/\Sigma)$ with the two maps $\iota_\Lambda : \Lambda_R \to \overline{A}(1)$ and $p_2 : A^{\mathrm{PD}} \to \overline{A}(1)$ describing the two projection maps.

4.2.2. The action of Γ'_R. As mentioned at the beginning of this subsection, N is a Wach module over A_R. Now, from Example 4.10, recall that we have an isomorphism of Λ_F-algebras $\Lambda_R = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge \xrightarrow{\sim} A^{\mathrm{PD}}$ (see the isomorphism ι in (3.13)) and $\overline{A}(1) = A_R(1)/p_1(\mu)$. Then, the $\overline{A}(1)$-module $\overline{N}(1) = \overline{A}(1) \otimes_{p_2, A_R} N$ is equipped with a Frobenius-semilinear endomorphism φ (after inverting p) and an action of $1 \times \Gamma_R$, which induces a Λ_R-linear flat q-connection $\nabla_q : \overline{N}(1) \to \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^1_{\overline{A}(1)/\Lambda_R}$. Moreover, from Proposition 4.13 recall that the data of the q-connection on $\overline{N}(1)$ is equivalent to the data of a connection $\nabla : \overline{N}(1) \to \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^1_{\overline{A}(1)/\Lambda_R}$. Now, we make the following claim:

Proposition 4.17. *Let N be a Wach module over A_R and set $M_\Lambda := (\overline{A}(1) \otimes_{p_2, A_R} N)^{\nabla=0}$ as a Λ_R-module via $\iota_\Lambda : \Lambda_R \xrightarrow{\sim} \overline{A}(1)^{\nabla=0}$ and equipped with an induced (φ, Γ_F)-action. Then we have a natural $(\varphi, \nabla, \Gamma_F)$-equivariant isomorphism*

$$\begin{aligned} \overline{A}(1) \otimes_{\Lambda_R} M_\Lambda &\xrightarrow{\sim} \overline{A}(1) \otimes_{p_2, A_R} N \\ a \otimes b \otimes x &\longmapsto ab \otimes x. \end{aligned} \tag{4.6}$$

Moreover, we have (φ, Γ_F)-equivariant isomorphisms of finitely generated p-adically complete modules over $\Lambda_R \underset{\sim}{\xrightarrow{(3.13)}} A^{\mathrm{PD}}$,

$$M_\Lambda \xrightarrow{\sim} A^{\mathrm{PD}} \otimes_{A_R} N \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N. \tag{4.7}$$

Furthermore, the de Rham complex $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega^\bullet_{\overline{A}(1)/\Lambda_R}$ is ayclic in positive degrees.

Proof. To use the results from Subsection 4.2.1, we will work with the notations in Remark 4.16, i.e. we set $\Sigma_m = \mathrm{Spec}\,(\Lambda_F/p^m)$ and $X_m = \mathrm{Spec}\,(A^{\mathrm{PD}}/p^m)$. Now, note that the A^{PD}-module N^{PD} is equipped with a Λ_F-linear p-adically quasi-nilpotent flat connection $\nabla : N^{\mathrm{PD}} \to N^{\mathrm{PD}} \otimes_{A^{\mathrm{PD}}} \Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$, in particular, N^{PD} is an object of $\mathrm{MIC}^\varphi(A^{\mathrm{PD}})$ (see Remark 4.14). Then from the equivalence in (4.5), there exists a finitely generated F-crystal $\mathcal{E}$ over $\mathrm{CRIS}(X/\Sigma)$ and we have $\mathcal{E}(A^{\mathrm{PD}}) = N^{\mathrm{PD}}$. Moreover, recall that we have a (φ, Γ_F)-equivariant isomorphism of rings $\iota : \Lambda_R \xrightarrow{\sim} A^{\mathrm{PD}}$ from (3.13). Next, from Proposition 3.25 and Remark 3.27 recall that $\overline{A}(1)$ is the p-adic completion of a PD-polynomial algebra over A^{PD} in variables $T_1, \ldots, T_d$. Moreover, $\overline{A}(1)$ is equipped with a Λ_R-linear flat connection $\nabla : \overline{A}(1) \to \Omega^1_{\overline{A}(1)/\Lambda_R} = \overline{A}(1) \otimes_{A^{\mathrm{PD}}} \Omega^1_{A^{\mathrm{PD}}/\Lambda_F}$ (see Proposition 3.37 and Remark 3.40). Now consider the injection $\mathrm{Spec}\,(\Lambda_R/p^m) \hookrightarrow \mathrm{Spec}\,(\overline{A}(1)/p^m)$ induced by the (φ, Γ_F)-equivariant surjection $\overline{A}(1) \twoheadrightarrow \Lambda_R$ from Remark 4.16 and note that the composition $\Lambda_R \xrightarrow{\iota_\Lambda} \overline{A}(1) \twoheadrightarrow \Lambda_R$ is the identity and (φ, Γ_F)-equivariant. So we have the following (φ, Γ_F)-equivariant morphisms in $\mathrm{CRIS}(X_m/\Sigma_m)$, for $m \geq 1$,

$$\begin{tikzcd} \mathrm{Spec}\,(\Lambda_R/p^m) \arrow[r, hook] \arrow[d, "\iota"] & \mathrm{Spec}\,(\overline{A}(1)/p^m) \arrow[d, "p_2"] \\ \mathrm{Spec}\,(A^{\mathrm{PD}}/p^m) \arrow[r, "id"] & \mathrm{Spec}\,(A^{\mathrm{PD}}/p^m) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}\,(\Lambda_R/p^m) \arrow[r, hook] \arrow[d, "id"] & \mathrm{Spec}\,(\overline{A}(1)/p^m) \arrow[d, "T_i^{[k_i]} \mapsto 0"] \\ \mathrm{Spec}\,(\Lambda_R/p^m) \arrow[r, hook] & \mathrm{Spec}\,(\overline{A}(1)/p^m). \end{tikzcd} \tag{4.8}$$

Evaluating the F-crystal $\mathcal{E}$ on $\mathrm{Spec}\,(\Lambda_R/p^m) \hookrightarrow \mathrm{Spec}\,(\overline{A}(1)/p^m)$ for each $m \geq 1$, and taking the limit over m, gives a finitely generated $\overline{A}(1)$-module $\mathcal{E}(\overline{A}(1))$ equipped with a (φ, Γ_F)-action and a Λ_R-linear p-adically quasi-nilpotent flat connection. Now, using the diagram on the left in (4.8) and the fact that $\mathcal{E}$ is an F-crystal, we get that $\mathcal{E}(\overline{A}(1)) \xleftarrow{\sim} \overline{A}(1) \otimes_{p_2, A^{\mathrm{PD}}} N^{\mathrm{PD}}$ compatible with the respective (φ, Γ_F)-actions and connections, where the right hand term is equipped with the tensor product (φ, Γ_F)-action and the tensor product of respective connections on $\overline{A}(1)$ and N^{PD}, described above. Next, note that the right vertical arrow of the right hand diagram of (4.8), factors through

$\mathrm{Spec}\,(\overline{A}(1)/p^m) \xrightarrow{p_1} \mathrm{Spec}\,(\Lambda_R/p^m) \hookrightarrow \mathrm{Spec}\,(\overline{A}(1)/p^m)$. Therefore, from the fact that $\mathcal{E}$ is an F-crystal, we obtain a (φ, Γ_F)-equivariant isomorphism

$$\overline{\varepsilon} : \overline{A}(1) \otimes_{\Lambda_R} \mathcal{E}(\Lambda_R) \xrightarrow{\sim} \mathcal{E}(\overline{A}(1)) \xleftarrow{\sim} \overline{A}(1) \otimes_{p_2, A^{\mathrm{PD}}} N^{\mathrm{PD}} = \overline{N}(1),$$

compatible with the respective Λ_R-linear connections, where the Λ_R-linear connection on the source is given as $\nabla \otimes 1$ with ∇ being the Λ_R-linear connection on $\overline{A}(1)$ described above. Explicity, for x in $\overline{A}(1) \otimes_{\Lambda_R} \mathcal{E}(\Lambda_R)$, the map $\overline{\varepsilon}$ is given by the formula $\overline{\varepsilon}(x) = \sum_{j_1,\ldots,j_d \geq 0} (-1)^{j_1+\cdots+j_d} \prod_{i=1}^{d} \nabla_i^{j_i}(x) \otimes T_i^{[j_i]}$, where $\nabla_i = [X_i^\flat]^{-1}\nabla_i^{\log}$ (see Proposition 4.13 for the definition of $\nabla_i^{\log}$). Now, using the (φ, Γ_R)-equivariant isomorphism $\iota : \Lambda_R \xrightarrow{\sim} A^{\mathrm{PD}}$, let us note that we have a (φ, Γ_F)-equivariant isomorphism $\mathcal{E}(\Lambda_R) \xrightarrow{\sim} N^{\mathrm{PD}}$. Then from the interpretation of $\overline{A}(1)$ as the self product of A^{PD} in $\mathrm{CRIS}(X/\Sigma)$ (see Remark 4.16), we see that the isomorphism $\overline{\varepsilon}$ is just the stratification of N^{PD} over $\overline{A}(1)$.

Next, consider the de Rham complex $\Omega^\bullet_{\overline{A}(1)/\Lambda_R} \xrightarrow{\sim} \overline{A}(1) \otimes_{P_R} \Omega^\bullet_{P_R/R}$ (see Remark 3.38 for the isomorphism and Remark 3.27 for the definition of P_R), regarded as a complex of P_R-modules via the map $P_R \to \overline{A}(1)$ induced by the inverse of the map (3.17). Then from the proof of [Ber74, Chapitre V, Lemme 2.1.2], it follows that the de Rham complex $\Omega^\bullet_{P_R/R}$ is acyclic in positive degrees. Hence, from the isomorphism of the de Rham complexes

$$(\overline{A}(1) \otimes_{\Lambda_R} \mathcal{E}(\Lambda_R)) \otimes_{\overline{A}(1)} \Omega^\bullet_{\overline{A}(1)/\Lambda_R} \xrightarrow{\sim} \overline{N}(1) \otimes_{\overline{A}(1)} \Omega^\bullet_{\overline{A}(1)/\Lambda_R},$$

it follows that the de Rham complex $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega^\bullet_{\overline{A}(1)/\Lambda_R}$ is acyclic in positive degrees.

Now, by taking the horizontal sections for the respective connections in the isomorphism $\overline{\varepsilon}$, we obtain (φ, Γ_F)-equivariant isomorphisms of finite Λ_R-modules $\mathcal{E}(\Lambda_R) \xrightarrow{\sim} \mathcal{E}(\overline{A}(1))^{\nabla=0} \xleftarrow{\sim} \overline{N}(1)^{\nabla=0}$. In particular, from the isomorphism $\overline{\varepsilon}$, we deduce that $\overline{A}(1)$-linearly extending the natural inclusion $\overline{N}(1)^{\nabla=0} \subset \overline{N}(1)$, we obtain the following $(\varphi, \nabla, \Gamma_F)$-equivariant diagram

$$\begin{tikzcd}
\overline{A}(1) \otimes_{\Lambda_R} \mathcal{E}(\Lambda_R) \arrow[r, "\overline{\varepsilon}", "\sim"'] \arrow[d, "\overline{\varepsilon}", "\wr"'] & \overline{N}(1) \arrow[d, equal] \\
\overline{A}(1) \otimes_{\Lambda_R} \overline{N}(1)^{\nabla=0} \arrow[r, "(4.6)"] & \overline{A}(1) \otimes_{p_2, A_R} N.
\end{tikzcd} \tag{4.9}$$

From the diagram, it follows that (4.6) is an isomorphism. Moreover, from the preceding discussions we also obtain (φ, Γ_F)-equivariant isomorphisms $N^{\mathrm{PD}} \xleftarrow{\sim} \mathcal{E}(\Lambda_R) \xrightarrow{\sim} \overline{N}(1)^{\nabla=0}$, i.e. the claimed isomorphisms in (4.7). This allows us to conclude. ■

Remark 4.18. In Subsection 3.4.1, we defined a $(\varphi, 1 \times \Gamma_F)$-equivariant map $\Delta' : A_R(1) \to \tilde{\Lambda}_R$ and from the proof of Lemma 3.50, we have a (φ, Γ_F)-equivariant diagram

$$\Lambda_R \xrightarrow[\sim]{\iota_\Lambda} A_R(1)/p_1(\mu) \xrightarrow{\Delta'} \tilde{\Lambda}_R/p_1(\mu) \xleftarrow[\sim]{(3.23)} \Lambda_R.$$

Now, base changing the $(\varphi, \nabla, \Gamma_F)$-equivariant diagram in (4.9) along Δ' mod $p_1(\mu)$, we obtain a (φ, Γ_F)-equivariant isomorphism $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$, which is precisely the isomorphism (4.7), proven in Proposition 4.17. In particular, we see that the (φ, Γ_F)-equivariant isomorphism in (4.7) is the base change of the $(\varphi, \nabla, \Gamma_F)$-equivariant isomorphism in (4.6) along the map Δ' from (3.25).

4.3. Arithmetic descent.

In this subsection, we will carry out the descent for the arithmetic part of Γ_R, i.e. Γ_F. From (1.6), recall that Γ_F fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1,$$

where, for $p \geq 3$, we have that $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ and for $p = 2$, we have that $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$. Moreover, for $p \geq 3$, we have that $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$ and the projection map $\Gamma_F \to \Gamma_{\mathrm{tor}}$, admits a section $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \to \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$, where the second map is given as $a \mapsto [a]$, the Teichmüller lift of a, and the final isomorphism is induced by the p-adic cyclotomic character. Finally, for $p = 2$, we have that $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \{\pm 1\}$, as groups.

4.3.1. The action of $\mathbb{F}_p^\times$. Our first goal is to carry out the descent of Wach modules, for the action of $\mathbb{F}_p^\times$, in the case of $p \geq 3$. Let N be a Wach module over A_R and let us consider it as a module over $R[\![\mu]\!]$ via the (φ, Γ_F)-equivariant isomorphism of rings $\iota : R[\![\mu]\!] \xrightarrow{\sim} A_R$ (see Subsection 3.1.1). In particular, by abusing notations, we will consider N as an $R[\![\mu]\!]$-module equipped with an $R[\![\mu]\!]$-linear Frobenius isomorphism $\varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ and an R-linear and continous action of Γ_F commuting with the Frobenius and such that the action of Γ_F is trivial on $N/\mu N$. Next, from Subsections 3.1.1 and 3.1.2, recall that we have $\mu_0 = -p + \sum_{a \in \mathbb{F}_p}(1+\mu)^{[a]}$ and $\tilde{p} = \mu_0 + p$, as elements of $R[\![\mu]\!]^{\mathbb{F}_p^\times}$. Moreover, from Lemma 3.4, we have a (φ, Γ_0)-equivariant isomorphism of rings $R[\![\mu_0]\!] \xrightarrow{\sim} R[\![\mu]\!]^{\mathbb{F}_p^\times}$. Then, we claim the following:

Proposition 4.19. *Let N be a Wach module over A_R. Then $N_0 := N^{\mathbb{F}_p^\times}$ is a finitely generated $R[\![\mu_0]\!]$-module, equipped with a continuous and semilinear action of Γ_0 such that the action of Γ_0 is trivial on $N_0/\mu_0 N_0 \xrightarrow{\sim} N/\mu N$, and we have a natural (φ, Γ_F)-equivariant isomorphism of $R[\![\mu]\!]$-modules $R[\![\mu]\!] \otimes_{R[\![\mu_0]\!]} N_0 \xrightarrow{\sim} N$. Moreover, N_0 is equipped with an $R[\![\mu_0]\!]$-linear isomorphism $\varphi^*(N_0)[1/\tilde{p}] \xrightarrow{\sim} N_0[1/\tilde{p}]$, compatible with the respective natural actions of Γ_0.*

Proof. From (A.1), note that we have an $\mathbb{F}_p^\times$-decomposition $N = \bigoplus_{i=0}^{p-2} N_i$, where each N_i is a (p, μ_0)-adically complete $R[\![\mu_0]\!]$-module equipped with a continuous and R-linear action of Γ_0. Moreover, recall that $R[\![\mu]\!]$ is flat and finite of degree $p-1$ over $R[\![\mu_0]\!]$ (see Remark 3.8), so it follows that N is finitely generated as an $R[\![\mu_0]\!]$-module. Since $R[\![\mu_0]\!]$ is noetherian, therefore, we get that the $R[\![\mu_0]\!]$-submodule $N_0 \subset N$ is finitely generated.

Next, by extending the natural $R[\![\mu_0]\!]$-linear and (φ, Γ_F)-equivariant inclusion $N_0 \subset N$, along the map $R[\![\mu_0]\!] \to R[\![\mu]\!]$, we obtain a natural (φ, Γ_F)-equivariant map

$$N' := R[\![\mu]\!] \otimes_{R[\![\mu_0]\!]} N_0 \longrightarrow N, \tag{4.10}$$

and our goal is to show that (4.10) is bijective. Recall that μ_0 is the product of μ^{p-1} with a unit in $R[\![\mu]\!]$ (see Lemma 3.7), so we see that N is μ_0-torsion free, and it follows that N_0 is μ_0-torsion free as well. As the natural map $R[\![\mu_0]\!] \to R[\![\mu]\!]$ is flat (see Remark 3.8), therefore, we get that N' is μ_0-torsion free $=$ μ^{p-1}-torsion free, hence, μ-torsion free. Moreover, as N_0 is (p, μ_0)-adically complete, it also follows that N' is (p, μ_0)-adically $=$ (p, μ)-adically complete. Furthermore, as N is (p, μ)-adically complete and μ-torsion free, therefore, to show that (4.10) is bijective it is enough to show that (4.10) is bijective modulo μ. We will first show that (4.10) is surjective. Indeed, note that we have an R-linear and Γ_F-equivariant surjective map $N \to N/\mu N$. Then, from the $\mathbb{F}_p^\times$-decomposition in (A.1), we can rewrite the preceding map as $\bigoplus_{i=0}^{p-2} N_i \to \bigoplus_{i=0}^{p-2}(N/\mu N)_i$ which is R-linear and Γ_F-equivariant, in particular, it is termwise surjective, i.e. the induced R-linear map $N_i \to (N/\mu N)_i$ is surjective for each $0 \leq i \leq p-2$. However, since the action of Γ_F is trivial on $N/\mu N$, therefore, we obtain that $N/\mu N = \bigoplus_{i=0}^{p-2}(N/\mu N)_i = (N/\mu N)_0$ and it follows that the natural R-linear map $N_0 \to N/\mu N$ is surjective. As μ is in the Jacobson radical of $R[\![\mu]\!]$, therefore, by using Nakayama Lemma, we see that the natural (φ, Γ_F)-equivariant map $R[\![\mu]\!] \otimes_{R[\![\mu_0]\!]} N_0 \to N$ is surjective as well. Next, let us consider the following diagram:

$$\begin{tikzcd} N'/\mu N' \arrow[r, two heads] \arrow[d, equal] & N/\mu N \\ N_0/\mu_0 N_0, \arrow[ur] & \end{tikzcd} \tag{4.11}$$

where the top arrow is surjective by the discussion above, the slanted arrow is the natural R-linear and Γ_F-equivariant map induced by the inclusion $N_0 \subset N$ and the left vertical equality follows because we have that

$$N'/\mu N' = (N'/\mu_0 N')/\mu N' = (R[\![\mu]\!]/\mu^{p-1} \otimes_R N_0/\mu_0 N_0)/\mu = N_0/\mu_0 N_0.$$

To show that (4.10) is injective modulo μ, it is enough to show that the slanted arrow in (4.11) is injective. So set $N'' := \mu N \cap N_0 \subset N$ as an $R[\![\mu_0]\!]$-module and note that we have a natural

(φ, Γ_F)-equivariant inclusion $N'' \to \mu N$. The preceding inclusion induces a Γ_F-equivariant map $N \to \mu N''/\mu^2 N''$, where the source admits a trivial action of $\mathbb{F}_p^\times$ and the target admits a non-trivial action of $\mathbb{F}_p^\times$ (see Remark 4.20). So it follows that $N'' = \mu^2 N \cap N_0 \subset N$. Iterating the preceding argument $p-2$ times, we obtain that $N'' = \mu^{p-1} N \cap N_0 = \mu_0 N \cap N_0 \subset N = \mu_0 N_0$, where the last equality follows because for any x in N and g in $\mathbb{F}_p^\times$, we have that $g(\mu_0 x) = \mu_0 x$ if and only if $g(x) = x$, i.e. x is in N_0. Thus, from the preceding observation, it follows that the natural map $N_0/\mu_0 N_0 \to N/\mu N$ is injective. Hence, from (4.11), we conclude that $N_0/\mu_0 N_0 \xrightarrow{\sim} N/\mu N$ as R-modules and (4.10) is bijective. In particular, we also get that the action of Γ_0 is trivial on $N_0/\mu_0 N_0$.

Finally, let us show the Frobenius finite height condition on N_0. Note that since $\tilde{p}$ the product of $[p]_q$ with a unit in $R[\![\mu]\!]$ (see Lemma 3.3), therefore, the Frobenius finite height condition on N can also be stated as an $R[\![\mu]\!]$-linear isomorphism $\varphi^*(N)[1/\tilde{p}] \xrightarrow{\sim} N[1/\tilde{p}]$. Now, recall that the Frobenius on N commutes with the action of Γ_F, so taking the invariants of the preceding isomorphism under the action of $\mathbb{F}_p^\times$ and using the (φ, Γ_F)-equivariant isomorphism in (4.10), we obtain that N_0 is equipped with an $R[\![\mu_0]\!]$-linear isomorphism $\varphi^*(N_0)[1/\tilde{p}] \xrightarrow{\sim} N_0[1/\tilde{p}]$, compatible with the natural action of Γ_0 on each side. This allows us to conclude. ∎

Remark 4.20. Let N be a Wach module over A_R. Then, as the action of Γ_R is trivial on $N/\mu N$, therefore, we see that for each $k \in \mathbb{N}$, over the quotient $\mu^k N/\mu^{k+1} N$, the action of Γ_R is given via the p-adic cyclotomic character. In particular, it follows that $\mathbb{F}_p^\times$ has a non-trivial action on $\mu^k N/\mu^{k+1} N$, for $1 \leq k \leq p-2$.

Remark 4.21. In Proposition 4.19, for $R = O_F$, note that the $O_F[\![\mu_0]\!]$-module N_0 is p-torsion free and μ_0-torsion free. Moreover, $N_0/\mu_0 N_0 \xrightarrow{\sim} N/\mu N$ is p-torsion free. Therefore, from [Abh23b, Lemma 3.5] and [Fon90, Proposition B.1.2.4], it follows that N_0 is finite free over $O_F[\![\mu_0]\!]$.

Proposition 4.22. *Let N be a Wach module over A_R and let $M_\Lambda = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R'}$ be the Λ_R-module from Proposition 4.17. Then $M_{\Lambda,0} := M_\Lambda^{1 \times \mathbb{F}_p^\times}$ is a finitely generated module over $\Lambda_{R,0} = \Lambda_R^{\mathbb{F}_p^\times}$, equipped with an induced semilinear and continuous action of Γ_0 such that the action of Γ_0 is trivial on $M_{\Lambda,0}/\mu_0 M_{\Lambda,0}$, and we have a natural (φ, Γ_F)-equivariant isomorphism of Λ_R-modules $\Lambda_R \otimes_{\Lambda_{R,0}} M_{\Lambda,0} \xrightarrow{\sim} M_\Lambda$. Moreover, $M_{\Lambda,0}$ is equipped with an $\Lambda_{R,0}$-linear isomorphism $\varphi^*(M_{\Lambda,0})[1/p] \xrightarrow{\sim} M_{\Lambda,0}[1/p]$, compatible with the natural action of Γ_0 on each side.*

Proof. From Proposition 4.17, let us recall that M_Λ is a finitely generated Λ_R-module and we have a (φ, Γ_F)-equivariant isomorphism of Λ_R-modules $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$ via the composition $A_R \to A^{\mathrm{PD}} \xrightarrow{\sim} \Lambda_R$, where the last isomorphism is the inverse of (3.13). Now, consider N as an $R[\![\mu]\!]$-module via the (φ, Γ_F)-equivariant isomorphism of rings $\iota : R[\![\mu]\!] \xrightarrow{\sim} A_R$ (see Subsection 3.1.1), equipped with a (φ, Γ_F)-action (see Subsection 4.3.1). Then, from Proposition 4.19, we have that $N_0 := N^{\mathbb{F}_p^\times}$ is a finitely generated $R[\![\mu_0]\!]$-module, equipped with an induced action of (φ, Γ_0). Moreover, from Proposition 4.19, we note that $R[\![\mu]\!]$-linearly extending the natural inclusion $N_0 \subset N$, induces a (φ, Γ_F)-equivariant isomorphism of $R[\![\mu]\!]$-modules $R[\![\mu]\!] \otimes_{R[\![\mu_0]\!]} N_0 \xrightarrow{\sim} N$. Combining this with the preceding discussion, it follows that we have a (φ, Γ_F)-equivariant isomorphism of Λ_R-modules $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{R[\![\mu_0]\!]} N_0$. Using that $\Lambda_{R,0} = \Lambda_R^{\mathbb{F}_p^\times}$ (see Construction 3.52), together with the preceding isomorphism, we obtain a (φ, Γ_0)-equivariant isomorphism of $\Lambda_{R,0}$-modules

$$M_{\Lambda,0} = M_\Lambda^{\mathbb{F}_p^\times} \xrightarrow{\sim} \Lambda_{R,0} \otimes_{R[\![\mu_0]\!]} N_0. \tag{4.12}$$

In particular, the $\Lambda_{R,0}$-module $M_{\Lambda,0}$ is finitely generated. Now, let g be any element of Γ_0. Then note that for any f in $\Lambda_{R,0}$ and y in N_0, we have that

$$(\gamma_0 - 1) f y = (\gamma_0 - 1) f \cdot y + \gamma_0(f)(\gamma_0 - 1) y \in \mu_0(\Lambda_{R,0} \otimes_{A_R} N).$$

From the (φ, Γ_0)-equivariant isomorphism in (4.12), it follows that for any x in $M_{\Lambda,0}$, we have that $(g-1)x$ is an element of $\mu_0 M_{\Lambda,0}$. Furthermore, from (4.12) it also follows that Λ_R-linearly extending the natural inclusion $M_{\Lambda,0} \subset M_\Lambda$, gives a (φ, Γ_F)-equivariant isomorphism of Λ_R-modules

$$\begin{aligned} \Lambda_R \otimes_{\Lambda_{R,0}} M_{\Lambda,0} &\xrightarrow{\sim} M_\Lambda \\ a \otimes x &\longmapsto ax. \end{aligned} \tag{4.13}$$

Finally, note that from Proposition 4.19, we have an $R[\![\mu_0]\!]$-linear isomorphism $\varphi^*(N_0)[1/\tilde{p}] \xrightarrow{\sim} N_0[1/\tilde{p}]$ compatible with the action of Γ_0 on each side. Then, by extending this isomorphism $\Lambda_{R,0}$-linearly, using the (φ, Γ_F)-equivariant isomorphism in (4.12) and noting that $\tilde{p}/p$ is a unit in $\Lambda_{F,0}$ from Lemma 3.24, we obtain a $\Lambda_{R,0}$-linear isomorphism $\varphi^*(M_{\Lambda,0})[1/p] \xrightarrow{\sim} M_{\Lambda,0}[1/p]$ compatible with the action of Γ_0 on each side. Hence, the proposition is proved. ∎

4.3.2. The action of $1 + p\mathbb{Z}_p$. In this subsubsection, we will assume $p \geq 3$ and show the descent step, for the action of $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + p\mathbb{Z}_p)$. Let N be a Wach module over A_R, let $M_\Lambda = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma'_R}$ be the Λ_R-module from Proposition 4.17 and let $M_{\Lambda,0} = M_\Lambda^{1 \times \mathbb{F}_p^\times}$ be the $\Lambda_{R,0}$-module from Proposition 4.22. Let γ_0 be a topological generator of Γ_0 such that $\chi(\gamma_0) = 1 + pa$, for a unit a in $\mathbb{Z}_p$. Then, from Proposition 4.22 note that for any x in $M_{\Lambda,0}$, we have that $(\gamma_0 - 1)x$ is an element of $\mu_0 M_{\Lambda,0}$. Set $s := \mu_0/p$ in $\Lambda_{R,0}$, and from Lemma 3.58 recall that $(\gamma_0 - 1)s = u\mu_0$, for some unit u in $\Lambda_{F,0}$. Therefore, we see that the following operator is well-defined

$$\begin{aligned} \nabla_{q,s} : M_{\Lambda,0} &\longrightarrow M_{\Lambda,0} \\ x &\mapsto \tfrac{(\gamma_0 - 1)x}{(\gamma_0 - 1)s}. \end{aligned} \tag{4.14}$$

As the operator $\nabla_{q,s}$ is an endomorphism of $M_{\Lambda,0}$, we can define the following two term Koszul complex:

$$K_{M_{\Lambda,0}}(\nabla_{q,s}) : [M_{\Lambda,0} \xrightarrow{\nabla_{q,s}} M_{\Lambda,0}]. \tag{4.15}$$

Remark 4.23. Considering s as a variable, similar to Remark 3.59, the operator $\nabla_{q,s}$ in (4.14), may also be considered as a q-connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32. Then, (4.15) is the q-de Rham complex arising from such a q-connection.

Proposition 4.24. *The series of operators $\nabla_0^{\log} = \frac{\log \gamma_0}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}}{k+1}$ converge p-adically on $M_{\Lambda,0}$. Let $z := t^{p-1}/p$ in $\Lambda_{R,0}$, then the operator $\nabla_0 := \frac{1}{(p-1)z} \nabla_0^{\log}$ defines an R-linear p-adically quasi-nilpotent flat connection on $M_{\Lambda,0}$, denoted $\nabla : M_{\Lambda,0} \to M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega^1_{\Lambda_{R,0}/R}$ and given as $x \mapsto \nabla_0(x) dz$. The data of the connection ∇ on $M_{\Lambda,0}$ is equivalent to the data of the q-connection $\nabla_{q,s}$ from (4.14), i.e. either may be recovered from the other. Moreover, the q-de Rham complex $K_{M_{\Lambda,0}}(\nabla_{q,s})$ in (4.15) is naturally quasi-isomorphic to the de Rham complex $M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega^\bullet_{\Lambda_{R,0}/R}$. In particular, we have that $M_{\Lambda,0}^{\nabla_{q,s}=0} \xrightarrow{\sim} M_{\Lambda,0}^{\nabla_0=0}$.*

Proof. Recall that $s = \mu_0/p = ut^{p-1}/p$, for a unit u in $\Lambda_{F,0}$ (see Construction 3.52, in particular, (3.30) and the discussion preceding it). Moreover, from Lemma 3.57 we have that $(\gamma_0 - 1)s = (\gamma_0 - 1)\frac{\mu_0}{p} = v\mu_0$, for a unit v in $\Lambda_{F,0}$. Now let $z = t^{p-1}/p$ and we write $(1 + pa)^{p-1} = 1 + pb$, where b is a unit in $\mathbb{Z}_p$. Then note that we have

$$(\gamma_0 - 1)z = (\gamma_0 - 1)\frac{t^{p-1}}{p} = (\chi(\gamma_0)^{p-1} - 1)\frac{t^{p-1}}{p} = ((1 + pa)^{p-1} - 1)\frac{t^{p-1}}{p} = bt^{p-1} = u^{-1}b\mu_0. \tag{4.16}$$

Therefore, it follows that the complex $K_{M_{\Lambda,0}}(\nabla_{q,s})$ is quasi-isomorphic to the following complex

$$K_{M_{\Lambda,0}}(\nabla_{q,z}) : [M_{\Lambda,0} \xrightarrow{\nabla_{q,z}} M_{\Lambda,0}]. \tag{4.17}$$

Rest of the proof is similar to the proof of Proposition 4.13, with some changes. To avoid confusion, we provide a sketch.

Let us first show that $\nabla_0^{\log} := \frac{\log(\gamma_0)}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))}\sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_0-1)^{k+1}}{k+1}$, converge as a series of operators on $M_{\Lambda,0}$. Indeed, let x be any element of $M_{\Lambda,0}$, then using Proposition 4.22, we can write $(\gamma_0-1)x = t^{p-1}x_1$, for some x_1 in $M_{\Lambda,0}$. Let us note that $\log(\chi(\gamma_0)) = \log(1+pa) = pc$, where c is a unit in $\mathbb{Z}_p$, and we also have $(\gamma_0-1)t^{p-1} = ((1+pa)^{p-1}-1)t^{p-1} = pbt^{p-1}$, where b is a unit in $\mathbb{Z}_p$. Therefore, an easy induction on $k \in \mathbb{N}$, shows that $(\gamma_0-1)^{k+1}x = p^kt^{p-1}x_{k+1}$, for some x_{k+1} in $M_{\Lambda,0}$. In particular, we get that

$$\begin{aligned}\nabla_0^{\log}(x) &= \tfrac{1}{\log(\chi(\gamma_0))}\textstyle\sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_0-1)^{k+1}(x)}{k+1}\\ &= \tfrac{1}{\log(\chi(\gamma_0))}\textstyle\sum_{k\in\mathbb{N}}(-1)^k\frac{p^kt^{p-1}x_{k+1}}{k+1}\\ &= \tfrac{t^{p-1}}{pc}\textstyle\sum_{k\in\mathbb{N}}(-1)^k\frac{p^kx_{k+1}}{k+1},\end{aligned}$$

which converges in $\frac{t^{p-1}}{p}M_{\Lambda,0} = zM_{\Lambda,0}$, since $M_{\Lambda,0}$ is p-adically complete. Moreover, it also follows that the operator $\nabla_0 := \frac{1}{(p-1)z}\nabla_0^{\log}$ is well-defined on $M_{\Lambda,0}$.

Next, similar to the case of $\nabla_i^{\log}$ in the proof of Proposition 4.13, it can be shown that $\nabla_0^{\log}$, and therefore $\nabla_0 = \frac{1}{(p-1)z}\nabla_0^{\log}$ satisfies a Leibniz rule, i.e. $\nabla_0(fx) = \nabla_0(f)x + f\nabla_0(x)$, where the first operator on the right is $\nabla_0 := \frac{\log(\gamma_0)}{(p-1)z\log(\chi(\gamma_0))} : \Lambda_{R,0} \to \Lambda_{R,0}$ whose well-definedness can be checked similar to above. Moreover, note that the operator ∇_0 is flat by definition. Furthermore, similar to the case of ∇_i in the proof of Proposition 4.13, it can be shown that the operator $\nabla_0 : \Lambda_{R,0} \to \Omega^1_{\Lambda_{R,0}/R}$ is the continuous de Rham differential operator $d : \Lambda_{R,0} \to \Omega^1_{\Lambda_{R,0}/R}$. So, in particular, the operator ∇ : $M_{\Lambda,0} \to M_{\Lambda,0}\otimes_{\Lambda_{R,0}}\Omega^1_{\Lambda_{R,0}/R}$, given as $x \mapsto \nabla_0(x)dz$, is a well defined flat connection. Now, let us show that the operator ∇_0 is p-adically quasi-nilpotent. Indeed, we first note that from the commutativity of φ and γ_0, it follows that $\nabla_0^{\log}\circ\varphi = \varphi\circ\nabla_0^{\log}$. Therefore, it is easy to see that $\nabla_0\circ\varphi = p^{p-1}\varphi\circ\nabla_0$. Recall that $M_{\Lambda,0}$ is equipped with a $\Lambda_{R,0}$-linear isomorphism $\varphi^*(M_{\Lambda,0})[1/p] \xrightarrow{\sim} M_{\Lambda,0}[1/p]$, compatible with action of Γ_0 on each side. In particular, for any x in $M_{\Lambda,0}$, there exists $r \in \mathbb{N}$ large enough such that p^rx belongs to $\varphi^*(M_{\Lambda,0})$. Then from the relation, $\nabla_0\circ\varphi = p^{p-1}\varphi\circ\nabla_0$, we see that $\nabla_0^k(p^rx)$ converges p-adically to 0 as $k \to +\infty$. Hence, it follows that $\nabla_0^k(x) = p^{-r}\nabla_0^k(p^rx)$ converges p-adically to 0 as $k \to +\infty$, in particular, ∇_0 is p-adically quasi-nilpotent.

So far, we have defined the p-adically quasi-nilpotent flat connection ∇ using the action of Γ_0 and conversely, we have shown that the action of Γ_0 can be recovered by the formula $\gamma_0 := \exp(\log(\chi(\gamma_0))\nabla_0^{\log})$. Again, similar to case of γ_i in the proof of Proposition 4.13, using the Leibniz rule for ∇_0, it can be checked that the action of γ_0 thus obtained, is semilinear.

Finally, it remains to compare the q-de Rham complex in (4.15) with the de Rham complex $M_{\Lambda,0}\otimes_{\Lambda_{R,0}}\Omega^\bullet_{\Lambda_{R,0}/R} = M_{\Lambda,0}\otimes_{\Lambda_{F,0}}\Omega^\bullet_{\Lambda_{F,0}/O_F}$. As ∇_0 is an endomorphism of $M_{\Lambda,0}$, let $K_{M_{\Lambda,0}}(\nabla_0)$ denote the corresponding Koszul complex in the sense of Definition A.8. Then we have an identification of complexes:

$$M_{\Lambda,0}\otimes_{\Lambda_{F,0}}\Omega^\bullet_{\Lambda_{F,0}/O_F} = K_{M_{\Lambda,0}}(\nabla_0) : [\Lambda_{R,0} \xrightarrow{\nabla_0} \Lambda_{R,0}].$$

Now, recall that we have $\gamma_0 = \exp(\log(\chi(\gamma_0))\nabla_0^{\log})$. Therefore, we can write

$$\nabla_{q,z} = \tfrac{\gamma_0-1}{(\gamma_0-1)z} = \tfrac{\log(\chi(\gamma_0))(p-1)z}{(\gamma_0-1)z}\nabla_0\big(1+\textstyle\sum_{k\geq 1}\frac{\log(\chi(\gamma_0))^k}{(k+1)!}(\nabla_0^{\log})^k\big). \tag{4.18}$$

Recall that $(\gamma_0-1)z = pbz$ and $\log(\chi(\gamma_0)) = pc$, for units b and c in $\mathbb{Z}_p$. Therefore, in (4.18), we have that $\frac{\log(\chi(\gamma_0))(p-1)z}{(\gamma_0-1)z} = \frac{c(p-1)}{b}$ is a unit and it is clear that the term inside the parentheses converges p-adically to a unit. Now, in the notation of Lemma A.9, let us set $i = 1$, $M = M_{\Lambda,0}$, $f_1 = \nabla_0$ and take h_1 to be the product of $\frac{c(p-1)}{b}$ with the formula in the parentheses of (4.18), in particular, $f_1h_1 = \nabla_{q,z}$. Then, from Lemma A.9, we obtain a natural quasi-isomorphism of complexes

$$K_{M_{\Lambda,0}}(\nabla_{q,s}) \xrightarrow{\sim} K_{M_{\Lambda,0}}(\nabla_{q,z}) \xrightarrow{\sim} K_{M_{\Lambda,0}}(\nabla_0).$$

In particular, we get that $M_{\Lambda,0}^{\nabla_{q,s}=0} \xrightarrow{\sim} M_{\Lambda,0}^{\nabla_0=0}$. This allows us to conclude. ■

Proposition 4.25. *Set $M := M_{\Lambda,0}^{\nabla_0=0}$ as an R-module via the isomorphism $R \xrightarrow{\sim} \Lambda_{R,0}^{\nabla_0=0}$. Then M is finitely generated over R and we have a natural (φ, ∇_0)-equivariant isomorphism*

$$\begin{aligned} \Lambda_{R,0} \otimes_R M &\xrightarrow{\sim} M_{\Lambda,0} \\ a \otimes x &\longmapsto ax. \end{aligned} \tag{4.19}$$

Moreover, the de Rham complex $M_{\Lambda,0} \otimes_{\Lambda_{F,0}} \Omega^\bullet_{\Lambda_{F,0}/O_F}$ is aylic in positive degrees. In particular, from Proposition 4.24, we have $H^1(K_{M_{\Lambda,0}}(\nabla_{q,s})) = 0$.

Proof. The proof is similar to the proof of Proposition 4.17. For the sake of completeness and to avoid confusion, we give the details.

We will use the results from Subsection 4.2.1, by setting $\Sigma_m = \mathrm{Spec}\,(R/p^m)$ and $X_m = \mathrm{Spec}\,(\Lambda_{R,0}/p^m)$. Let us first note that from Proposition 4.24, the $\Lambda_{R,0}$-module $M_{\Lambda,0}$ is equipped with a Frobenius endomorphism (after inverting p) and an R-linear p-adically quasi-nilpotent flat connection $\nabla : M_{\Lambda,0} \to M_{\Lambda,0} \otimes_{\Lambda_{F,0}} \Omega^1_{\Lambda_{F,0}/O_F}$, in particular, $M_{\Lambda,0}$ is an object of $\mathrm{MIC}^\varphi(\Lambda_{R,0})$. Then, from the equivalence in (4.5), there exists a finitely generated F-crystal $\mathcal{E}$ over $\mathrm{CRIS}(X/\Sigma)$ and we have $\mathcal{E}(\Lambda_{R,0}) = M_{\Lambda,0}$. Next, from Construction 3.52 recall that $\Lambda_{R,0}$ is the p-adic completion of a PD-polynomial algebra over R in the variable s (see (3.30)). Moreover, $\Lambda_{R,0}$ is equipped with an R-linear flat connection $\nabla : \Lambda_{R,0} \to \Omega^1_{\Lambda_{R,0}/R} = \Lambda_{R,0} \otimes_{\Lambda_{F,0}} \Omega^1_{\Lambda_{F,0}/O_F}$ (see the proofs of Proposition 4.24 and Lemma 3.60). Now, consider the injective map $\mathrm{Spec}\,(R/p^m) \hookrightarrow \mathrm{Spec}\,(\Lambda_{R,0}/p^m)$ induced by the φ-equivariant surjection $\Lambda_{R,0} \twoheadrightarrow R$, sending $s^{[k]} \mapsto 0$ for each $k \geq 1$, and note that the composition $R \to \Lambda_{R,0} \twoheadrightarrow R$ is the identity and φ-equivariant. So we have the following φ-equivariant diagram in $\mathrm{CRIS}(X_m/\Sigma_m)$, for $m \geq 1$,

$$\begin{tikzcd} \mathrm{Spec}\,(R/p^m) \arrow[r, hook] \arrow[d, "id"] & \mathrm{Spec}\,(\Lambda_{R,0}/p^m) \arrow[d] \\ \mathrm{Spec}\,(R/p^m) \arrow[r, "id"] & \mathrm{Spec}\,(R/p^m). \end{tikzcd} \tag{4.20}$$

Evaluating the F-crystal $\mathcal{E}$ on $\mathrm{Spec}\,(R/p^m) \hookrightarrow \mathrm{Spec}\,(\Lambda_{R,0}/p^m)$ for each $m \geq 1$, and taking the limit over m, gives a finitely generated $\Lambda_{R,0}$-module $\mathcal{E}(\Lambda_{R,0}) = M_{\Lambda,0}$, equipped with a Frobenius and an R-linear p-adically quasi-nilpotent flat connection, described in Proposition 4.24 and the discussion preceding it. Now, using the fact that $\mathcal{E}$ is an F-crystal and the diagram (4.20), we obtain a φ-equivariant isomorphism

$$\bar{\varepsilon} : \Lambda_{R,0} \otimes_R \mathcal{E}(R) \xrightarrow{\sim} \mathcal{E}(\Lambda_{R,0}) = M_{\Lambda,0},$$

compatible with respective R-linear connections, where the R-linear connection on the source is given as $\nabla \otimes 1$ with ∇ being the R-linear connection on $\Lambda_{R,0}$ described in the proof of Proposition 4.24. Explicity, for x in $\Lambda_{R,0} \otimes_R \mathcal{E}(R)$, the map $\bar{\varepsilon}$ is given by the formula $\bar{\varepsilon}(x) = \sum_j (-1)^j \nabla_0^j(x) \otimes z^{[j]}$, where $\nabla_0 = \frac{1}{(p-1)z}\nabla_0^{\log}$ (see Proposition 4.24 for the definition of $\nabla_0^{\log}$).

Next, consider the de Rham complex $\Omega^\bullet_{\Lambda_{R,0}/R} = \Lambda_{R,0} \otimes_{\Lambda_{F,0}} \Omega^\bullet_{\Lambda_{F,0}/O_F}$, regarded as a complex of $\Lambda_{F,0}$-modules via the natural map $\Lambda_{F,0} \to \Lambda_{R,0}$. Recall that $\Lambda_{F,0} = O_F[s^{[k]}, k \in \mathbb{N}]^\wedge_p$, therefore it follows that the de Rham complex $\Omega^\bullet_{\Lambda_{R,0}/R} = \Lambda_{R,0} \otimes_{\Lambda_{F,0}} \Omega^\bullet_{\Lambda_{F,0}/O_F}$ is acyclic in positive degrees (also see [Ber74, Chapitre V, Lemme 2.1.2]). Hence, from the isomorphism of the de Rham complexes

$$(\Lambda_{R,0} \otimes_R \mathcal{E}(R)) \otimes_{\Lambda_{R,0}} \Omega^\bullet_{\Lambda_{R,0}/R} \xrightarrow{\sim} M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega^\bullet_{\Lambda_{R,0}/R},$$

it follows that the de Rham complex $M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega^\bullet_{\Lambda_{R,0}/R}$ is acyclic in positive degrees.

Now, taking horizontal sections for respective connections in the isomorphism $\bar{\varepsilon}$, we obtain a φ-equivariant isomorphism of finite R-modules $\mathcal{E}(R) \xrightarrow{\sim} \mathcal{E}(\Lambda_{R,0})^{\nabla=0} = M_{\Lambda,0}^{\nabla_0=0}$. In particular, from the isomorphism $\bar{\varepsilon}$, we deduce that $\Lambda_{R,0}$-linearly extending the natural inclusion $M_{\Lambda,0}^{\nabla_0=0} \subset M_{\Lambda,0}$, we obtain the following (φ, ∇_0)-equivariant diagram

$$\begin{tikzcd} \Lambda_{R,0} \otimes_R \mathcal{E}(R) \arrow[r, "\overline{\varepsilon}", "\sim"'] \arrow[d, "\overline{\varepsilon}", "\wr"'] & \mathcal{E}(\Lambda_{R,0}) \arrow[d, equal] \\ \Lambda_{R,0} \otimes_{\Lambda_R} M_{\Lambda,0}^{\nabla_0=0} \arrow[r, "(4.19)"] & M_{\Lambda,0}. \end{tikzcd}$$

Hence, it follows that (4.19) is an isomorphism. This concludes our proof. ∎

4.3.3. The case $p = 2$. In this subsubsection, we will prove statements analogous to Proposition 4.22, Proposition 4.24 and Proposition 4.25, for $p = 2$. Let N be a Wach module over A_R and let $M_\Lambda = (A_R(1)/p_1(\mu) \otimes_{p_2,A_R} N)^{1\times\Gamma'_R}$ be the Λ_R-module from Proposition 4.17 equipped with an induced action of (φ, Γ_F). First, we will look at the action of Γ_{tor} on M_Λ. Let σ denote a generator of Γ_{tor}. Then from (A.2), recall that by setting $M_{\Lambda,+} := \{x \in M_\Lambda \text{ such that } \sigma(x) = x\}$ and $M_{\Lambda,-} := \{x \in M_\Lambda \text{ such that } \sigma(x) = -x\}$, we have a natural injective map of modules over $\Lambda_{R,+}$ (see Subsection 3.4.4),

$$M_{\Lambda,+} \oplus M_{\Lambda,-} \longrightarrow M_\Lambda, \tag{4.21}$$

given as $(x, y) \mapsto x+y$. Note that the action of $1\times\Gamma_F$ is continuous for the $(p, p_1(\mu))$-adic topology on M_Λ, so it follows that $M_{\Lambda,+}$ is a $(p, p_1(\mu))$-adically complete $\Lambda_{R,+}$-submodule of M_Λ, stable under the action of (φ, Γ_F) on M_Λ, and similarly, $M_{\Lambda,-}$ is a complete $\Lambda_{R,+}$-submodule, stable under the action of (φ, Γ_F). Equipping $M_{\Lambda,+}$ and $M_{\Lambda,-}$ with induced structures, we see that (4.21) is (φ, Γ_F)-equivariant.

Proposition 4.26. *The natural map in (4.21) is bijective. Moreover, we have that $M_{\Lambda,-} = (t/2)M_{\Lambda,+}$. In particular, we have a natural Λ_R-linear and (φ, Γ_F)-equivariant isomorphism*

$$\Lambda_R \otimes_{\Lambda_{R,+}} M_{\Lambda,+} \xrightarrow{\sim} M_\Lambda. \tag{4.22}$$

Proof. Let us first note that by Λ_R-linearly extending the $\Lambda_{R,+}$-linear and (φ, Γ_F)-equivariant injective map $M_{\Lambda,+} \to M_\Lambda$ from (4.21), we obtain the Λ_R-linear and (φ, Γ_F)-equivariant map in (4.22). Recall that $\Lambda_R = \Lambda_{R,+} \oplus (t/2)\Lambda_{R,+}$ from Lemma 3.61 and Lemma 3.62, and $(t/2)M_{\Lambda,+} \subset M_{\Lambda,-}$, so from the injectivity of (4.21), it follows that (4.22) is injective. To prove the claims, it is enough to show that (4.22) is surjective as well.

Let I denote the kernel of the surjective map $\Lambda_R \twoheadrightarrow R$. Then from the explicit desciption of Λ_R in Proposition 3.21, it follows that I is the ideal of Λ_R generated by the divided powers of $t/2$. As the divided powers of $t/2$ are topologically niplotent, it follows that I is contained in the Jacobson radical of Λ_R. Moreover, since the ideal I is stable under the action of (φ, Γ_F) on Λ_R, it follows that the map $\Lambda_R \to R$ is (φ, Γ_F)-equivariant. Next, from (4.7) in Proposition 4.17, recall that we have a (φ, Γ_F)-equivariant isomorphism of Λ_R-modules $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$. Base changing this isomorphism along the surjective map $\Lambda_R \twoheadrightarrow R$, we obtain a (φ, Γ_F)-equivariant isomorphism of R-modules $M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N$. Precomposing it with (4.22) gives a Λ_R-linear map

$$\Lambda_R \otimes_{\Lambda_{R,+}} M_{\Lambda,+} \longrightarrow M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N. \tag{4.23}$$

Since I is in the Jacobson radical of Λ_R, therefore, by Nakayama Lemma, to show that (4.22) is surjective, it is enough to show that the the first map in (4.23) is surjective. So let $\overline{x}$ be any element of M_Λ/IM_Λ and let x be a lift of $\overline{x}$ in M_Λ. We first claim that $(\sigma+1)x$ is an element of $2M_\Lambda$. Indeed, let us first write $x = \sum_i a_i \otimes x_i$, for some a_i in Λ_R and x_i in N. Then, from the description of Λ_R in Lemma 3.61 and Lemma 3.62 we have that $(\sigma-1)a_i = tb_i$, for some b_i in $\Lambda_{R,+}$ and from the triviality of the action of Γ_R on $N/\mu N$, we can write $(\sigma-1)x_i = \mu y_i$, for some y_i in N. So we have the following:

$$\begin{aligned}(\sigma+1)x = (\sigma-1)x + 2x &= (\sigma-1)(\textstyle\sum_i a_i \otimes x_i) + 2x \\ &= \textstyle\sum_i(\sigma-1)a_i \otimes x_i + \sum_i \sigma(a_i) \otimes (\sigma-1)x_i + 2x \\ &= \textstyle\sum_i tbi \otimes x_i + \sum_i \sigma(a_i) \otimes \mu y_i + 2x,\end{aligned}$$

which is clearly in $2M_\Lambda$ (since t/μ is a unit in Λ_R, see Lemma 3.24). As M_Λ is p-torsion free, therefore, we set $x' := \frac{(\sigma+1)}{2}(x)$ in M_Λ and note that $\sigma(x') = x'$, i.e. x' is in $M_{\Lambda,+}$. From the computation of $(\sigma+1)x$ above and the fact that $M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N$ (see (4.23)) is p-torsion free, we see that $x' = \overline{x} \bmod IM_\Lambda$. In particular, we conclude that the first map of (4.23) is surjective, hence, (4.22) is bijective. Finally, from the decompositon $\Lambda_R = \Lambda_{R,+} \oplus (t/2)\Lambda_{R,+}$, the inclusion $(t/2)M_{\Lambda,+} \subset M_{\Lambda,-}$ and from the bijectivity of (4.22), it follows that (4.21) is bijective and $M_{\Lambda,-} = (t/2)M_{\Lambda,+}$. This completes our proof. ∎

Next, we will look at the action of $\Gamma_0 \xrightarrow{\sim} 1+4\mathbb{Z}_2$ on $M_{\Lambda,+}$. From (3.40), recall that $\nu = \frac{\mu^2}{1+\mu}$ is an element of $\Lambda_{F,+}$ and we claim the following:

Lemma 4.27. *The action of Γ_0 is trivial on $M_{\Lambda,+}/\nu M_{\Lambda,+}$.*

In the following, we provide an explicit proof of the claim. Note that the claim in Lemma 4.27 also follows from Remark 5.27. Moreover, the arguments in Remark 5.27 do not depend on the claim proven in Lemma 4.27 and the subsequent results of the current Subsection 4.3.3. In particular, Remark 5.27 gives a more conceptual proof of the claim.

Proof. Note that the action of Γ_0 is continuous on $M_{\Lambda,+}$, so it is enough to show the claim for a topological generator of Γ_0. So let us fix γ_0 to be a topological generator of Γ_0 such that $\chi(\gamma_0) = 1+4a$, for a unit a in $\mathbb{Z}_2$. Moreover, using Lemma 3.24, it is easy to see that ν is the product of t^2 with a unit in $\Lambda_{F,+}$. So for any m in $M_{\Lambda,+}$, it is enough to show that $(\gamma_0-1)m$ is an element of $t^2M_{\Lambda,+}$. Now, using Lemma 4.28, let us write $(\gamma_0-1)m = 2tm'$, for some m' in M_Λ. Since, $\sigma(m) = m$, we get that $\sigma(m') = -m'$, i.e. m' belongs to $M_{\Lambda,-}$. Using Proposition 4.26, we can write $m' = (t/2)n$, for some n in $M_{\Lambda,+}$. Hence, we get that $(\gamma_0-1)m = t^2n$, as claimed. ∎

The following observation was used above:

Lemma 4.28. *Let γ_0 be a topological generator of Γ_0 such that $\chi(\gamma_0) = 1+4a$, for a unit a in $\mathbb{Z}_2$. Then for any m in M_Λ, the element $(\gamma_0-1)m$ belongs to $2tM_\Lambda$.*

Proof. Let us start with some observations. Let f be an element of Λ_R. Then using that $\Lambda_R = \Lambda_{R,+} \oplus (t/2)\Lambda_{R,+}$ from Lemma 3.61 and Lemma 3.62 and the fact that the action of γ_0 is trivial on $\Lambda_{R,+}/\nu\Lambda_{R,+}$, where ν is the product of t^2 with a unit in $\Lambda_{F,+}$, it follows that $(\gamma_0-1)f$ is an element of $2t\Lambda_R$.

Next, let x be any element of $M_{\Lambda,+} \subset M_\Lambda$, then from the (φ, Γ_F)-equivariant isomorphism $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$ (see (4.7) in Proposition 4.17), the triviality of the action of Γ_F on Λ_R/μ (see Lemma 3.48) and $N/\mu N$ (see Definition 4.1) and the fact that t/μ is a unit in Λ_F (see Lemma 3.24), it follows that we can write$(\gamma_0-1)x = tx_1$, for some x_1 in M_Λ. As x is in $M_{\Lambda,+}$, we have $\sigma(x) = x$ and since Γ_F is commutative and M_Λ is t-torsion free, it follows that $\sigma(x_1) = -x_1$, i.e. x_1 is an element of $M_{\Lambda,-}$. Then using Proposition 4.26, we can write $x_1 = (t/2)x_2$, for some x_2 in $M_{\Lambda,+}$, and get that $(\gamma_0-1)x$ is an element of $(t^2/2)M_{\Lambda,+}$.

Next, let y be any element of $M_{\Lambda,-}$ and using Proposition 4.26, write $y = (t/2)y_1$, for some y_1 in $M_{\Lambda,+}$. From the preceding discussion, note that we have $(\gamma_0-1)y_1 = (t^2/2)y_2$, for some y_2 in $M_{\Lambda,+}$. So, we get that

$$\begin{aligned}(\gamma_0-1)y = (\gamma_0-1)(\tfrac{t}{2}y_1) &= \tfrac{1}{2}(y_1(\gamma_0-1)t + \gamma_0(t)(\gamma_0-1)y_1)\\ &= \tfrac{1}{2}(4aty_2 + (1+4a)ty_2\tfrac{t^2}{2}) = 2t(ay_1 + (1+4a)y_2\tfrac{t^2}{8}),\end{aligned}$$

is an element of $2tM_{\Lambda,+}$.

Next, note that since $N[1/p]$ is finite projective over $A_R[1/p]$ (see Remark 4.2) and the map $A_R \to \Lambda_R$ is injective, therefore, it follows that the A_R-linear and (φ, Γ_F)-equivariant map $N \to M_\Lambda$ is injective. Now, let z be an element of N and let us denote its image in M_Λ, again by z. Then by Definition 4.1, we have that $(\gamma_0-1)z = \mu z_0$, for some z_0 in $N \subset M_\Lambda$. Using Proposition 4.26, let us write $z = x+y$, for some x in $M_{\Lambda,+}$ and y in $M_{\Lambda,-} = (t/2)M_{\Lambda,+}$. Then from the preceding discussions and the fact that t/μ is a unit in Λ_F, we can write $(\gamma_0-1)x = (\mu^2/2)x'$ and $(\gamma_0-1)y = 2\mu y'$, for some x' and y' in M_Λ. In particular, since M_Λ is μ-torsion free, we get that $z_0 = (\mu/2)x' + 2y'$. Reducing the preceding equality modulo IM_Λ, where I is the kernel of the surjective map $\Lambda_R \to R$ (see the proof of Proposition 4.26), we see that $z_0 = 2y' \bmod IM_\Lambda$, in $M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N$ (see (4.23)). Since $y' \bmod IM_\Lambda$ is an element of $M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N$ and z_0 is in N, we get that $z_0 \bmod \mu N$ is an element of $2(N/\mu N)$. In particular, we can write $z_0 = 2z_1 + \mu z_2$ for some z_1, z_2 in N. So, we have that,

$$(\gamma_0 - 1)z = \mu z_0 = 2\mu z_1 + \mu^2 z_2 = 2\mu(z_1 + \tfrac{\mu}{2} z_2),$$

is an element of $2tM_\Lambda$, since t/μ is a unit in Λ_F.

Now, let $f \otimes z$ be an element of $\Lambda_R \otimes_{A_R} N$, and using the discussion at the beginning of the proof, let us write $(\gamma_0 - 1)f = 2te$, for some e in Λ_R. Moreover, from the discussion above we can write $(\gamma_0 - 1)z = 2tz'$, for some z' in M_Λ. In particular, we see that

$$(\gamma_0 - 1)(f \otimes z) = ((\gamma_0 - 1)f) \otimes z + \gamma_0(f) \otimes (\gamma_0 - 1)z = 2te + \gamma_0(f)2tz' = 2t(e + \gamma_0(f)z'),$$

is an element of $2tM_\Lambda$. Using the (φ, Γ_F)-equivariant isomorphism $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$ (see (4.7) in Proposition 4.17) and the preceding observation, we conclude that for any m in M_Λ, the element $(\gamma_0 - 1)m$ belongs to $2tM_\Lambda$. Hence, the lemma is proved. ∎

In the rest of this subsubsection, let us fix γ_0 to be a topological generator of Γ_0 such that $\chi(\gamma_0) = 1 + 4a$, for a unit a in $\mathbb{Z}_2$. Then from Lemma 4.27, note that for any x in $M_{\Lambda,+}$, we have that $(\gamma_0 - 1)x$ is an element of $\nu M_{\Lambda,+}$. Set $\tau := \nu/8$ in $\Lambda_{R,+}$, then from Lemma 3.72 we know that $(\gamma_0 - 1)\tau = u\nu$, for some unit u in $\Lambda_{F,+}$. Therefore, we see that the following operator is well-defined:

$$\begin{aligned} \nabla_{q,\tau} : M_{\Lambda,+} &\longrightarrow M_{\Lambda,+} \\ x &\mapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)\tau}. \end{aligned} \tag{4.24}$$

As the operator $\nabla_{q,\tau}$ is an endomorphism of $M_{\Lambda,+}$, we can define the following two term Koszul complex:

$$K_{M_{\Lambda,+}}(\nabla_{q,\tau}) : [M_{\Lambda,+} \xrightarrow{\nabla_{q,\tau}} M_{\Lambda,+}]. \tag{4.25}$$

Remark 4.29. Considering τ as a variable, similar to Remark 3.73, the operator $\nabla_{q,\tau}$ in (4.24), may be also considered as a q-connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32. Then, (4.25) is the q-de Rham complex arising from such a q-connection.

Proposition 4.30. *The series of operators $\nabla_0^{\log} = \frac{\log \gamma_0}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))} \sum_{k\in\mathbb{N}} (-1)^k \frac{(\gamma_0-1)^{k+1}}{k+1}$ converge p-adically on $M_{\Lambda,+}$. Let $w := t^2/8$ in $\Lambda_{R,+}$, then the operator $\nabla_0 := \frac{1}{2w}\nabla_0^{\log}$ defines an R-linear p-adically quasi-nilpotent flat connection on $M_{\Lambda,+}$, denoted $\nabla : M_{\Lambda,+} \to M_{\Lambda,+} \otimes_{\Lambda_{R,+}} \Omega^1_{\Lambda_{R,+}/R}$ and given as $x \mapsto \nabla_0(x)dw$. The data of the connection ∇ on $M_{\Lambda,+}$ is equivalent to the data of the q-connection $\nabla_{q,\tau}$ from (4.24), i.e. either may be recovered from the other. Moreover, the q-de Rham complex $K_{M_{\Lambda,+}}(\nabla_{q,\tau})$ in (4.25) is naturally quasi-isomorphic to the de Rham complex $M_{\Lambda,+} \otimes_{\Lambda_{R,+}} \Omega^\bullet_{\Lambda_{R,+}/R}$. In particular, we have that $M_{\Lambda,+}^{\nabla_{q,\tau}=0} \xrightarrow{\sim} M_{\Lambda,+}^{\nabla_0=0}$.*

Proof. The idea of the proof is similar to the proof of Proposition 4.24, with slightly different computations. We sketch it below. Recall that $\tau = et^2/8$, for a unit e in $\Lambda_{F,+}$ (see Lemma 3.62). Moreover, from Lemma 3.72 we have that $(\gamma_0 - 1)\tau = (\gamma_0 - 1)\frac{\nu}{8} = u\nu$, for a unit u in $\Lambda_{F,+}$. Now let $w = t^2/8$ and we write $(1 + 4a)^2 = 1 + 8b$, noting that $b = a(2a + 1)$ is a unit in $\mathbb{Z}_2$. So we have

$$(\gamma_0 - 1)w = (\gamma_0 - 1)\tfrac{t^2}{8} = (\chi(\gamma_0)^2 - 1)\tfrac{t^2}{8} = ((1 + 4a)^2 - 1)\tfrac{t^2}{8} = bt^2 = e^{-1}b(2a + 1)\nu. \tag{4.26}$$

Therefore, it follows that the complex $K_{M_{\Lambda,+}}(\nabla_{q,\tau})$ is quasi-isomorphic to the following complex

$$K_{M_{\Lambda,+}}(\nabla_{q,w}) : [M_{\Lambda,+} \xrightarrow{\nabla_{q,w}} M_{\Lambda,+}]. \tag{4.27}$$

Rest of the proof is similar to the proof of Proposition 4.13, with some changes. To avoid confusion, we provide a sketch.

Let us first show that $\nabla_0^{\log} := \frac{\log(\gamma_0)}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))}\sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_0-1)^{k+1}}{k+1}$, converge as a series of operators on $M_{\Lambda,+}$. Indeed, let x be any element of $M_{\Lambda,+}$, then using Lemma 4.27, we can write $(\gamma_0-1)x = t^2x_1$, for some x_1 in $M_{\Lambda,+}$. Let us note that $\log(\chi(\gamma_0)) = \log(1+4a) = 4c$, where c is a unit in $\mathbb{Z}_2$, and we also have $(\gamma_0-1)t^2 = ((1+4a)^2-1)t^{p-1} = 8bt^2$, where b is a unit in $\mathbb{Z}_2$. Therefore, an easy induction on $k \in \mathbb{N}$, shows that $(\gamma_0-1)^{k+1}x = 8^kt^2x_{k+1}$, for some x_{k+1} in $M_{\Lambda,+}$. In particular, we get that

$$\begin{aligned}\nabla_0^{\log}(x) &= \tfrac{1}{\log(\chi(\gamma_0))}\textstyle\sum_{k\in\mathbb{N}}(-1)^k\frac{(\gamma_0-1)^{k+1}(x)}{k+1}\\ &= \tfrac{1}{\log(\chi(\gamma_0))}\textstyle\sum_{k\in\mathbb{N}}(-1)^k\frac{8^kt^2x_{k+1}}{k+1}\\ &= \tfrac{t^2}{4c}\textstyle\sum_{k\in\mathbb{N}}(-1)^k\frac{8^kx_{k+1}}{k+1},\end{aligned}$$

which converges in $\frac{t^2}{4}M_{\Lambda,+} = 2wM_{\Lambda,+}$, since $M_{\Lambda,+}$ is 2-adically complete. Then, it also follows that the operator $\nabla_0 := \frac{1}{2w}\nabla_0^{\log}$ is well-defined on $M_{\Lambda,+}$.

Next, similar to the case of $\nabla_i^{\log}$ in the proof of Proposition 4.13, it can be shown that $\nabla_0^{\log}$, and therefore $\nabla_0 = \frac{1}{2w}\nabla_0^{\log}$ satisfies a Leibniz rule, i.e. $\nabla_0(fx) = \nabla_0(f)x + f\nabla_0(x)$, where the first operator on the right is $\nabla_0 := \frac{\log(\gamma_0)}{2w\log(\chi(\gamma_0))} : \Lambda_{R,+} \to \Lambda_{R,+}$ whose well-definedness can be checked similar to above. Moreover, note that the operator ∇_0 is flat by definition. Furthermore, similar to the case of ∇_i in the proof of Proposition 4.13, it can be shown that the operator $\nabla_0 : \Lambda_{R,+} \to \Omega^1_{\Lambda_{R,+}/R}$ is the usual de Rham differential $d : \Lambda_{R,+} \to \Omega^1_{\Lambda_{R,+}/R}$. So, in particular, the map $\nabla : M_{\Lambda,+} \to M_{\Lambda,+} \otimes_{\Lambda_{R,+}} \Omega^1_{\Lambda_{R,+}/R}$, given as $x \mapsto \nabla_0(x)dw$, is a well defined connection. Now, let us show that the operator ∇_0 is p-adically quasi-nilpotent. Indeed, we first note that from the commutativity of φ and γ_0, it follows that $\nabla_0^{\log} \circ \varphi = \varphi \circ \nabla_0^{\log}$. Therefore, it is easy to see that $\nabla_0 \circ \varphi = 4\varphi \circ \nabla_0$. Next, recall that M_Λ is equipped with a Λ_R-linear isomorphism $\varphi^*(M_\Lambda)[1/p] \xrightarrow{\sim} M_\Lambda[1/p]$, compatible with the action of Γ_F on each side. Then using the isomorphism (4.22) in Proposition 4.26, we easily obtain that $M_{\Lambda,+}$ is equipped with a $\Lambda_{R,+}$-linear isomorphism $\varphi^*(M_{\Lambda,+})[1/p] \xrightarrow{\sim} M_{\Lambda,+}[1/p]$, compatible with action of Γ_0 on each side. Using this observation and the relation $\nabla_0 \circ \varphi = 4\varphi \circ \nabla_0$, similar to the proof of Proposition 4.24, it can be shown that for any x in $M_{\Lambda,+}$,the sequence $\nabla_0^k(x)$ converges p-adically to 0 as $k \to +\infty$, in particular, ∇_0 is p-adically quasi-nilpotent. Furthermore, note that so far we have defined the p-adically quasi-nilpotent flat connection ∇ using the action of Γ_0 and conversely, we have shown that the action of Γ_0 can be recovered by the formula $\gamma_0 := \exp(\log(\chi(\gamma_0))\nabla_0^{\log})$. Again, similar to case of γ_i in the proof of Proposition 4.13, using the Leibniz rule for ∇_0, it can be checked that the action of γ_0 thus obtained, is semilinear.

Finally, it remains to compare the q-de Rham complex in (4.15) with the de Rham complex $M_{\Lambda,+} \otimes_{\Lambda_{R,+}} \Omega^\bullet_{\Lambda_{R,+}/R} = M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega^\bullet_{\Lambda_{F,+}/O_F}$. As ∇_0 is an endomorphism of $M_{\Lambda,+}$, let $K_{M_{\Lambda,+}}(\nabla_0)$ denote the corresponding Koszul complex in the sense of Definition A.8. Then we have an identification of complexes:

$$M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega^\bullet_{\Lambda_{F,+}/O_F} = K_{M_{\Lambda,+}}(\nabla_0) : [\Lambda_{R,+} \xrightarrow{\nabla_0} \Lambda_{R,+}].$$

Now, recall that we have $\gamma_0 = \exp(\log(\chi(\gamma_0))\nabla_0^{\log})$. Therefore, we can write

$$\nabla_{q,w} = \frac{\gamma_0-1}{(\gamma_0-1)w} = \frac{\log(\chi(\gamma_0))2w}{(\gamma_0-1)w}\nabla_0\big(1 + \textstyle\sum_{k\geq 1}\frac{\log(\chi(\gamma_0))^k}{(k+1)!}(\nabla_0^{\log})^k\big). \tag{4.28}$$

Recall that $(\gamma_0-1)w = 8bw$ and $\log(\chi(\gamma_0)) = 4c$, for units b and c in $\mathbb{Z}_2$. Therefore, in (4.28), we have that $\frac{\log(\chi(\gamma_0))2w}{(\gamma_0-1)w} = \frac{c}{b}$ is a unit and it is clear that the term inside the parentheses converges p-adically to a unit. Now, in the notation of Lemma A.9, let us set $i = 1$, $M = M_{\Lambda,+}$, $f_1 = \nabla_0$ and take h_1 to be the product of $\frac{c}{b}$ with the formula in the parentheses in (4.28), in particular, $f_1h_1 = \nabla_{q,w}$. Then, from Lemma A.9, we obtain a natural quasi-isomorphism of complexes

$$K_{M_{\Lambda,+}}(\nabla_{q,\tau}) \xrightarrow{\sim} K_{M_{\Lambda,+}}(\nabla_{q,w}) \xrightarrow{\sim} K_{M_{\Lambda,+}}(\nabla_0).$$

In particular, we get that $M_{\Lambda,+}^{\nabla_{q,\tau}=0} \xrightarrow{\sim} M_{\Lambda,+}^{\nabla_0=0}$. This allows us to conclude. ■

Proposition 4.31. *Set $M := M_{\Lambda,+}^{\nabla_0=0}$ as an R-module via the isomorphism $R \xrightarrow{\sim} \Lambda_{R,+}^{\nabla_0=0}$. Then M is finitely generated over R and we have a natural (φ, ∇_0)-equivariant isomorphism*

$$\begin{aligned} \Lambda_{R,+} \otimes_R M &\xrightarrow{\sim} M_{\Lambda,+} \\ a \otimes x &\longmapsto ax. \end{aligned} \tag{4.29}$$

Moreover, the de Rham complex $M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega^\bullet_{\Lambda_{F,+}/O_F}$ is ayclic in positive degrees. In particular, from Proposition 4.30, we have $H^1(K_{M_{\Lambda,+}}(\nabla_{q,\tau})) = 0$.

Proof. The proof works in exactly the same manner as the proof of Proposition 4.25 by changing the notations as follows: replace $\Lambda_{R,0}$ with $\Lambda_{R,+} = R[\tau^{[k]}, k \in \mathbb{N}]_p^\wedge$ and replace $M_{\Lambda,0}$ with $M_{\Lambda,+}$ equipped with a Frobenius endomorphism (after inverting p) and a p-adically quasi-nilpotent flat connection as in Proposition 4.30, where in the definition of the connection we replace the parameter z with w and the operator $\nabla_0 = \frac{1}{(p-1)z}\nabla_0^{\log}$ with $\nabla_0 = \frac{1}{2w}\nabla_0^{\log}$. Then the proof of the isomorphism (4.29) and the cyclicity of the de Rham complex $M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega^\bullet_{\Lambda_{F,+}/O_F}$ in positive degrees follows by arguments similar to the ones given, respectively, for the isomorphism (4.19) and the acyclicity of the de Rham complex $M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega^\bullet_{\Lambda_{F,+}/O_F}$ in positive degrees, in Proposition 4.25. ∎

4.4. Proof of Theorem 4.5. Let N be a Wach module over A_R as above and consider the $A_R(1)$-module $A_R(1) \otimes_{p_2, A_R} N$ equipped with the tensor product Frobenius and tensor product action of Γ_R^2, where Γ_R^2 acts on N via projection onto the second coordinate. Then, in Example 4.10, using the action of $1 \times \Gamma_R'$, we equipped $A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$ with a q-connection, denoted as ∇_q. Moreover, in Proposition 4.13, we equipped $A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$ with a connection, denoted as ∇, and showed that we have $(A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla=0} = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla_q=0}$. Since the action of $1 \times \Gamma_R'$ on $A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$ is continuous, we deduce that

$$(A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R'} = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla_q=0} = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla=0}.$$

Next, from Proposition 4.17, let us recall that $M_\Lambda := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla=0}$ is a finitely generated Λ_R-module, equipped with an induced action of (φ, Γ_F). Then from (4.6), we have that by $A_R(1)/p_1(\mu)$-linearly extending the natural inclusion $M_\Lambda \subset A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$, induces a $(\varphi, \nabla, \Gamma_F)$-equivariant, or equivalently, a (φ, Γ_R)-equivariant isomorphism

$$A_R(1)/p_1(\mu) \otimes_{\Lambda_R} M_\Lambda \xrightarrow{\sim} A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N.$$

Now, let us set $M := M_\Lambda^{\Gamma_F}$ as an R-module. Then, by using Proposition 4.22 and Proposition 4.25 for $p \geq 3$ and using Proposition 4.26 and Proposition 4.31 for $p = 2$, we see that M is a finitely generated R-module. Moreover, since $A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$ is p-torsion free by Lemma 4.11, therefore, we get that M is p-torsion free as well. Furthermore, for $p \geq 3$, using (4.13) in the proof of Proposition 4.22 and (4.19) in Proposition 4.25, and for $p = 2$, using (4.22) in Proposition 4.26 and (4.29) in Proposition 4.31, we see that by Λ_R-linearly extending the natural inclusion $M \subset M_\Lambda$, we obtain a (φ, Γ_F)-equivariant isomorphism of Λ_R-modules

$$\begin{aligned} \Lambda_R \otimes_R M &\xrightarrow{\sim} M_\Lambda \\ a \otimes x &\longmapsto ax. \end{aligned} \tag{4.30}$$

Putting everything together from the discussion above, we have the following diagram with $(\varphi, 1 \times \Gamma_R)$-equivariant arrows:

$$\begin{tikzcd} A_R(1)/p_1(\mu) \otimes_{p_1, R} M \arrow[r, "(4.1)"] \arrow[d, "\wr"', "(4.30)"] & A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N \\ A_R(1)/p_1(\mu) \otimes_{\Lambda_R} M_\Lambda. \arrow[ur, "\sim", "(4.6)"'] & \end{tikzcd}$$

The diagram commutes by the definition of the arrows. Hence, it follows that (4.1) is an isomorphism.

Finally, it remains to show that we have a φ-equivariant isomorphism of R-modules $M \xrightarrow{\sim} N/\mu N$. To show this claim, let us note that the multiplication map $\Delta : A_R(1) \to A_R$ is $(\varphi, \Gamma_R \times 1)$-equivariant, where $\Gamma_R \times 1$ acts on A_R via projection onto the first coordinate (see Lemma 3.78). Then the multiplication map Δ induces a φ-equivariant map $\Delta_N : A_R(1) \otimes_{p_2, A_R} N \to N$. Reducing it modulo $p_1(\mu)^n$ we obtain an A_R/μ^n-linear and φ-equivariant map $\Delta_N : A_R(1)/p_1(\mu)^n \otimes_{p_2, A_R} N \to N/\mu^n N$. For $n = 1$, we claim the following:

Proposition 4.32. *Let $M := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R}$ as a finitely generated φ-module over R. Then the map Δ_N restricts to a φ-equivariant R-linear isomorphism*

$$M = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} \xrightarrow{\sim} N/\mu N. \tag{4.31}$$

Proof. By definition, we have that Δ_N is φ-equivariant, so we only need to check the bijectivity of the map in claim. Let us note that the composition $R \xrightarrow{p_1} A_R(1)/p_1(\mu) \xrightarrow{\Delta} R$ is identity. Moreover, the composition $A_R \xrightarrow{p_2} A_R(1) \twoheadrightarrow A_R(1)/p_1(\mu) \xrightarrow{\Delta} R$ coincides with the map $A_R \twoheadrightarrow A_R/\mu = R$. So we have that,

$$\begin{aligned} \Delta_N : M &= R \otimes_{\Delta, A_R(1)/p_1(\mu)} (A_R(1)/p_1(\mu) \otimes_{p_1, R} M) \\ &\xrightarrow{\sim} R \otimes_{\Delta, A_R(1)/p_1(\mu)} (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N) \xrightarrow{\sim} A_R/\mu \otimes_{A_R} N = N/\mu N, \end{aligned}$$

where the first isomorphism follows from the isomorphism in Theorem 4.5 and the second isomorphism follows from the discussion above. Hence, we get the claim. ∎

This completes the proof of Theorem 4.5.

Remark 4.33. Let N be a Wach module over A_R. Then we have a φ-equivariant commutative diagram,

$$\begin{tikzcd} A_R(1)/p_1(\mu) \otimes_{p_1, A_R} N \arrow[r, "\sim"] \arrow[d, equal] & A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N \\ A_R(1)/p_1(\mu) \otimes_{p_1, R} N/\mu N \arrow[r, "\sim"] & A_R(1)/p_1(\mu) \otimes_{p_1, R} M, \arrow[u, "\wr"'] \end{tikzcd} \tag{4.32}$$

where the bottom horizontal arrow is the extension along $p_1 : R \to A_R(1)/p_1(\mu)$ of the inverse of the isomorphism Δ_N in (4.31), the right vertical arrow is (4.1) and the top horizontal arrow is the composition of left vertical, bottom horizontal and left vertical arrows. Let us note that in Proposition 5.29, we will show that the map Δ_N is equivariant for the action of $1 \times \Gamma_R$ on the source and the action of Γ_R on the target. Then it will follow that the bottom arrow in the diagram (4.32) is $(\varphi, 1 \times \Gamma_R)$-equivariant and therefore, the diagram (4.32) is also $(\varphi, 1 \times \Gamma_R)$-equivariant.

5. Prismatic F-crystals and Wach modules

In this section, let $X := \mathrm{Spf}\, R$ denote a p-adic formal scheme, where R is the ring from Subsection 1.6. Our main goal in this section is to relate analytic/completed prismatic F-crystals on the absolute prismatic site $(\mathrm{Spf}\, R)_{\Delta}$ (see Subsection 2.3) to Wach modules over A_R (see Definition 4.1). We start with the following construction:

Proposition 5.1. *Let $\mathcal{E}$ be an object of $\mathrm{Vect}^{\mathrm{an}}(X_{\Delta})$ and set $N := \mathcal{E}(A_R, [p]_q)$. Then N is a finitely generated A_R-module equipped with a continuous action of Γ_R such that the action of Γ_R is trivial on $N/\mu N$. Moreover, if $\mathcal{E}$ is an object of $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta})$, then N is a Wach module over A_R.*

Proof. First, by using Lemma 5.3, with $\mathcal{E}(A_R, [p]_q)[1/p]$ as a finite projective $A_R[1/p]$-module and $\mathcal{E}(A_R, [p]_q)[1/\mu]$ as a finite projective $A_R[1/\mu]$-module, we see that the A_R-module

$$N := \mathcal{E}(A_R, [p]_q) = \mathcal{E}(A_R, [p]_q)[1/p] \cap \mathcal{E}(A_R, [p]_q)[1/\mu] \subset \mathcal{E}(A_R, [p]_q)[1/p, 1/\mu],$$

is finitely generated and the sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N. Next, let g be any element of Γ_R and from Lemma 3.12 recall that g is an automorphism of $(A_R, [p]_q)$ in X_{Δ}, i.e. we have that $g : (A_R, [p]_q) \xrightarrow{\sim} (A_R, [p]_q)$. Since, $\mathcal{E}$ is a crystal, it follows that base changing $N = \mathcal{E}(A_R, [p]_q)$ along g, induces an A_R-semilinear isomorphism $g : A_R \otimes_{g,A_R} N \xrightarrow{\sim} N$, in particular, a semilinear action of Γ_R. Note that $A_R/\mu \xrightarrow{\sim} R$, the pair (R, p) is a prism and an object of $(\mathrm{Spf}\ R)_{\Delta}$ and Γ_R acts trivially on R. Therefore, reducing the isomorphism $g : A_R \otimes_{g,A_R} N \xrightarrow{\sim} N$ modulo μ, we obtain an isomorphism $g \bmod \mu : R \otimes_{g,R} N/\mu N \xrightarrow{\sim} N/\mu N$, which is the identity, in particular, we see that the action of Γ_R is trivial on $N/\mu N$. Now, additionally assume that $\mathcal{E}$ is an F-crystal, i.e. an object of $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta})$. Then, from Definition 2.22 we have an A_R-linear isomorphism $\varphi_N : (\varphi^* N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Again, using that $\mathcal{E}$ is a crystal and any g in Γ_R is an automorphism of $(A_R, [p]_q)$ in X_{Δ}, it is easy to show that φ_N is equivariant for the action of Γ_R on N described above. So, we see that $N = \mathcal{E}(A_R, [p]_q)$ satisfies all the axioms of Definition 4.1, i.e. it is a Wach module over A_R. ∎

Remark 5.2. A claim similar to Proposition 5.1 holds for completed prismatic F-crystals. More precisely, let $\mathcal{E}$ be an object of $\mathrm{CR}^{\wedge}(X_{\Delta})$ and set $N := \mathcal{E}(A_R, [p]_q)$. Then, N is a finitely generated A_R-module equipped with a continuous action of Γ_R such that the action of Γ_R is trivial on $N/\mu N$. Moreover, if $\mathcal{E}$ is an object of $\mathrm{CR}^{\wedge,\varphi}(X_{\Delta})$, then N is a Wach module over A_R.

Proof. By Definition 2.16, note that $N := \mathcal{E}(A_R, [p]_q)$ is a finitely generated $(p, [p]_q) = (p, \mu)$-adically complete A_R-module. Then, similar to the proof of Proposition 5.1, using that $\mathcal{E}$ is a crystal, we see that any g in Γ_R is an automorphism of $(A_R, [p]_q)$ in X_{Δ} and the action of Γ_R is trivial on $A_R/\mu \xrightarrow{\sim} R$, in particular, N is equipped with a semilinear action of Γ_R such that the action of Γ_R is trivial on $N/\mu N$. Now additionally assume that $\mathcal{E}$ is an F-crystal, i.e. an object of $\mathrm{CR}^{\wedge,\varphi}(X_{\Delta})$. So from Definition 2.18, we have an A_R-linear isomorphism $\varphi_N : (\varphi^* N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ and similar to above, it is easy to see that φ_N is Γ_R-equivariant. It remains to show that the sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N. Now, let $\mathfrak{S} := R[\![u]\!]$, then the Breuil-Kisin prism $(\mathfrak{S}, u - p)$ from Subsection 2.16 is an object of X_{Δ} and a cover of the final object of the topos $\mathrm{Shv}(X_{\Delta})$ (see [DLMS24, Subsection 3.3]). Since $(\mathfrak{S}, u - p)$ is a cover of the final object of $\mathrm{Shv}(X_{\Delta})$, there exists a prism (B, J) in X_{Δ} so that it is $(p, [p]_q)$-completely faithfully flat over $(A_R, [p]_q)$ and admits a map from $(\mathfrak{S}, u - p)$ in X_{Δ}. Moreover, as A_R is noetherian, it follows that the map $A_R \to B$ is faithfully flat. Now using an argument similar to [DLMS24, Lemma 3.19] we have that $B \otimes_{A_R} N \xrightarrow{\sim} \mathcal{E}(B, J)$, therefore,

$$(B \otimes_{A_R} N)[1/p] \xrightarrow{\sim} \mathcal{E}(B, J)[1/p] \xleftarrow{\sim} B \otimes_{\mathfrak{S}} \mathcal{E}(\mathfrak{S}, u - p)[1/p],$$

where the last isomorphism follows from [DLMS24, Lemma 3.24]. As $\mathcal{E}(\mathfrak{S}, u-p)[1/p]$ is finite projective over $\mathfrak{S}[1/p]$ and $A_R \to B$ is faithfully flat, it follows that $N[1/p]$ is finite projective over $A_R[1/p]$. A similiar argument (by inverting the prismatic ideal instead of p) shows that $N[1/[p]_q]$ is finite projective over $A_R[1/[p]_q]$. Then, by using Lemma 5.3, we see that the sequences $\{p, [p]_q\}$ and $\{[p]_q, p\}$ are regular on N, and therefore, the sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N by [Abh23b, Lemma 3.6]. Hence, it follows that N satisfies all the axioms of Definition 4.1, in particular, it is a Wach module over A_R. ∎

Let a be an element of A_R such that the sequences $\{p, a\}$ and $\{a, p\}$ are regular on A_R and we have an equality of zero loci $V(p, a) = V(p, [p]_q)$ inside $\mathrm{Spec}\,(A_R)$. For example, we may take $a = \mu$. Then, we have the following:

Lemma 5.3. *Let M be a finite projective $A_R[1/p]$-module and D a finite projective $A_R[1/a]$-module, equipped with an $A_R[1/p, 1/a]$-linear isomorphism $f : A_R[1/p, 1/a] \otimes_{A_R[1/p]} M \xrightarrow{\sim} A_R[1/p, 1/a] \otimes_{A_R[1/a]} D$. Then $N := M \cap D \subset A_R[1/p, 1/a] \otimes_{A_R[1/a]} D$ is a finitely generated A_R-module, the sequences $\{p, a\}$ and $\{a, p\}$ are regular on N, and $N[1/p] \xrightarrow{\sim} M$ and $N[1/a] \xrightarrow{\sim} D$.*

Proof. Let $S = \mathrm{Spec}\,(A_R)$, $Z = V(p, a) \subset S$ as the zero locus of the ideal $(p, a) \subset A_R$ (the same as the zero locus of $(p, [p]_q) \subset A_R$) and set $j : U = S \setminus Z \subset S$. Then, note that the natural map $\mathrm{Spec}\,(A_R[1/a]) \cup \mathrm{Spec}\,(A_R[1/p]) \to \mathrm{Spec}\,(A_R) \setminus V(p, a) = U$ is a flat cover. Therefore, from the data (M, D, f) and faithfully flat descent we obtain a vector bundle $\mathcal{F}$ over $U \subset S$. Since S is irreducible and $\mathcal{F}$ is a vector bundle on it, therefore, the associated point of $\mathcal{F}$ is the generic point of U. Moreover, by definition Z is of codimension 2, so for any z in Z and $\mathfrak{p}$ the associated point of $\mathcal{F}$ in $\mathcal{O}^{\wedge}_{S,z}$, we have that $\dim(\mathcal{O}^{\wedge}_{S,z}/\mathfrak{p}) \geq 2$. Then, from [Sta23, Tag 0BK1] it follows that $j_*\mathcal{F}$ is coherent, i.e. $N := H^0(S, j_*\mathcal{F}) = H^0(U, \mathcal{F}) = M \cap D \subset A_R[1/p, 1/a] \otimes_{A_R[1/a]} D$ is a finitely presented A_R-module. Furthermore, [Sta23, Tag 0BK0] implies that $j^*j_*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$ on U and $H^0_Z(N) = H^1_Z(N) = 0$. Now, an easy argument (for example, see [Abh23b, Lemma 3.3] for $a = \mu$) shows that the sequences $\{p, a\}$ and $\{a, p\}$ are regular on N, and $N[1/p] \xrightarrow{\sim} M$ and $N[1/a] \xrightarrow{\sim} D$ (see [Abh23b, Lemma 3.5] for $a = \mu$). This allows us to conclude. ■

Remark 5.4. Let N be a finitely generated A_R-module such that the sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N. Then, from [Abh23b, Lemma 3.5], we have that $N = N[1/p] \cap N[1/\mu] \subset N[1/p, 1/\mu]$. Moreover, if N is equipped with an A_R-linear isomorphism $\varphi_N : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$, then, by [Abh23b, Proposition 3.11 & Remark 3.12] we have that the $A_R[1/p]$-module $N[1/p]$ is finite projective, the $A_R[1/\mu]$-module $N[1/\mu]$ is finite projective and the $A_R[1/[p]_q]$-module $N[1/[p]_q]$ is finite projective.

From Proposition 5.1, note that we have a well-defined evaluation functor for analytic F-crystals:

$$\begin{aligned} \mathrm{ev}^{\Delta}_{A_R} : \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) &\longrightarrow (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{A_R} \\ \mathcal{E} &\longmapsto \mathcal{E}(A_R, [p]_q). \end{aligned} \tag{5.1}$$

Similarly, from Remark 5.2, we also obtain a well-defined evaluation functor for completed F-crystals:

$$\begin{aligned} \mathrm{ev}^{\Delta}_{A_R} : \mathrm{CR}^{\wedge,\varphi}(X_{\Delta}) &\longrightarrow (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{A_R} \\ \mathcal{E} &\longmapsto \mathcal{E}(A_R, [p]_q). \end{aligned} \tag{5.2}$$

Recall that from Lemma 3.11 we have that $(A_R, [p]_q)$ covers the final object of the topos $\mathrm{Shv}(X_{\Delta})$. Then we claim the following:

Theorem 5.5. *The evaluation functors in (5.1) and (5.2) induce equivalences of categories.*

Proof. The claim follows from Proposition 5.11 and Theorem 5.12 shown below. ■

In the rest of this section we will build the theory needed to state and prove Proposition 5.11 and Theorem 5.12.

5.1. A_R-modules with stratification.

In order to prove Theorem 5.5 we will interpret crystals in terms of modules with stratification. We begin with the definition of some cosimplicial objects in $\mathrm{Shv}(X_{\Delta})$.

5.1.1. Stratifications.

Using the cosimplicial object $A_R(\bullet)$ in $(\mathrm{Spf}\ R)_{\Delta}$ as described in Construction 3.13, we define stratifications as follows:

Definition 5.6 (Prismatic stratification)**.** A *stratification* on an A_R-module N with respect to $A_R(\bullet)$ is an $A_R(1)$-linear isomorphism $\varepsilon : A_R(1) \otimes_{p_1, A_R} N \xrightarrow{\sim} A_R(1) \otimes_{p_2, A_R} N$ satisfying the following conditions:

(1) The scalar extension $\Delta^*(\varepsilon)$ of ε by $\Delta : A_R(1) \to A_R$ is the identity map on N.

(2) Cocycle condition, i.e. we have an isomorphism $p_{23}^*(\varepsilon) \circ p_{12}^*(\varepsilon) = p_{13}^*(\varepsilon) : A_R(2) \otimes_{r_1, A_R} N \xrightarrow{\sim} A_R(2) \otimes_{r_3, A_R} N$.

Let $\mathrm{Strat}(A_R(\bullet))$ denote the category of A_R-modules equipped with a stratification with respect to $A_R(\bullet)$. Additionally, we will say that N is *analytic* if the following holds:

(3) The sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N.

Let $\mathrm{Strat}^{\mathrm{an}}(A_R(\bullet))$ denote the full subcategory of analytic objects in $\mathrm{Strat}(A_R(\bullet))$. Furthermore, we will say that N is a φ-module over A_R equipped with a stratification if N is finitely generated and equipped with a Frobenius, i.e.

(4) An A_R-linear isomorphism $\varphi_N : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ compatible with the stratification ε.

Let $\mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet))$ denote the category of analytic φ-modules over A_R equipped with a stratification.

Now, we will relate the category $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta})$ of analytic prismatic F-crystals over X_{Δ} to the category $\mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet))$ of analytic φ-modules over A_R equipped with a stratification.

Construction 5.7. Let $\mathcal{E}$ be an object of $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta})$ and set $N := \mathcal{E}(A_R(0))$ as an A_R-module. Then, by using Proposition 5.1, we have that $N = \mathcal{E}(A_R, [p]_q)$ is a finitely generated A_R-module such that the sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N. Moreover, N is equipped with an A_R-linear isomorphism $\varphi_N : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Next, set $D := \mathcal{E}[1/\mathcal{I}_{\Delta}](A_R, [p]_q) = N[1/[p]_q]$ as a finite projective $A_R[1/[p]_q]$-module and note that D is equipped with a stratitification with respect to $A_R(\bullet)$ given as the composition

$$\varepsilon_D : A_R(1) \otimes_{p_1, A_R} D \xrightarrow[\sim]{p_1} \mathcal{E}[1/\mathcal{I}_{\Delta}](A_R(1)) \xrightarrow[\sim]{p_2^{-1}} A_R(1) \otimes_{p_2, A_R} D, \tag{5.3}$$

such that $\Delta^*(\varepsilon_D) = id$ and ε_D satisfies the cocycle condition over $A_R(2)$. Similarly, set $M := \mathcal{E}[1/p](A_R, [p]_q) = N[1/p]$ as a finite projective $A_R[1/p]$-module and note that M is equipped with a stratitification with respect to $A_R(\bullet)$ given as the composition

$$\varepsilon_M : A_R(1) \otimes_{p_1, A_R} M \xrightarrow[\sim]{p_1} \mathcal{E}[1/p](A_R(1)) \xrightarrow[\sim]{p_2^{-1}} A_R(1) \otimes_{p_2, A_R} M, \tag{5.4}$$

such that $\Delta^*(\varepsilon_M) = id$ and ε_M satisfies the cocycle condition over $A_R(2)$. Note that by definition, we have an $A_R[1/p, 1/[p]_q]$-linear isomorphism $f : A_R[1/p, 1/[p]_q] \otimes_{A_R[1/p]} M \xrightarrow{\sim} A_R[1/p, 1/[p]_q] \otimes_{A_R[1/[p]_q]} D$, and (5.3) and (5.4) are compatible with the preceding isomorphism, i.e. $f \circ (\varepsilon_M[1/[p]_q]) = (\varepsilon_D[1/p]) \circ f$. Therefore, by taking the intersection of (5.3) with (5.4) inside $A_R[1/p, 1/[p]_q] \otimes_{A_R[1/[p]_q]} D$, and noting that $N = M \cap D$ and that the maps $p_1, p_2 : A_R \to A_R(1)$ are faithfully flat (see Lemma 3.15), we obtain an isomorphism

$$\varepsilon : A_R(1) \otimes_{p_1, A_R} N \xrightarrow[\sim]{p_1} \mathcal{E}(A_R(1)) \xrightarrow[\sim]{p_2^{-1}} A_R(1) \otimes_{p_2, A_R} N, \tag{5.5}$$

such that $\Delta^*(\varepsilon) = id$ and ε satisfies the cocycle condition over $A_R(2)$. Hence, N is an analytic φ-module over A_R equipped with a stratification and functorial in $\mathcal{E}$. In particular, we have described a well-defined natural functor

$$\mathrm{ev}^{\Delta}_{A_R(\bullet)} : \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) \longrightarrow \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)), \tag{5.6}$$

by sending a crystal $\mathcal{E}$ to the finitely generated A_R-module $N = \mathcal{E}(A_R(0))$ equipped with a stratification as in (5.5). ■

Proposition 5.8. *The functor in (5.6) induces a natural equivalence of categories.*

Proof. Let us first set $U := \mathrm{Spec}\,(A_R) \setminus V(p, [p]_q)$, $U(1) := \mathrm{Spec}\,(A_R(1)) \setminus V(p, [p]_q)$ equipped with projection maps $p_1, p_2 : U(1) \to U$ and $U(2) := \mathrm{Spec}\,(A_R(2)) \setminus V(p, [p]_q)$ equipped with projection maps $p_{12}, p_{23}, p_{13} : U(2) \to U(1)$ (see Construction 3.13 for the projection maps). We define the category $\mathrm{Vect}^{\mathrm{an},\varphi}(A_R(\bullet))$ of analytic prismatic F-crystals over $(A_R(\bullet), I(\bullet))$ as follows (also see Definition 2.22): an object is a vector bundle $\mathcal{E}$ on U equipped with an isomorphism $\varepsilon : p_1^*\mathcal{E} \xrightarrow{\sim} p_2^*\mathcal{E}$ of vector bundles over $U(1)$ such that $\Delta^*(\varepsilon) = id$ on $\mathcal{E}$ and ε satisfies the cocycle condition over $U(2)$, i.e. $p_{23}^*(\varepsilon) \circ p_{12}^*(\varepsilon) = p_{13}^*(\varepsilon)$. Moreover, $\mathcal{E}$ is equipped with an isomorphism $\varphi_{\mathcal{E}} : (\varphi^*\mathcal{E})[1/I] \xrightarrow{\sim} \mathcal{E}[1/I]$.

Then, from Lemma 3.11, Construction 3.13 and the well-known result describing the category of crystals of vector bundles as modules equipped with a stratification (see [Ber74, Chapitre IV, Subsection 1.6]), it follows that the top horizontal arrow in the following diagram is a natural equivalence of categories:

$$\begin{tikzcd} \mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) \arrow[r, "\sim"] \arrow[dr, "(5.6)"'] & \mathrm{Vect}^{\mathrm{an},\varphi}(A_R(\bullet)) \arrow[d] \\ & \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)). \end{tikzcd} \tag{5.7}$$

In (5.7), the right vertical arrow is defined by sending an object $\mathcal{E}$ to the finite A_R-module $N := H^0(\mathcal{E}, U)$ equipped with a Frobenius structure and a stratification as in (5.5) of Construction 5.7. It is clear that the diagram (5.7) commutes by definition. We claim that the right vertical arrow in (5.7) is an equivalence. Indeed, let us describe its quasi-inverse. Let N be an analytic φ-module over A_R equipped with a stratification. Note that the $A_R[1/p]$-module $M = N[1/p]$ is finite projective, the $A_R[1/\mu]$-module $D = N[1/\mu]$ is finite projective and $N = M \cap D \subset D[1/p]$ (see Remark 5.4). Then, from the proof of Lemma 5.3 we obtain a vector bundle $\mathcal{E}$ on U such that $N \xrightarrow{\sim} H^0(\mathcal{E}, U)$ and the stratification on N induces a stratification on $\mathcal{E}$ satisfying the desired properties. Hence, it follows that this gives a quasi-inverse to the right vertical arrow in (5.7) and induces an equivalence of categories. In particular, we get that the slanted vertical arrow in (5.7), i.e. (5.6) induces an equivalence of categories. ∎

Remark 5.9. Following arguments similar to Construction 5.7 and Proposition 5.8, it can also be shown that we have a natural equivalence of categories,

$$\mathrm{ev}^{\Delta}_{A_R(\bullet)} : \mathrm{CR}^{\wedge,\varphi}(X_{\Delta}) \xrightarrow{\sim} \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)). \tag{5.8}$$

obtained by sending a crystal $\mathcal{E}$ to the finitely generated A_R-module $N = \mathcal{E}(A_R(0))$ equipped with a stratification similar to (5.5) in Construction 5.7.

In the rest of this subsection, we will show that the functor $\mathrm{ev}^{\Delta}_{A_R(\bullet)}$ in (5.6) (resp. (5.8)) is suitably compatible with the functor $\mathrm{ev}^{\Delta}_{A_R}$ in (5.1) (resp. (5.2)), similar to [MT20, Lemma 3.16]. From Remark 3.20, recall that for $n \in \mathbb{N}$, the product $\Gamma_R^{\times(n+1)}$ of $n+1$ copies of Γ_R naturally acts on $(A_R(n), [p]_q)$. Moreover, from Proposition 3.17 we have that the action of the i^{th} component of $\Gamma_R^{\times(n+1)}$ is trivial on $A_R(n)/(n_i(\mu))$.

Construction 5.10. Let (N, ε) be an object of $\mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet))$, i.e. N is a finitely generated A_R-module on which the sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular, N is equipped with a stratification ε with respect to $A_R(\bullet)$ and an A_R-linear isomorphism $\varphi_N : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. We define a functor,

$$\mathrm{ev}^{\mathrm{Strat}}_{A_R} : \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)) \longrightarrow (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{A_R}, \tag{5.9}$$

by setting the underlying A_R-module to be N satisfying (1) and (3) of Definition 4.1. We equip N with an action of Γ_R as follows: Note that the Γ_R^2-action on $A_R(1)$ induces a semilinear-action of Γ_R on N. Indeed, for each g in Γ_R, the base change of the stratification ε along $A_R(1) \xrightarrow{(g,1)} A_R(1) \xrightarrow{\Delta} A_R$ defines an isomorphism $A_R \otimes_{g, A_R} N \xrightarrow{\sim} N$, i.e. a semilinear action of the element g on N. Moreover, for any g' in Γ_R, base change of the aforementioned isomorphism along $g' : A_R \xrightarrow{\sim} A_R$ is the base change of ε along $g' \circ \Delta \circ g = \Delta \circ (g'g, g')$. So, from Definition 5.6 it follows that the isomorphisms $A_R \otimes_{g, A_R} N \xrightarrow{\sim} N$, for g in Γ_R, define a semilinear action of Γ_R on N. Finally, since the action of $\Gamma_R \times 1$ is trivial on $A_R(1)/(p_1(\mu))$, therefore, it follows that the induced action of Γ_R on $N/\mu N$ is also trivial, and therefore, continuous by [Abh23b, Lemma 3.7]. Hence, N is a Wach module over A_R in the sense of Definition 4.1. ∎

We have the following compatibility between the functors of (5.1), (5.6) and (5.9) (resp. (5.2), (5.8) and (5.9))

Proposition 5.11. *The following diagram is commutative up to canonical isomorphisms:*

$$\begin{tikzcd}
\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta}) \arrow[r, "\mathrm{ev}^{\Delta}_{A_R(\bullet)}\,(5.6)", "\sim"'] \arrow[dr, "\mathrm{ev}^{\Delta}_{A_R}\,(5.1)"'] & \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)) \arrow[d, "(5.9)\ \mathrm{ev}^{\mathrm{Strat}}_{A_R}"] & \mathrm{CR}^{\wedge,\varphi}(X_{\Delta}) \arrow[l, "\mathrm{ev}^{\Delta}_{A_R(\bullet)}\,(5.8)"', "\sim"] \arrow[dl, "\mathrm{ev}^{\Delta}_{A_R}\,(5.1)"] \\
 & (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{A_R}. &
\end{tikzcd}$$

Proof. We will only prove the commutativity of the left triangle; commutativity of the right triangle follows by a similar argument. Let $\mathcal{E}$ be an object of $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\Delta})$ and set $(N, \varepsilon) := \mathrm{ev}^{\Delta}_{A_R(\bullet)}(\mathcal{E})$. Since $N = \mathcal{E}(A_R(0))$ and $A_R(0) = A_R$, we note that the A_R-module obtained by composing the top horizontal arrow and the left vertical arrow is isomorphic (as an A_R-module) to the module obtained via the slanted vertical arrow. From Lemma 5.3, we know that the sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N. The conditions on φ_N as explained in Proposition 5.1 and Construction 5.10 ensure that the Frobenius on N is of finite $[p]_q$-height. So it remains to check the compatibility of the action of Γ_R. Note that for each g in Γ_R, the following diagrams commute:

$$\begin{tikzcd}
A_R(1) \arrow[r, "(g,1)"] & A_R(1) \arrow[d, "\Delta"] \\
A_R(0) \arrow[u, "p_1"] \arrow[r, "g"'] & A_R(0),
\end{tikzcd}
\qquad
\begin{tikzcd}
A_R(1) \arrow[r, "(g,1)"] & A_R(1) \arrow[d, "\Delta"] \\
A_R(0) \arrow[u, "p_2"] \arrow[r, "id"'] & A_R(0).
\end{tikzcd}$$

Therefore, the action of g on $N = \mathcal{E}(A_R(0))$ induced by the stratification ε (see Construction 5.10) coincides with the action of g on $\mathcal{E}(A_R(0))$ induced by the action on $A_R(0)$ and crystal property of $\mathcal{E}$ (see Proposition 5.1). Hence, $\mathrm{ev}^{\mathrm{Strat}}_{A_R}(N, \varepsilon) \xrightarrow{\sim} \mathcal{E}(A_R(0)) = \mathcal{E}(A_R) = \mathrm{ev}^{\Delta}_{A_R}(\mathcal{E})$, allowing us to conclude. ∎

From Proposition 5.11, we note that in order to show that the functor $\mathrm{ev}^{\Delta}_{A_R}$ induces an equivalence of categories, it is enough to show that the functor $\mathrm{ev}^{\mathrm{Strat}}_{A_R}$ induces an equivalence of categories.

5.2. Constructing stratifications on Wach modules.

In this subsection we will show the following claim:

Theorem 5.12. *The functor in* (5.9) *induces a natural equivalence of categories*

$$\mathrm{ev}^{\mathrm{Strat}}_{A_R} : \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{A_R}.$$

Proof. In (5.31), we will define a functor $\mathrm{Strat}_{A_R(\bullet)}$ from the category of Wach modules to the category of φ-modules over A_R equipped with a stratification with respect to $A_R(\bullet)$. Moreover, in Proposition 5.31, we wil show that $\mathrm{Strat}_{A_R(\bullet)}$ is a quasi-inverse to the functor $\mathrm{ev}^{\mathrm{Strat}}_{A_R}$, thus extablishing the desired catgeorical equivalence ∎

In the rest of this subsection, we will construct the promised quasi-inverse to the functor $\mathrm{ev}^{\mathrm{Strat}}_{A_R}$, where the non-trivial step is the construction of a stratification on a Wach module. Our strategy will be similar to that of [MT20, Subsection 3.2]. However, as we are working with the Galois group Γ_R, which has "arithmetic $= \Gamma_F$" and "geometric $= \Gamma'_R$" parts, instead of beign a truely "geometric" Galois group as considered in loc. cit., therefore, our arguments are of different nature and require different computations. More precisely, recall that Γ_R fits into the following exact sequence:

$$1 \longrightarrow \Gamma'_R \longrightarrow \Gamma_R \longrightarrow \Gamma_F \longrightarrow 1.$$

Furthermore, from (1.6), recall that Γ_F fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1,$$

where, for $p \geq 3$, we have that $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ and for $p = 2$, we have that $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$. Moreover, for $p \geq 3$, we have that $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$ and the projection map $\Gamma_F \to \Gamma_{\text{tor}}$, admits a section $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \to \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$, where the second map is given as $a \mapsto [a]$, the Teichmüller lift of a, and the final isomorphism is induced by the p-adic cyclotomic character. Finally, for $p = 2$, we have that $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$, as groups.

Let us set $N(1) := A_R(1) \otimes_{p_2, A_R} N$ equipped with the tensor product Frobenius and tensor product action of Γ_R^2, where Γ_R^2 acts on N via projection to the second coordinate. We will start by proving some results on the cohomology of $N(1)$ for the action of $1 \times \Gamma_R$. The steps will be similar to the "3-step" argument presented in Subsection 3.4. In particular, let us note that the results for the action of the geometric part of Γ_R in Subsection 5.2.1, are applicable for all primes p. However, for $p = 2$, since $\mathbb{F}_p^\times$ is the trivial group, in Subsections 5.2.2 and 5.2.3, we will assume that $p \geq 3$. For $p = 2$, the arithmetic action of Γ_R will be handled in Subsection 5.2.4. Finally, we will put everything together in Subsection 5.2.5 to construct a stratification using the action of Γ_R on Wach modules (see Proposition 5.31).

5.2.1. The action of Γ'_R.

In this subsubsection, our first goal is to show the following claim:

Lemma 5.13. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant sequence is exact:*

$$0 \longrightarrow (N(1)/p_1(\mu))^{1\times\Gamma'_R} \xrightarrow{p_1(\mu)^n} (N(1)/p_1(\mu)^{n+1})^{1\times\Gamma'_R} \longrightarrow (N(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \longrightarrow 0. \tag{5.10}$$

Proof. The proof is similar to the proof of Lemma 3.43. To lighten notations, let us denote by $A(1) := A_R(1)$, $\overline{A}(1) := A_R(1)/p_1(\mu)$ and $\overline{N}(1) := N(1)/p_1(\mu)$. Instead of working with the action of $1 \times \Gamma'_R$, we will work with the q-connection arising from this action. More precisely, in the notation of Definition 3.31, take D to be $\tilde{\Lambda}_R \xrightarrow{\sim} A(1)^{1\times\Gamma'_R}$ (see Lemma 3.47), and A to be $A(1)$ equipped with a $\tilde{\Lambda}_R$-linear action of $1 \times \Gamma'_R$ and let $\{\gamma_1, \ldots, \gamma_d\}$ be the topological generators of Γ'_R (see Subsection 3.1). Then, by setting $q = 1 + p_2(\mu)$ and $U_i = p_2([X_i^\flat])$, for $1 \leq i \leq d$, we know that $A(1)$ satisfies the hypothesis of Definition 3.31 (see the proof of Lemma 3.43). In particular, $A(1)$ is equipped with a $\tilde{\Lambda}_R$-linear q-connection $\nabla_q : A(1) \to q\Omega^1_{A(1)/\tilde{\Lambda}_R}$, given as $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)}\, d\log(p_2([X_i^\flat]))$.

Next, we have that N is a Wach module over A_R and $N(1) = A(1) \otimes_{p_2, A_R} N$ is equipped with the tensor product Frobenius and the tensor product action of Γ_R^2. Note that for any $f \otimes y$ in $N(1)$ and g in $1 \times \Gamma_R$, we have that $(g-1)(f \otimes y) = (g-1)y \otimes y + g(f) \otimes (g-1)y$ is in $p_2(\mu)N(1)$. Therefore, the operator

$$\begin{aligned} \nabla_q : N(1) &\longrightarrow N(1) \otimes_{A(1)} \Omega^1_{A(1)/\tilde{\Lambda}_R} \\ x &\longmapsto \sum_{i=1}^d \tfrac{\gamma_i(x)-x}{p_2(\mu)}\, d\log([X_i^\flat]), \end{aligned}$$

satisfies the assumptions of Definition 4.7. Moreover, from the proof of Lemma 3.47 and Example 4.8, we see that the q-connection ∇_q on $N(1)$ is (p,μ)-adically quasi-nilpotent, and it is flat because γ_i commute with each other. Let us also note that the action of $1 \times \Gamma'_R$ is trivial on $p_1(\mu)$ and $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R \xrightarrow{\sim} \overline{A}(1)^{1\times\Gamma'_R}$ (see (3.23), Example 3.36 and the proof of Lemma 3.47). Therefore, we see that the q-connection on $\overline{N}(1)$, induced by reducing modulo $p_1(\mu)$ the q-connection on $N(1)$, coincides with the q-connection on $\overline{N}(1)$ described in Example 4.10. Now consider the following exact sequence of q-de Rham complexes:

$$\begin{aligned} 0 \longrightarrow \overline{N}(1) \otimes_{\overline{A}(1)} q\Omega^\bullet_{\overline{A}(1)/\Lambda_R} &\xrightarrow{p_1(\mu)^n} N(1)/p_1(\mu)^{n+1} \otimes_{A(1)} q\Omega^\bullet_{A(1)/\tilde{\Lambda}_R} \longrightarrow \\ &\longrightarrow N(1)/p_1(\mu)^n \otimes_{A(1)} q\Omega^\bullet_{A(1)/\tilde{\Lambda}_R} \longrightarrow 0. \end{aligned}$$

Since the action of $1 \times \Gamma'_R$ is continuous for the $(p, p_1(\mu))$-adic topology on $N(1)$, therefore, we see that $(N(1)/p_1(\mu)^n)^{1\times\Gamma'_R} = (N(1)/p_1(\mu)^n)^{\nabla_q=0}$. In particular, showing that (5.10) is exact, is equivalent to showing that $H^1(\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega^\bullet_{\overline{A}(1)/\Lambda_R}) = 0$. Now, from Proposition 4.13, recall that the q-de Rham complex $q\Omega^{\bullet}_{\overline{A}(1)/\Lambda_R}$ is naturally quasi-isomorphic to the de Rham complex $\Omega^{\bullet}_{\overline{A}(1)/\Lambda_R}$. Moreover, in Proposition 4.17, we have shown that the de Rham complex $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega^{\bullet}_{\overline{A}(1)/\Lambda_R}$ is acyclic in positive degrees, in particular, we see that $H^1(\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega^{\bullet}_{\overline{A}(1)/\Lambda_R}) = H^1(\overline{N}(1) \otimes_{\overline{A}(1)} \Omega^{\bullet}_{\overline{A}(1)/\Lambda_R}) = 0$. Hence, it follows that (5.10) is exact. ■

Let $M_{\tilde{\Lambda}} := N(1)^{1\times\Gamma'_R}$ as a module over $A(1)^{1\times\Gamma'_R} \xleftarrow{\sim} \tilde{\Lambda}_R$ (see Lemma 3.47), equipped with the induced Frobenius and an induced semilinear and continuous action of $\Gamma_R \times \Gamma_F$. Let us first note the following:

Lemma 5.14. *The sequence $\{p_2(\mu), p\}$ is regular on $M_{\tilde{\Lambda}}$.*

Proof. From Definition 4.1 recall that $\{\mu, p\}$ is a regular sequence on N and $p_2 : A_R \to A_R(1)$ is faithfully flat from Lemma 3.15. Therefore, it follows that $\{p_2(\mu), p\}$ is a regular sequence on $A_R(1) \otimes_{p_2, A_R} N$. Since $p_2(\mu)$ is invariant under the action of $1 \times \Gamma'_R$, it follows that $p_2(\mu)N(1) \cap M_{\tilde{\Lambda}} = p_2(\mu)M_{\tilde{\Lambda}}$. In particular, the natural map $M_{\tilde{\Lambda}}/p_2(\mu)M_{\tilde{\Lambda}} \to N(1)/p_2(\mu)N(1)$ is injective. Hence, $M_{\tilde{\Lambda}}/p_2(\mu)M_{\tilde{\Lambda}}$ is p-torsion free, as claimed. ■

Our next goal is to describe the action of $1 \times \Gamma_F \subset \Gamma_R \times \Gamma_F$ on $M_{\tilde{\Lambda}}$ more explicitly. So let us consider the following $(\varphi, 1 \times \Gamma_F)$-equivariant diagram

$$\begin{array}{ccc} A_R & \xrightarrow{p_2} & A_R(1) \\ \iota \uparrow \wr & & \downarrow \Delta' \\ R[\![\mu]\!] & \xrightarrow{\rho} & \tilde{\Lambda}_R, \end{array}$$

where the map ι is described in Subsection 3.1, the map p_2 is described in Subsection 3.2.2, the map Δ' is described in Subsection 3.4.1 and we set $\rho := \Delta' \circ p_2 \circ \iota$. In particular, for the bottom horizontal map we have that $\rho(\mu) = p_2(\mu)$ and $\rho(X_i) = p_1([X_i^\flat])$, for all $1 \leq i \leq d$. Tensoring the right vertical arrow with the Wach module N over A_R, we obtain the following $(\varphi, 1 \times \Gamma_F)$-equivariant map

$$\Delta'_N : N(1) = A_R(1) \otimes_{p_2, A_R} N \longrightarrow \tilde{\Lambda}_R \otimes_{\Delta' \circ p_2, A_R} N = \tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N,$$

where we consider N as an $R[\![\mu]\!]$-module via the isomorphism $\iota : R[\![\mu]\!] \xrightarrow{\sim} A_R$, equipped with a $(\varphi, 1 \times \Gamma_F)$-action (see Subsection 4.3.1). For each $n \geq 1$, reducing the map Δ'_N modulo $p_1(\mu)^n$ and taking $(1 \times \Gamma'_R)$-invariants of the source, we have the following $(\varphi, 1 \times \Gamma_F)$-equivariant composition:

$$\Delta'_N : (N(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \longrightarrow (\tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N)/p_1(\mu)^n. \tag{5.11}$$

Then we claim the following:

Lemma 5.15. *For each $n \geq 1$, the map Δ'_N in (5.11) is a $(\varphi, 1 \times \Gamma_F)$-equivariant isomorphism of $\tilde{\Lambda}_R$-modules. Moreover, (5.11) induces a $\tilde{\Lambda}_R$-linear and $(\varphi, 1 \times \Gamma_F)$-equivariant isomorphism*

$$\Delta'_N : M_{\tilde{\Lambda}} := N(1)^{1\times\Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N. \tag{5.12}$$

In particular, for each $n \geq 1$, we have natural $\tilde{\Lambda}_R$-linear and $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphisms

$$M_{\tilde{\Lambda}}/p_1(\mu)^n \xrightarrow{\sim} (N(1)/p_1(\mu)^n)^{1\times\Gamma'_R}. \tag{5.13}$$

Proof. Note that for $n = 1$, from (5.11), we have the following Λ_R-linear and (φ, Γ_F)-equivariant composition

$$M_\Lambda = \overline{N}(1)^{1\times\Gamma'_R} \longrightarrow \Lambda_R \otimes_{\rho, R[\![\mu]\!]} N,$$

where $\overline{N}(1) := N(1)/p_1(\mu)$, and we have used that $\overline{N}(1)^{1\times\Gamma'_R} = \overline{N}(1)^{\nabla_q=0} = \overline{N}(1)^{\nabla=0} = M_\Lambda$ (in the notations of Proposition 4.13 and Proposition 4.17), since the action of $1 \times \Gamma'_R$ is continuous on $\overline{N}(1)$. Now, from Proposition 4.17 and Remark 4.18, we have that the composition above is precisely the isomorphism in (4.7), in particular, for $n = 1$, (5.11) is an isomorphism. Now consider the following $\tilde{\Lambda}_R$-linear and $(\varphi, 1 \times \Gamma_F)$-equivariant diagram

$$\begin{tikzcd}
0 \arrow[r] & M_{\Lambda} \arrow[r, "p_1(\mu)^n"] \arrow[d, "\wr", "(5.11)"'] & (N(1)/p_1(\mu)^{n+1})^{1\times\Gamma'_R} \arrow[r] \arrow[d, "(5.11)"] & (N(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \arrow[r] \arrow[d, "(5.11)"] & 0\\
0 \arrow[r] & \Lambda_R \otimes_{\rho, R[\![\mu]\!]} N \arrow[r, "p_1(\mu)^n"] & (\tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N)/p_1(\mu)^{n+1} \arrow[r] & (\tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N)/p_1(\mu)^n \arrow[r] & 0,
\end{tikzcd}$$

where the first exact sequence is from (5.10). Using the diagram, an easy induction on $n \geq 1$, shows that the natural $\tilde{\Lambda}_R$-linear and $(\varphi, 1 \times \Gamma_F)$-equivariant map in (5.11) is an isomorphism, i.e. $\Delta'_N : M_{\tilde{\Lambda}}/p_1(\mu)^n \xrightarrow{\sim} (\tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N)/p_1(\mu)^n$. Moreover, as both $N(1)$ and $(\tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N)$ are $p_1(\mu)$-adically complete, taking the limit over $n \geq 1$ and noting that limit commutes with right adjoint functors, in particular, taking $(1\times\Gamma'_R)$-invariants, we obtain the $\tilde{\Lambda}_R$-linear and $(\varphi, 1\times\Gamma_F)$-equivariant isomorphism in (5.12), i.e. $\Delta'_N : M_{\tilde{\Lambda}} = N(1)^{1\times\Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N$. Finally, note that for each $n \geq 1$, it is clear that we have a natural $\tilde{\Lambda}_R$-linear and $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant inclusions $M_{\tilde{\Lambda}}/p_1(\mu)^n \subset (N(1)/p_1(\mu)^n)^{1\times\Gamma'_R}$, in particular, the map in (5.13) is injective. Now consider the following $\tilde{\Lambda}_R$-linear and $(\varphi, 1 \times \Gamma_F)$-equivariant morphisms

$$M_{\tilde{\Lambda}}/p_1(\mu)^n \xrightarrow{(5.13)} (N(1)/p_1(\mu)^n)^{1\times\Gamma'_R} \xrightarrow[\sim]{(5.11)} (\tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N)/p_1(\mu)^n.$$

where it is easy to see that the composition is reduction modulo $p_1(\mu)^n$ of the isomorphism $M_{\tilde{\Lambda}} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho, R[\![\mu]\!]} N$. Since the last arrow is bijective, therefore, it follows that (5.13) is bijective as well. This allows us to conclude. ∎

5.2.2. The action of $\mathbb{F}_p^\times$. In this subsubsection, we will assume $p \geq 3$ and consider the invariants of the exact sequence (5.10), for the action of $1 \times \mathbb{F}_p^\times$. More precisely, we claim the following:

Lemma 5.16. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant sequence is exact:*

$$0 \longrightarrow M_{\Lambda}^{\mathbb{F}_p^\times} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1\times\mathbb{F}_p^\times} \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1\times\mathbb{F}_p^\times} \longrightarrow 0. \tag{5.14}$$

Proof. Using the $\tilde{\Lambda}_R$-linear and $(\varphi, \Gamma_R\times\Gamma_F)$-equivariant isomorphism (5.13), the exact sequence (5.10) can be written as the following $\tilde{\Lambda}_R$-linear and $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant exact sequence:

$$0 \longrightarrow M_{\Lambda} \xrightarrow{p_1(\mu)^n} M_{\tilde{\Lambda}}/p_1(\mu)^{n+1} \longrightarrow M_{\tilde{\Lambda}}/p_1(\mu)^n \longrightarrow 0.$$

By considering the associated long exact sequence for the cohomology of $(1 \times \mathbb{F}_p^\times)$-action and noting that $H^1(1 \times \mathbb{F}_p^\times, M_\Lambda) = 0$, since $p-1$ is invertible in $\mathbb{Z}_p$, we obtain that the sequence in (5.14) is exact. ∎

Let us describe the $\tilde{\Lambda}_R$-modules in Lemma 5.16 more explicitly. Recall that, from Construction 3.52, we have the ring $\tilde{\Lambda}_{R,0} = \tilde{\Lambda}_R^{1\times\mathbb{F}_p^\times}$ equipped with an induced Frobenius and an induced continuous action of $\Gamma_R \times \Gamma_F$. Moreover, $M_{\tilde{\Lambda}} = N(1)^{1\times\Gamma'_R}$ is a $\tilde{\Lambda}_R$-module equipped with an induced continuous action of $(\varphi, \Gamma_R \times \Gamma_F)$. Set $M_{\tilde{\Lambda},0} := M_{\tilde{\Lambda}}^{1\times\mathbb{F}_p^\times}$ as a $\tilde{\Lambda}_{R,0}$-module, equipped with an induced Frobenius and an induced semilinear and continuous action of $\Gamma_R \times \Gamma_0$. Furthermore, from (5.13) in Lemma 5.15, we have a natural Λ_R-linear and $(\varphi, 1 \times \Gamma_F)$-equivariant isomorphism $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} M_\Lambda$, and we set $M_{\Lambda,0} := M_\Lambda^{1\times\mathbb{F}_p^\times}$ as a module over $\Lambda_{R,0} \xleftarrow{\sim} \tilde{\Lambda}_{R,0}/p_1(\mu)$ (see (3.29)), equipped with an induced Frobenius and an induced semilinear and continuous action of Γ_0. Then from the discussion before (4.13), we have that $M_{\Lambda,0}$ is a finitely generated $\Lambda_{R,0}$-module and Λ_R-linearly extending the natural inclusion $M_{\Lambda,0} \subset M_\Lambda$, induces a (φ, Γ_F)-equivariant isomorphism of Λ_R-modules $\Lambda_R \otimes_{\Lambda_{R,0}} M_{\Lambda,0} \xrightarrow{\sim} M_\Lambda$. More generally, we have the following:

Lemma 5.17. *The $\tilde{\Lambda}_{R,0}$-module $M_{\tilde{\Lambda},0}$ is finitely generated. Moreover, by $\tilde{\Lambda}_R$-linearly extending the natural inclusion $M_{\tilde{\Lambda},0} \subset M_{\tilde{\Lambda}}$, we obtain a $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism of $\tilde{\Lambda}_R$-modules*

$$\begin{aligned} \tilde{\Lambda}_R \otimes_{\tilde{\Lambda}_{R,0}} M_{\tilde{\Lambda},0} &\xrightarrow{\sim} M_{\tilde{\Lambda}} \\ a \otimes x &\longmapsto ax. \end{aligned} \tag{5.15}$$

Furthermore, we have a natural (φ, Γ_0)-equivariant isomorphism $M_{\tilde{\Lambda},0}/p_1(\mu) = M_{\tilde{\Lambda}}^{1\times\mathbb{F}_p^\times}/p_1(\mu) \xrightarrow{\sim} M_{\Lambda}^{1\times\mathbb{F}_p^\times} = M_{\Lambda,0}$.

Proof. From (5.12) in Lemma 5.15, we have a $(\varphi, 1\times\Gamma_F)$-equivariant isomorphism of $\tilde{\Lambda}_R$-modules $\Delta'_N : M_{\tilde{\Lambda}} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho,R[\![\mu]\!]} N$, in particular, $M_{\tilde{\Lambda}}$ is finitely generated over $\tilde{\Lambda}_R$. Moreover, from Proposition 4.19, recall that $N_0 := N^{\mathbb{F}_p^\times}$ is a finitely generated module over $R[\![\mu_0]\!] = R[\![\mu]\!]^{\mathbb{F}_p^\times}$ (see Lemma 3.4), equipped with the induced action of (φ, Γ_0), and we have a natural (φ, Γ_F)-equivariant isomorphism of $R[\![\mu]\!]$-modules $R[\![\mu]\!] \otimes_{R[\![\mu_0]\!]} N_0 \xrightarrow{\sim} N$. Then, taking invariants of (5.12) under the action of $1 \times \mathbb{F}_p^\times$, induces a $(\varphi, 1\times\Gamma_0)$-equivariant isomorphism of $\tilde{\Lambda}_{R,0}$-modules,

$$M_{\tilde{\Lambda},0} \xrightarrow{\sim} \tilde{\Lambda}_{R,0} \otimes_{\rho,R[\![\mu_0]\!]} N_0. \tag{5.16}$$

In particular, it follows that $M_{\tilde{\Lambda},0}$ is finitely generated over $\tilde{\Lambda}_{R,0}$. Now consider the following natural $\tilde{\Lambda}_R$-linear diagram

$$\begin{tikzcd} \tilde{\Lambda}_R \otimes_{\tilde{\Lambda}_{R,0}} M_{\tilde{\Lambda},0} \arrow[r, "(5.15)"] \arrow[d, "\wr", "(5.16)"'] & M_{\tilde{\Lambda}} \arrow[d, "\wr", "(5.11)"'] \\ \tilde{\Lambda}_R \otimes_{\rho,R[\![\mu_0]\!]} N_0 \arrow[r, "\sim"] & \tilde{\Lambda}_R \otimes_{\rho,R[\![\mu]\!]} N, \end{tikzcd}$$

where the bottom horizontal arrow is the extension along the $(\varphi, 1\times\Gamma_F)$-equivariant map $\rho : R[\![\mu]\!] \to \tilde{\Lambda}_R$ (see the discussion before Lemma 5.15) of the (φ, Γ_F)-equivariant isomorphism $R[\![\mu]\!] \otimes_{R[\![\mu_0]\!]} N_0 \xrightarrow{\sim} N$ from Proposition 4.19. The diagram commutes by definition and it follows that the top horizontal arrow, i.e. (5.15) is also an isomorphism. Finally, it is easy to see that the isomorphism (5.15) induces a natural (φ, Γ_0)-equivariant isomorphism $M_{\tilde{\Lambda},0}/p_1(\mu) = M_{\tilde{\Lambda}}^{1\times\mathbb{F}_p^\times}/p_1(\mu) \xrightarrow{\sim} M_{\Lambda}^{1\times\mathbb{F}_p^\times} = M_{\Lambda,0}$. ∎

Let us note an important observation for the action of $\Gamma_R \times \Gamma_F$ on $M_{\tilde{\Lambda}}$.

Lemma 5.18. *The action of $1 \times \Gamma_F$ is trivial on $M_{\tilde{\Lambda}}/p_2(\mu)$ and the action of $\Gamma_R \times 1$ is trivial on $M_{\tilde{\Lambda}}/p_1(\mu)$.*

Proof. From (5.12) in Lemma 5.15, recall that we have a $(\varphi, 1\times\Gamma_F)$-equivariant isomorphism $\Delta'_N : M_{\tilde{\Lambda}} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho,R[\![\mu]\!]} N$. Now let g be any element of $\Gamma_F = 1 \times \Gamma_F$, then for any $f \otimes y$ in $\tilde{\Lambda}_R \otimes_{\rho,R[\![\mu]\!]} N$, we have that

$$(g-1)(f \otimes y) = ((g-1)f) \otimes y + g(f) \otimes (g-1)y.$$

Since the action of $1\times\Gamma_F$ is trivial on $\tilde{\Lambda}_R/p_2(\mu)$ from Lemma 3.48 as well as on $N/\mu N$ by definition, it follows that $(g-1)(f\otimes x)$ is an element of $p_2(\mu)\tilde{\Lambda}_R \otimes_{\rho,R[\![\mu]\!]} N$. Then using the $(\varphi, 1\times\Gamma_F)$-equivariant isomorphism (5.12), it follows that for any x in $M_{\tilde{\Lambda}}$, we have that $(g-1)x$ is an element of $p_2(\mu)M_{\tilde{\Lambda}}$, in particular, the action of $1 \times \Gamma_F$ is trivial on $M_{\tilde{\Lambda}}/p_2(\mu)$. Next, since the action of $\Gamma_R \times 1$ is trivial on $N(1)/p_1(\mu)$ (see the proof of Lemma 5.13), therefore, it easily follows that the induced action of $\Gamma_R \times 1$ trivial on $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} (N(1)/p_1(\mu))^{1\times\Gamma'_R}$ (see (5.13) in Lemma 5.15). This concludes our proof. ∎

5.2.3. The action of $1 + p\mathbb{Z}_p$.

In this subsection, we will assume that $p \geq 3$ and consider the invariants of the exact sequence (5.14), for the action of $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + p\mathbb{Z}_p)$, and show the following:

Lemma 5.19. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times 1)$-equivariant sequence is exact:*

$$0 \longrightarrow M_{\Lambda}^{1\times\Gamma_F} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1\times\Gamma_F} \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1\times\Gamma_F} \longrightarrow 0. \tag{5.17}$$

For each $n \geq 1$, note that by reducing modulo $p_1(\mu)^n$ the $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism in (5.15) and taking its $(1 \times \mathbb{F}_p^\times)$-invariants, we obtain a $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism $M_{\Lambda,0}/p_1(\mu)^n \xrightarrow{\sim} (M_\Lambda/p_1(\mu)^n)^{1\times\mathbb{F}_p^\times}$, because $p_1(\mu)$ is invariant under the action of $1 \times \Gamma_F$. Consequently, the sequence in (3.31) can be rewritten as the following $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant sequence:

$$0 \longrightarrow M_{\tilde{\Lambda},0}^{\Gamma_0} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1})^{1\times\Gamma_0} \longrightarrow (M_{\tilde{\Lambda},0}/p_1(\mu)^n)^{1\times\Gamma_0} \longrightarrow 0. \tag{5.18}$$

In order to prove that (5.18) is exact, we will now look at the action of $\Gamma_R \times \Gamma_F$ on $M_{\tilde{\Lambda}}$ and $M_{\tilde{\Lambda},0}$, respectively. We start with the following observation:

Lemma 5.20. *The action of* $1 \times \Gamma_0$ *is trivial on* $M_{\tilde{\Lambda},0}/p_2(\mu_0)$ *and the action of* $\Gamma_R \times 1$ *is trivial on* $M_{\tilde{\Lambda},0}/p_1(\mu)$.

Proof. Using the triviality of the action of $1 \times \Gamma_F$ on $M_{\tilde{\Lambda}}/p_2(\mu)$ from Lemma 5.18, the $(\varphi, 1 \times \Gamma_0)$-equivariant isomorphism $M_{\tilde{\Lambda},0} \xrightarrow{\sim} \tilde{\Lambda}_{R,0} \otimes_{\rho, R[\![\mu_0]\!]} N_0$ (see (5.15) in Lemma 5.16) and the fact that the action of $1 \times \Gamma_0$ is trivial on $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$ (see Lemma 3.55) as well as on $N_0/\mu_0 N_0$ (see Proposition 4.19), it follows that the action of $1 \times \Gamma_0$ is trivial on $M_{\tilde{\Lambda},0}/p_2(\mu_0)$. Next, since the action of $\Gamma_R \times 1$ is trivial on $N(1)/p_1(\mu)$ (see the proof of Lemma 5.13), therefore, it easily follows that the induced action of $\Gamma_R \times 1$ trivial on $M_{\tilde{\Lambda},0}/p_1(\mu) \xrightarrow{\sim} M_\Lambda^{1\times\mathbb{F}_p^\times}$ (see Lemma 5.17). ■

Remark 5.21. From Lemma 5.20, note that the action of $1 \times \Gamma_0$ is trivial on $M_{\tilde{\Lambda},0}/p_2(\mu_0)$ and the element $p_1(\mu)$ is invariant under the action of $1 \times \Gamma_0$ on $\tilde{\Lambda}_{R,0}$. Therefore, it follows that for any g in Γ_0 and any x in $M_{\tilde{\Lambda},0}/p_1(\mu)^n$ we have that $(g-1)x$ is an element of $p_2(\mu_0)M_{\tilde{\Lambda},0}/p_1(\mu)^n$. In particular, for $n = 1$, using the isomorphism $M_{\tilde{\Lambda},0}/p_1(\mu) \xleftarrow{\sim} M_{\Lambda,0}$ from Lemma 5.17, we get that for any g in Γ_0 and any x in $M_{\Lambda,0}$ the element $(g-1)x$ belongs to $\mu_0 M_{\Lambda,0}$.

Using the action of $1 \times \Gamma_0$ on $M_{\tilde{\Lambda},0}$, we will define a q-connection (see Definition 4.7). Recall that in Subsection 3.4.3, in order to define a q-de Rham complex over $\tilde{\Lambda}_{R,0}$, we fixed the following element in $\tilde{\Lambda}_{R,0}$ as a parameter (see (3.33)):

$$\tilde{s} = \tfrac{1\otimes\tilde{p}-\tilde{p}\otimes 1}{\tilde{p}\otimes 1} = \tfrac{p_2(\tilde{p})-p_1(\tilde{p})}{p_1(\tilde{p})}$$

Moreover, if γ_0 is any element of $1 \times \Gamma_0$, then from Lemma 3.57 we have that $(\gamma_0 - 1)\tilde{s} = u\,p_2(\mu_0)$, for some unit u in $\tilde{\Lambda}_{R,0}$.

In the rest of this subsubsection, we will fix the choice of a topological generator γ_0 of $1 \times \Gamma_0$ such that $\chi(\gamma_0) = 1 + pa$, for a unit a in $\mathbb{Z}_p$. Let us now consider the following operator on $M_{\tilde{\Lambda},0}$:

$$\begin{aligned} \nabla_{q,\tilde{s}} : M_{\tilde{\Lambda},0} &\longrightarrow M_{\tilde{\Lambda},0} \\ x &\mapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)\tilde{s}}. \end{aligned} \tag{5.19}$$

From the triviality of the action of $1 \times \Gamma_0$ on $M_{\tilde{\Lambda},0}/p_2(\mu_0)$ (see Lemma 5.20) and from Lemma 3.57, it follows that the operator $\nabla_{q,\tilde{s}}$ is well-defined. For each $n \in \mathbb{N}$, using Remark 5.21, the operator in (5.19), induces well-defined operators $\nabla_{q,\tilde{s}} : M_{\tilde{\Lambda},0}/p_1(\mu)^n \longrightarrow M_{\tilde{\Lambda},0}/p_1(\mu)^n$. As the operator $\nabla_{q,\tilde{s}}$ is an endomorphism of $M_{\tilde{\Lambda},0}/p_1(\mu)^n$, we can define the following two term Koszul complex:

$$K_{M_{\tilde{\Lambda},0}/p_1(\mu)^n}(\nabla_{q,\tilde{s}}) : [M_{\tilde{\Lambda},0}/p_1(\mu)^n \xrightarrow{\nabla_{q,\tilde{s}}} M_{\tilde{\Lambda},0}/p_1(\mu)^n]. \tag{5.20}$$

In particular, for $n = 1$, we set $s := \mu_0/p$ in $\Lambda_{R,0}$, then using Remark 5.21 and the fact that $(\gamma_0 - 1)s = v\mu_0$, for some unit v in $\Lambda_{F,0}$ (see Lemma 3.58), we have a well-defined operator

$$\begin{aligned} \nabla_{q,s} : M_{\Lambda,0} &\longrightarrow M_{\Lambda,0} \\ x &\mapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)s}. \end{aligned}$$

Note that the operator above coincides with the operator defined in (4.14) and the complex from (5.20) for $n = 1$, coincides with the complex from (4.15). Therefore, from Proposition 4.25, we have that the cohomology of the Koszul complex $K_{M_{\Lambda,0}}(\nabla_{q,s})$ vanishes in degree 1, i.e. $H^1(K_{M_{\Lambda,0}}(\nabla_{q,s})) = 0$.

Remark 5.22. Considering $\tilde{s}$ as a parameter, similar to Remark 3.59, the operator $\nabla_{q,\tilde{s}}$ in (5.19), may be considered as a q-connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32. Then, (4.15) is the q-de Rham complex arising from such a q-connection. Similarly, considering s as a parameter, the operator $\nabla_{q,s}$ on $M_{\Lambda,0}$, may be also considered as a q-connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32.

Proof of Lemma 5.19. Note that using the $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism $M_{\Lambda,0}/p_1(\mu)^n \xrightarrow{\sim} (M_\Lambda/p_1(\mu)^n)^{1\times\mathbb{F}_p^\times}$ (see the discussion before (5.18)), we can rewrite the exact sequence in (5.14) as follows:

$$0 \longrightarrow M_{\Lambda,0} \xrightarrow{p_1(\mu)^n} M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1} \longrightarrow M_{\tilde{\Lambda},0}/p_1(\mu)^n \longrightarrow 0.$$

Then, using the operator $\nabla_{q,\tilde{s}}$ in (5.19) and the Koszul complex defined in (5.20), we obtain an exact sequence of Koszul complexes:

$$0 \longrightarrow K_{M_{\Lambda,0}}(\nabla_{q,s}) \xrightarrow{p_1(\mu)^n} K_{M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1}}(\nabla_{q,\tilde{s}}) \longrightarrow K_{M_{\tilde{\Lambda},0}/p_1(\mu)^n}(\nabla_{q,\tilde{s}}) \longrightarrow 0.$$

Considering the associated long exact sequence, and noting that $H^1(K_{M_{\Lambda,0}}(\nabla_{q,s})) = 0$ from Proposition 4.25, we obtain the following exact sequence:

$$0 \longrightarrow M_{\Lambda,0}^{\nabla_{q,s}=0} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{s}}=0} \longrightarrow (M_{\tilde{\Lambda},0}/p_1(\mu)^n)^{\nabla_{q,\tilde{s}}=0} \longrightarrow 0.$$

Since the action of $1 \times \Gamma_0$ is continuous on $M_{\tilde{\Lambda},0}$ for the $(p, p_1(\mu))$-adic topology, therefore, we see that $(M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{s}}=0} = (M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1})^{1\times\Gamma_0}$, for each $n \in \mathbb{N}$. Hence, from the preceding exact sequence we obtain that the sequence in (5.18) is exact, and therefore, the sequence (5.17) is exact as well. ∎

5.2.4. The case $p = 2$. In this subsubsection, our goal is to prove a statement similar to Lemma 5.19, for $p = 2$. From (1.6), recall that Γ_F fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\text{tor}} \longrightarrow 1,$$

where, we have that $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$ and $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$, as groups.

We will first look at the action of Γ_{tor} on $M_{\tilde{\Lambda}}$. Let σ denote a generator of Γ_{tor}. Then, from (A.2), recall that by setting $M_{\tilde{\Lambda},+} := \{x \in M_{\tilde{\Lambda}} \text{ such that } \sigma(x) = x\}$ and $M_{\tilde{\Lambda},-} := \{x \in M_{\tilde{\Lambda}} \text{ such that } \sigma(x) = -x\}$, we have a natural injective map of $\tilde{\Lambda}_{R,+}$-modules

$$M_{\tilde{\Lambda},+} \oplus M_{\tilde{\Lambda},-} \longrightarrow M_{\tilde{\Lambda}}, \tag{5.21}$$

given as $(x, y) \mapsto x + y$. Note that the action of $1 \times \Gamma_F$ is continuous for the $(p, p_1(\mu))$-adic topology on $M_{\tilde{\Lambda}}$, so it follows that $M_{\tilde{\Lambda},+}$ is a $(p, p_1(\mu))$-adically complete $\tilde{\Lambda}_{R,+}$-submodule of $M_{\tilde{\Lambda}}$, stable under the action of $(\varphi, \Gamma_R \times \Gamma_F)$ on $M_{\tilde{\Lambda}}$, and similarly, $M_{\tilde{\Lambda},-}$ is a complete $\tilde{\Lambda}_{R,+}$-submodule of $M_{\tilde{\Lambda}}$, stable under the action of $(\varphi, \Gamma_R \times \Gamma_F)$. Equipping $M_{\tilde{\Lambda},+}$ and $M_{\tilde{\Lambda},-}$ with induced structures, we see that (5.21) is $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant.

Now, from (5.13), recall that $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} (N(1)/p_1(\mu))^{1\times\Gamma'_R} = M_\Lambda$. Similar to above, in Subsection 4.3.3, we defined $\Lambda_{R,+}$-modules $M_{\Lambda,+}$ and $M_{\Lambda,-}$ and showed that their natural inclusion in M_Λ induces a natural (φ, Γ_F)-equivariant isomorphism of $\Lambda_{R,+}$-modules $M_{\Lambda,+} \oplus M_{\Lambda,-} \xrightarrow{\sim} M_{\Lambda,+}$ (see (4.21) in Lemma 4.26).

Lemma 5.23. *For each $n \geq 1$, reduction modulo $p_1(\mu)^n$ of (5.21), induces a natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism*

$$M_{\tilde{\Lambda},+}/p_1(\mu)^n \xrightarrow{\sim} (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1\times\Gamma_{\text{tor}}}. \tag{5.22}$$

Moreover, for $n = 1$, the (φ, Γ_F)-equivariant isomorphism $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} M_\Lambda$ from (5.13), induces a natural (φ, Γ_F)-equivariant isomorphism

$$M_{\tilde{\Lambda},+}/p_1(\mu) \xrightarrow{\sim} (M_{\tilde{\Lambda}}/p_1(\mu))^{1\times\Gamma_{\text{tor}}} \xrightarrow{\sim} M_{\Lambda,+}. \tag{5.23}$$

Proof. Let us consider the following natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & M_{\tilde{\Lambda},+}/p_1(\mu) \arrow[r, "p_1(\mu)^n"] \arrow[d, "(5.21)"] & M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1} \arrow[r] \arrow[d, "(5.21)"] & M_{\tilde{\Lambda},+}/p_1(\mu)^n \arrow[r] \arrow[d, "(5.21)"] & 0 \\ 0 \arrow[r] & (M_{\tilde{\Lambda}}/p_1(\mu))^{1\times\Gamma_{\mathrm{tor}}} \arrow[r, "p_1(\mu)^n"] & (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1\times\Gamma_{\mathrm{tor}}} \arrow[r] & (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}} \arrow[r] & 0, \end{tikzcd} \tag{5.24}$$

where the top row is exact and the vertical maps are injective because we have $p_1(\mu)^n M_{\tilde{\Lambda}} \cap M_{\tilde{\Lambda},+} = p_1(\mu)^n M_{\tilde{\Lambda},+}$, as $p_1(\mu)$ is invariant under the action of $1 \times \Gamma_F$. As the action of $1 \times \Gamma_{\mathrm{tor}} \subset 1 \times \Gamma_F$ is trivial on $M_{\tilde{\Lambda}}/p_2(\mu)$ from Lemma 5.18, therefore, for each $n \geq 1$ and x in $M_{\tilde{\Lambda}}/p_1(\mu)^n$, we see that $(\sigma - 1)x$ is an element of $p_2(\mu)(M_{\tilde{\Lambda}}/p_1(\mu)^n)$. Then from Lemma A.11, it follows that the bottom right horizontal arrow in (5.24) is surjective, in particular, the bottom row is exact.

Next, by composing the left vertical arrow in (5.24) with the (φ, Γ_F)-equivariant isomorphism $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} M_\Lambda$ from (5.13), we obtain a natural (φ, Γ_F)-equivariant injective map $M_{\tilde{\Lambda},+}/p_1(\mu) \rightarrow M_{\Lambda,+}$, and we will show that it is surjective as well. Indeed, since the action of $1 \times \Gamma_F$ is trivial on $M_{\tilde{\Lambda}}/p_2(\mu)$ from Lemma 5.18 and the action of Γ_F is trivial on $M_\Lambda/\mu M_\Lambda$ using (4.7) in Proposition 4.17, therefore, by using Lemma A.11 it follows that the $\tilde{\Lambda}_R$-linear and $(\varphi, 1 \times \Gamma_F)$-equivariant surjective map $M_{\tilde{\Lambda}} \twoheadrightarrow M_\Lambda$ from (5.13), induces a $\tilde{\Lambda}_{R,+}$-linear and $(\varphi, 1 \times \Gamma_0)$-equivariant surjective map $M_{\tilde{\Lambda},+} \twoheadrightarrow M_{\Lambda,+}$, which factors through (5.23). In particular, we get that the composition in (5.23) is bijective, therefore, the left vertical arrow in (5.24) is also bijective. Now, using the diagram (5.24), an easy induction on $n \geq 1$, gives that for each $n \geq 1$, the right vertical arrow is bijective. Hence, it follows that the natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant map $M_{\tilde{\Lambda},+}/p_1(\mu)^n \rightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}}$, induced by (5.21), is bijective for each $n \geq 1$. ∎

From Lemma 5.23, we obtain the following:

Lemma 5.24. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant sequence is exact:*

$$0 \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu))^{1\times\Gamma_{\mathrm{tor}}} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1\times\Gamma_{\mathrm{tor}}} \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}} \longrightarrow 0. \tag{5.25}$$

Proof. The sequence (5.25) is the same as the second row of the diagram (5.22), which was shown to be exact in the proof of Lemma 5.23. ∎

Next, we will look at the action of $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + 4\mathbb{Z}_2)$ on $M_{\tilde{\Lambda},+}$ and show the following:

Lemma 5.25. *For each $n \geq 1$, the following natural $(\varphi, \Gamma_R \times 1)$-equivariant sequence is exact:*

$$0 \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu))^{1\times\Gamma_F} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1\times\Gamma_F} \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1\times\Gamma_F} \longrightarrow 0. \tag{5.26}$$

For each $n \geq 1$, note that by reducing modulo $p_1(\mu)^n$ the $(\varphi, \Gamma_R \times \Gamma_F)$-equivariant isomorphism in (5.15) and taking its $(1 \times \Gamma_{\mathrm{tor}})$-invariants, we obtain a $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism $M_{\Lambda,+}/p_1(\mu)^n \xrightarrow{\sim} (M_\Lambda/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}}$, because $p_1(\mu)$ is invariant under the action of $1 \times \Gamma_F$. Consequently, the sequence in (5.17) can be rewritten as the following $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant sequence:

$$0 \longrightarrow M_{\tilde{\Lambda},+}^{\Gamma_0} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1})^{1\times\Gamma_0} \longrightarrow (M_{\tilde{\Lambda},+}/p_1(\mu)^n)^{1\times\Gamma_0} \longrightarrow 0. \tag{5.27}$$

In order to prove that (5.27) is exact, we will now look at the action of $\Gamma_R \times \Gamma_F$ on $M_{\tilde{\Lambda}}$ and $M_{\tilde{\Lambda},+}$, respectively. From (3.40) recall that we set $\nu = \frac{\mu^2}{1+\mu}$ in A_F, and we make the following observation:

Lemma 5.26. *The action of $1 \times \Gamma_0$ is trivial on $M_{\tilde{\Lambda},+}/p_2(\nu)$ and the action of $\Gamma_R \times 1$ is trivial on $M_{\tilde{\Lambda},+}/p_1(\mu)$.*

Proof. For the first claim, note that ν is invariant under the action of Γ_{tor} and we have that $p_2(\nu)M_{\tilde{\Lambda}} \cap M_{\tilde{\Lambda},+} = p_2(\nu)M_{\tilde{\Lambda},+}$. So, if x is an element of $M_{\tilde{\Lambda},+}$ and g any element of $1 \times \Gamma_0$, then it is enough to show that $(g-1)x$ is an element of $p_2(\nu)M_{\tilde{\Lambda}}$. Moreover, note that ν and μ^2 differ by a unit in A_F. Therefore, we are reduced to showing that $(g-1)x$ is an element of $p_2(\mu)^2M_{\tilde{\Lambda}}$. Now, using Lemma 5.18, we can write $(g-1)x = p_2(\mu)y$, for some y in $M_{\tilde{\Lambda}}$. Let σ be a generator of $1 \times \Gamma_{\mathrm{tor}}$ and note that $\sigma(x) = x$. Then, we have that $\sigma(p_2(\mu))\sigma(y) = p_2(\mu)y$, in particular, $(\sigma - 1)y = -(2 + p_2(\mu))y$. Again using Lemma 5.18, we can write $-p_2([p]_q)y = (\sigma-1)y = p_2(\mu)z$, for some z in $M_{\tilde{\Lambda}}$. So we get that $-py = 0 \bmod p_2(\mu)M_{\tilde{\Lambda}}$. Note that $(p_2(\mu), p)$ is a regular sequence on $M_{\tilde{\Lambda}}$ (see Lemma 5.14). Therefore, we conclude that $y = 0 \bmod p_2(\mu)\tilde{\Lambda}_R$, i.e. y is an element of $p_2(\mu)\tilde{\Lambda}_R$ and $(g-1)x = p_2(\mu)y$ is an element of $p_2(\mu)^2\tilde{\Lambda}_R$, as claimed. The second claim easily follows from Lemma 3.48. ∎

Remark 5.27. From Lemma 5.26, note that the action of $1 \times \Gamma_0$ is trivial on $M_{\tilde{\Lambda},+}/p_2(\nu)$ and the element $p_1(\mu)$ is invariant under this action. Therefore, it follows that for any g in $1 \times \Gamma_0$ and any x in $M_{\tilde{\Lambda},+}/p_1(\mu)^n$, we have that $(g-1)x$ is an element of $p_2(\nu)M_{\tilde{\Lambda},+}/p_1(\mu)^n$. In particular, for $n = 1$, using the isomorphism $M_{\tilde{\Lambda},+}/p_1(\mu) \xrightarrow{\sim} M_{\Lambda,+}$ from (5.23), we get that for any g in Γ_0 and any x in $M_{\Lambda,+}$, the element $(g-1)x$ belongs to $\nu M_{\Lambda,+}$.

Using the action of $1 \times \Gamma_0$ on $M_{\tilde{\Lambda},+}$, we will define a q-connection (see Definition 4.7). Recall that in Subsection 3.4.4, in order to define a q-de Rham complex over $\tilde{\Lambda}_R$, we fixed the following element in $\tilde{\Lambda}_{R,+}$ as a parameter (see (3.42)):

$$\tilde{\tau} = \tfrac{1}{p_2([p]_q)}\delta\big(\tfrac{p_2([p]_q)}{p_1([p]_q)}\big).$$

Moreover, if γ_0 is any element of $1 \times \Gamma_0$, then from Lemma 3.71 we have that $(\gamma_0 - 1)\tilde{\tau} = u\,p_2(\nu)$, for some unit u in $\tilde{\Lambda}_{R,+}$.

In the rest of this subsubsection, we will fix the choice of a topological generator γ_0 of $1 \times \Gamma_0$ such that $\chi(\gamma_0) = 1 + 4a$, for a unit a in $\mathbb{Z}_2$. Let us now consider the following operator on $M_{\tilde{\Lambda},+}$:

$$\begin{aligned} \nabla_{q,\tilde{\tau}} : M_{\tilde{\Lambda},+} &\longrightarrow M_{\tilde{\Lambda},+} \\ x &\longmapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)\tilde{\tau}}. \end{aligned} \tag{5.28}$$

From the triviality of the action of $1 \times \Gamma_0$ on $M_{\tilde{\Lambda},+}/p_2(\nu)$ (see Lemma 5.26) and from Lemma 3.71, it follows that the operator $\nabla_{q,\tilde{\tau}}$ is well-defined. For each $n \in \mathbb{N}$, using Remark 5.27, the operator in (5.28), induces well-defined operators $\nabla_{q,\tilde{\tau}} : M_{\tilde{\Lambda},+}/p_1(\mu)^n \longrightarrow M_{\tilde{\Lambda},+}/p_1(\mu)^n$. As the operator $\nabla_{q,\tilde{\tau}}$ is an endomorphism of $M_{\tilde{\Lambda},+}/p_1(\mu)^n$, we can define the following two term Koszul complex:

$$K_{M_{\tilde{\Lambda},+}/p_1(\mu)^n}(\nabla_{q,\tilde{\tau}}) : \big[M_{\tilde{\Lambda},+}/p_1(\mu)^n \xrightarrow{\nabla_{q,\tilde{\tau}}} M_{\tilde{\Lambda},+}/p_1(\mu)^n\big]. \tag{5.29}$$

In particular, for $n = 1$, we set $\tau := \nu/8$ in $\Lambda_{R,+}$, then using Remark 5.27 and the fact that $(\gamma_0 - 1)s = v\nu$, for some unit v in $\Lambda_{F,+}$ (see Lemma 3.72), we have a well-defined operator

$$\begin{aligned} \nabla_{q,\tau} : M_{\Lambda,+} &\longrightarrow M_{\Lambda,+} \\ x &\longmapsto \tfrac{(\gamma_0-1)x}{(\gamma_0-1)\tau}. \end{aligned}$$

Note that the operator above coincides with the operator defined in (4.24) and the complex from (5.29) for $n = 1$, coincides with the complex from (4.25). Therefore, from Proposition 4.31, we have that the cohomology of the Koszul complex $K_{M_{\Lambda,+}}(\nabla_{q,s})$ vanishes in degree 1, i.e. $H^1(K_{M_{\Lambda,+}}(\nabla_{q,\tau})) = 0$.

Remark 5.28. Considering $\tilde{\tau}$ as a parameter, similar to Remark 3.73, the operator $\nabla_{q,\tilde{\tau}}$ in (5.28), may be considered as a q-connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32. Then, (4.25) is the q-de Rham complex arising from such a q-connection. Similarly, considering s as a parameter, the operator $\nabla_{q,\tau}$ on $M_{\Lambda,+}$, may be also considered as a q-connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32.

Proof of Lemma 5.25. Note that by using the $(\varphi, \Gamma_R \times \Gamma_0)$-equivariant isomorphism $M_{\Lambda,+}/p_1(\mu)^n \xrightarrow{\sim} (M_\Lambda/p_1(\mu)^n)^{1\times\Gamma_{\mathrm{tor}}}$ (see the discussion before (5.27)), we can rewrite the exact sequence in (5.25) as follows:

$$0 \longrightarrow M_{\Lambda,+} \xrightarrow{p_1(\mu)^n} M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1} \longrightarrow M_{\tilde{\Lambda},+}/p_1(\mu)^n \longrightarrow 0.$$

Then, using the operator $\nabla_{q,\tilde{\tau}}$ in (5.28) and the Koszul complex defined in (5.29), we obtain an exact sequence of Koszul complexes:

$$0 \longrightarrow K_{M_{\Lambda,+}}(\nabla_{q,\tau}) \xrightarrow{p_1(\mu)^n} K_{M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1}}(\nabla_{q,\tilde{\tau}}) \longrightarrow K_{M_{\tilde{\Lambda},+}/p_1(\mu)^n}(\nabla_{q,\tilde{\tau}}) \longrightarrow 0.$$

Considering the associated long exact sequence, and noting that $H^1(K_{M_{\Lambda,+}}(\nabla_{q,\tau})) = 0$ from Proposition 4.31, we obtain the following exact sequence:

$$0 \longrightarrow M_{\Lambda,+}^{\nabla_{q,\tau}=0} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{\tau}}=0} \longrightarrow (M_{\tilde{\Lambda},+}/p_1(\mu)^n)^{\nabla_{q,\tilde{\tau}}=0} \longrightarrow 0.$$

Since the action of $1 \times \Gamma_0$ is continuous on $M_{\tilde{\Lambda},+}$ for the $(p, p_1(\mu))$-adic topology, therefore, we see that $(M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{\tau}}=0} = (M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1})^{1\times\Gamma_0}$, for each $n \in \mathbb{N}$. Hence, from the preceding exact sequence we obtain that the sequence in (5.27) is exact, and therefore, the sequence (5.26) is exact as well. ■

5.2.5. Stratification on Wach modules. In this subsubsection we will construct a stratification on a Wach module over A_R and prove Proposition 5.31 stated below. The most important input for our arguments is Theorem 4.5.

Let N be the Wach module over A_R and set recall that $N(1) = A_R(1) \otimes_{p_2, A_R} N$ is equipped with a (φ, Γ_R^2)-action (see the discussion before Subsection 5.2.1). Moreover, from the discussion before Proposition 4.32, recall that we have a φ-equivariant homomorphism $\Delta_N : N(1) \to N$ induced by tensoring $\Delta : A_R(1) \to A_R$ with N. After reducing Δ_N modulo $p_1(\mu)^n$, we claim the following:

Proposition 5.29. *Let $n \in \mathbb{N}_{\geq 1}$ then Δ_N modulo $p_1(\mu)^n$ restricts to a $(\varphi, \Gamma_R \times 1)$-equivariant isomorphism of A_R/μ^n-modules*

$$\Delta_N : (N(1)/p_1(\mu)^n)^{1\times\Gamma_R} \xrightarrow{\sim} N/\mu^n N.$$

where the A_R-module structure on the source is defined via the map $p_1 : A_R \to A_R(1)$.

Proof. Let us first consider the following $(\varphi, 1\times\Gamma_R)$-equivariant short exact sequence of $A_R(1)$-modules:

$$0 \longrightarrow N(1)/p_1(\mu) \xrightarrow{p_1(\mu)^n} N(1)/p_1(\mu)^{n+1} \longrightarrow N(1)/p_1(\mu)^n \longrightarrow 0$$

Then for $p \geq 3$ using Lemma 5.13, Lemma 5.16 and Lemma 5.19, and for $p = 2$ using Lemma 5.13, Lemma 5.24 and Lemma 5.25, it follows that the following $(\varphi, \Gamma_R \times 1)$-equivariant sequence is exact:

$$0 \longrightarrow (N(1)/p_1(\mu))^{1\times\Gamma_R} \xrightarrow{p_1(\mu)^n} (N(1)/p_1(\mu)^{n+1})^{1\times\Gamma_R} \longrightarrow (N(1)/p_1(\mu)^n)^{1\times\Gamma_R} \longrightarrow 0. \tag{5.30}$$

Now, consider the following φ-equivariant commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & (N(1)/p_1(\mu))^{1\times\Gamma_R} \arrow[r, "p_1(\mu)^n"] \arrow[d, "\Delta_N", "\wr"'] & (N(1)/p_1(\mu)^{n+1})^{1\times\Gamma_R} \arrow[r] \arrow[d, "\Delta_N"] & (N(1)/p_1(\mu)^n)^{1\times\Gamma_R} \arrow[r] \arrow[d, "\Delta_N"] & 0 \\
0 \arrow[r] & N/\mu N \arrow[r, "\mu^n"] & N/\mu^{n+1}N \arrow[r] & N/\mu^n N \arrow[r] & 0,
\end{tikzcd}$$

where the top row is the exact sequence in (5.30). For $n = 1$, from Theorem 4.5, in particular, from Proposition 4.32, recall that we have the isomorphism $\Delta_N : (N(1)/p_1(\mu))^{1\times\Gamma_R} \xrightarrow{\sim} N/\mu N$. Then, by using the diagram, an easy induction on $n \geq 1$, gives the φ-equivariant isomorphism

$$\Delta_N : (A_R(1)/p_1(\mu)^{n+1} \otimes_{p_2, A_R} N)^{1\times\Gamma_R} \xrightarrow{\sim} N/\mu^{n+1}N.$$

Finally, we need to check the $(\Gamma_R \times 1)$-equivariance of Δ_N, the proof of which is similar to that of [MT20, Lemma 3.19]. From Lemma 3.78 recall that for any g in Γ_R and a in $A_R(1)/p_1(\mu)^{n+1}$, we have that $\Delta((g,g)a) = g(a)$, which implies that $\Delta_N((g,g)(x)) = g(\Delta_N(x))$, for any g in Γ_R and x in $N(1)/p_1(\mu)^{n+1}$. So if x is $(1 \times \Gamma_R)$-invariant, then for g_1, g_2 in Γ_R, we have that $\Delta_N((g_1, g_2)x) = \Delta_N((g_1, g_1)x) = g_1(\Delta_N(x))$. This concludes our proof. ∎

An immediate consequence of Proposition 5.29 is the following:

Proposition 5.30. *Let N be a Wach module over A_R. Then the (φ, Γ_R)-equivariant homomorphism $\Delta_N : A_R(1) \otimes_{p_2, A_R} N \to N$ induced by $\Delta : A_R(1) \to A_R$ restricts to a $(\varphi, \Gamma_R \times 1)$-equivariant isomorphism of A_R-modules*

$$\Delta_N : (A_R(1) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} \xrightarrow{\sim} N,$$

where the A_R-module structure on the source is defined via the map $p_1 : A_R \to A_R(1)$.

Proof. Note that by construction Δ_N is $(\varphi, \Gamma_R \times 1)$-equivariant. Moreover, A_R is μ-adically complete and N is a finite A_R-module, in particular, we have that $N = \lim_n N/\mu^n N$. Similarly, $A_R(1)$ is $p_1(\mu)$-adically complete, so we have that $A_R(1) \otimes_{p_2, A_R} N = \lim_n \big(A_R(1)/p_1(\mu)^n \otimes_{p_2, A_R} N\big)$. Now, recall that inverse limit commutes with right adjoint functors, in particular, with taking $(1 \times \Gamma_R)$-invariants. So we get that

$$\begin{aligned}(A_R(1) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} &= \big(\lim_n A_R(1)/p_1(\mu)^n \otimes_{p_2, A_R} N\big)^{1 \times \Gamma_R} \\ &= \lim_n \big(A_R(1)/p_1(\mu)^n \otimes_{p_2, A_R} N\big)^{1 \times \Gamma_R} \xrightarrow[\Delta]{\sim} \lim_n N/\mu^n N = N,\end{aligned}$$

where the isomorphism follows from Proposition 5.29. This concludes our proof. ∎

Using Proposition 5.30 we can define a natural stratification on a Wach module over A_R as follows:

Proposition 5.31. *Let N be a Wach module over A_R and let $\varepsilon : A_R(1) \otimes_{p_1, A_R} N \to A_R(1) \otimes_{p_2, A_R} N$ be the $A_R(1)$-linear homomorphism induced by the inverse of the isomorphism in Proposition 5.30. Then we have the following:*

(1) *The homomorphism ε is a stratification on N with respect to $A_R(\bullet)$.*

(2) *The action of Γ_R on $\mathrm{ev}_{A_R}^{\mathrm{Strat}}(N, \varepsilon)$, whose underlying A_R-module is N, coincides with the original action of Γ_R on N, i.e. $\mathrm{ev}_{A_R}^{\mathrm{Strat}}(N, \varepsilon) \xrightarrow{\sim} N$ as Wach modules over A_R.*

Let us first note that the construction of the stratification ε in Proposition 5.31 is functorial in N. In particular, we have defined a functor

$$\mathrm{Strat}_{A_R(\bullet)} : (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{[p]_q} \longrightarrow \mathrm{Strat}^{\mathrm{an}, \varphi}(A_R(\bullet)). \tag{5.31}$$

Moreover, from the statement of Proposition 5.31 it is clear that the functor $\mathrm{Strat}_{A_R(\bullet)}$ in (5.31) is a quasi-inverse to the functor $\mathrm{ev}_{A_R}^{\mathrm{Strat}}$ in (5.9). We note the following:

Lemma 5.32. *The $A_R(1)$-linear homomorphism ε in Proposition 5.31 is Γ_R^2-equivariant.*

Proof. It is enough to show that $\varepsilon(1 \otimes g_1(y)) = (g_1, g_2)(\varepsilon(1 \otimes y))$, for all y in N and g_1, g_2 in Γ_R. Similar to the last part of the proof of Proposition 5.29 we note that for any x in $A_R(1) \otimes_{p_1, A_R} N$, we have that $\Delta_N((g_1, g_2)x) = g_1(\Delta_N(x))$. Then, by setting $y = \Delta_N(x)$ we get the claim. ∎

The goal of the rest of this subsubsection is to prove Proposition 5.31. To this end, we need a result analogous to Proposition 5.30 over $A_R(2)$. Recall that we have (φ, Γ_R^3)-equivariant maps $r_i : A_R \to A_R(2)$, for $i = 1, 2, 3$, where $A_R(2)$ is equipped with an action of Γ_R^3 as discussed before Construction 5.10 and A_R is equipped with an action of Γ_R^3 via projection onto the i^{th}-coordinate. Similarly, we have natural (φ, Γ_R^3)-equivariant maps $p_{ij} : A_R(1) \to A_R(2)$ for $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$ and where

$A_R(2)$ is equipped with an action of Γ_R^3 via projection onto the $(i,j)^{\text{th}}$-coordinate. Now, let N be a Wach module over A_R as above and consider the $A_R(2)$-module $A_R(2) \otimes_{r_3,A_R} N$ equipped with the tensor product Frobenius and the tensor product action of Γ_R^3, where Γ_R^3 acts on N via projection onto the third coordinate. Note that the multiplication map $\Delta : A_R(2) \to A_R$ is $(\varphi, \Gamma_R \times 1 \times 1)$-equivariant, where $\Gamma_R \times 1 \times 1$ acts on A_R via projection onto the first coordinate. The multiplication map Δ induces a $(\varphi, \Gamma_R \times 1 \times 1)$-equivariant map $\Delta_N : A_R(2) \otimes_{r_3,A_R} N \to N$, where $\Gamma_R \times 1 \times 1$ acts on N (in the target) via projection onto the first coordinate. Then we claim the following:

Proposition 5.33. *Let N be a Wach module over A_R. Then the homomorphism $\Delta_N : A_R(2) \otimes_{r_3,A_R} N \to N$ induced by $\Delta : A_R(2) \to A_R$ restricts to an injective map*

$$\Delta_N : (A_R(2) \otimes_{r_3,A_R} N)^{1\times\Gamma_R\times\Gamma_R} \longrightarrow N.$$

Proof. Recall that inverse limit commutes with right adjoint functors, in particular, with taking $(1 \times \Gamma_R \times \Gamma_R)$-invariants. Moreover, A_R is μ-adically complete, N is a finite A_R-module and $A_R(2)$ is $p_1(\mu)$-adically complete. Therefore, we have that,

$$\begin{aligned}(A_R(2) \otimes_{r_3,A_R} N)^{1\times\Gamma_R\times\Gamma_R} &= (\lim_n A_R(2)/p_1(\mu)^n \otimes_{r_3,A_R} N)^{1\times\Gamma_R\times\Gamma_R}\\ &= \lim_n (A_R(2)/p_1(\mu)^n \otimes_{r_3,A_R} N)^{1\times\Gamma_R\times\Gamma_R}\\ &\xrightarrow{\Delta_N} \lim_n N/\mu^n N = N.\end{aligned}$$

Since limit is a left exact functor, to show the claim, it is enough to show that the following map is injective

$$(A_R(2)/p_1(\mu)^n \otimes_{r_3,A_R} N)^{1\times\Gamma_R\times\Gamma_R} \longrightarrow N/\mu^n N.$$

We will show this by induction on $n \in \mathbb{N}_{\geq 1}$. For $n = 1$, from Theorem 4.5, in particular, from Remark 4.33, recall that we have a (φ, Γ_R^2)-equivariant isomorphism (see (4.32)),

$$A_R(1)/p_1(\mu) \otimes_{p_1,A_R} N \xrightarrow{\sim} A_R(1)/p_1(\mu) \otimes_{p_2,A_R} N,$$

where Γ_R^2 acts on N in the left hand term via projection onto the first coordinate and on N in the right hand term via projection onto the second coordinate. Moreover, the composition $A_R \xrightarrow{p_1} A_R(1) \xrightarrow{p_{13}} A_R(2)$ coincides with the composition $A_R \xrightarrow{r_1} A_R(2)$. Similarly, the composition $A_R \xrightarrow{p_2} A_R(1) \xrightarrow{p_{13}} A_R(2)$ coincides with the composition $A_R \xrightarrow{r_3} A_R(2)$. So, by base changing the top horizontal isomorphism in (4.32) along $A_R(1) \xrightarrow{p_{13}} A_R(2)$, we obtain a (φ, Γ_R^3)-equivariant isomorphism

$$A_R(2)/p_1(\mu) \otimes_{r_1,A_R} N \xrightarrow{\sim} A_R(2)/p_1(\mu) \otimes_{r_3,A_R} N, \tag{5.32}$$

where Γ_R^3 acts on N in the source via projection onto the first coordinate and on N in the target via projection onto the third coordinate. Now consider the following diagram,

$$\begin{tikzcd} (A_R(2)/p_1(\mu) \otimes_{r_1,A_R} N)^{1\times\Gamma_R\times\Gamma_R} \arrow[r, "\sim"] & (A_R(2)/p_1(\mu) \otimes_{r_3,A_R} N)^{1\times\Gamma_R\times\Gamma_R} \arrow[d, "\Delta_N"] \\ N/\mu N \arrow[u, "r_1", "\wr"'] \arrow[r, "\sim", "id"'] & N/\mu N, \end{tikzcd}$$

where the top horizontal arrow is obtained as $(1 \times \Gamma_R \times \Gamma_R)$-invariant of (5.32) and the left vertical arrow is the natural isomorphism from Lemma 5.34. The commutativity of the diagram follows from (4.32) and the observation that the composition $A_R \xrightarrow{r_1} A_R(2) \xrightarrow{\Delta} A_R$ is the identity. Therefore, the right vertical arrow is bijective as well, i.e. we obtain a (φ, Γ_R)-equivariant isomorphism

$$(A_R(2)/p_1(\mu) \otimes_{r_3,A_R} N)^{1\times\Gamma_R\times\Gamma_R} \xrightarrow{\sim} N/\mu N.$$

To prove our claim, we will now proceed by induction on $n \geq 1$. From the discussion above, we see that the claim is true for $n = 1$, so let $N(2) := A_R(2)/p_1(\mu) \otimes_{r_3,A_R} N$ and assume that $\Delta_N \bmod p_1(\mu)^n$ is injective for some $n \in \mathbb{N}_{\geq 1}$. Now consider the following diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & (N(2)/p_1(\mu))^{1\times\Gamma_R\times\Gamma_R} \arrow[r, "p_1(\mu)^n"] \arrow[d, "\Delta_N", "\wr"'] & (N(2)/p_1(\mu)^{n+1})^{1\times\Gamma_R\times\Gamma_R} \arrow[r] \arrow[d, "\Delta_N"] & Q \arrow[r] \arrow[d, "\Delta_N"] & 0 \\
0 \arrow[r] & N/\mu N \arrow[r] & N/\mu^{n+1}N \arrow[r] & N/\mu^n N \arrow[r] & 0,
\end{tikzcd}$$

where Q is the cokernel of the top left horizontal arrow. It is easy to see that we have an injective map $Q \to (N(2)/p_1(\mu)^n)^{1\times\Gamma_R\times\Gamma_R} \xrightarrow{\Delta_N} N/\mu^n N$, where the injectivity of the first map is obtained by considering the long exact sequence for the $(1 \times \Gamma_R \times \Gamma_R)$-cohomology of the short exact sequence $0 \to N(2)/p_1(\mu) \xrightarrow{p_1(\mu)^n} N(2)/p_1(\mu)^{n+1} \to N(2)/p_1(\mu)^n$, and the second map is injective by the induction assumption. Therefore, it follows that the middle vertical arrow in the diagram above is injective as well. This proves the claim, allowing us to conclude. ■

The following result was used above:

Lemma 5.34. *Extending scalars along $r_1 : A_R \to A_R(1)$, gives a (φ, Γ_R^3)-equivariant map $r_1 : N \to A_R(2) \otimes_{r_1,A_R} N$. Then, reduction modulo μ, restricts r_1 to an R-linear φ-equivariant isomorphism $r_1 : N/\mu N \xrightarrow{\sim} (A_R(2)/p_1(\mu) \otimes_{r_1,A_R} N)^{1\times\Gamma_R\times\Gamma_R}$.*

Proof. Note that the map $r_1 : R \xrightarrow{\sim} (A_R(2)/p_1(\mu))^{1\times\Gamma_R\times\Gamma_R}$ is a φ-equivariant isomorphism by Remark 3.76. Moreover, we have that,

$$(A_R(2)/p_1(\mu) \otimes_{r_1,A_R} N)^{1\times\Gamma_R\times\Gamma_R} = (A_R(2)/p_1(\mu))^{1\times\Gamma_R\times\Gamma_R} \otimes_{r_1,A_R} N.$$

Hence, we get the claimed isomorphism $r_1 : N/\mu N \xrightarrow{\sim} (A_R(2)/p_1(\mu) \otimes_{r_1,A_R} N)^{1\times\Gamma_R\times\Gamma_R}$. ■

We have all the necessary input to prove Proposition 5.31 similar to that the proof of [MT20, Proposition 3.18].

Proof of Proposition 5.31. For the first claim, note that from the definition of ε it is clear that its base change along $\Delta : A_R(1) \to A_R$ is the identity. Moreover, by using Lemma 5.32 we get that the base changes $p_{ij}^*(\varepsilon) : A_R(2) \otimes_{r_i,A_R} N \to A_R(2) \otimes_{r_j,A_R} N$, for $(i,j) \in \{(1,2),(2,3),(1,3)\}$ are Γ_R^3-equivariant. Therefore, restrictions of $p_{13}^*(\varepsilon)$ and $p_{23}^*(\varepsilon) \circ p_{12}^*(\varepsilon)$ to N have images in $(A_R(2) \otimes_{r_3,A_R} N)^{1\times\Gamma_R\times\Gamma_R}$, and their composition with the injective map Δ_N in Proposition 5.33 is the identity. So it follows that $p_{13}^*(\varepsilon) = p_{23}^*(\varepsilon) \circ p_{12}^*(\varepsilon)$, since both sides are $A_R(2)$-linear. We also get that ε is an isomorphism since we can give its inverse as its base change along the involution $A_R(1) \xrightarrow{\sim} A_R(1)$ swapping the two factors. This proves the first claim.

To show the second claim, let g in Γ_R. From Lemma 5.32 we have the following diagram

$$\begin{tikzcd}
A_R(1) \otimes_{(\gamma,1),A_R(1)} (A_R(1) \otimes_{p_1,A_R} N) \arrow[r, "\sim", "1\otimes\varepsilon"'] \arrow[d, "1\otimes(g,1)", "\wr"'] & A_R(1) \otimes_{(\gamma,1),A_R(1)} (A_R(1) \otimes_{p_2,A_R} N) \arrow[d, "1\otimes(g,1)", "\wr"'] \\
A_R(1) \otimes_{p_1,A_R} N \arrow[r, "\sim", "\varepsilon"'] & A_R(1) \otimes_{p_2,A_R} N.
\end{tikzcd}$$

From the diagram it is clear that the image of any x in N under the left vertical arrow is $g(x)$ while its image under the right vertical arrow is x. Base changing the diagram along the $(\Gamma_R \times 1)$-equivariant map $\Delta : A_R(1) \to A_R$, we obtain the following diagram:

$$\begin{tikzcd}
A_R \otimes_{g,A_R} N \arrow[r, "\sim", "\Delta^*((g,1)^*(1\otimes\varepsilon))"'] \arrow[d, "1\otimes g", "\wr"'] & N \arrow[d, "id", "\wr"'] \\
N \arrow[r, "\sim", "id"'] & N,
\end{tikzcd}$$

where the left vertical arrow is the action of g on N. This concludes our proof. ■

A. Some basic definitions and lemmas

In this section, we collect some standard definitions to enhance the readability of the text. For more details, the reader should look at the cited references. Let p be a fixed prime number.

A.1. Basic definitions.

Let R be a commutative ring and let $D(R)$ denote the derived ∞-category of R-modules.

Definition A.1. Let S denote a commutative R-algebra. Take $P_\bullet \to S$ to be a simplicial resolution of S by polynomial R-algebras. Define the *cotangent complex of* $R \to S$ to be the simplicial S-module $L_{S/R} := \Omega^1_{P_\bullet/R} \otimes_{P_\bullet} S$. Its wedge powers will be denoted by $\wedge^i_S L_{S/R} = \Omega^i_{P_\bullet/R} \otimes_{P_\bullet} R$ for $i \geq 1$. This definition is independent of the choice of the resolution $P_\bullet$.

Remark A.2. The object in $D(R)$ defined by $L_{S/R}$ coincides with that attached to the simplicial R-module $\Omega^1_{P_\bullet/A}$ via the Dold-Kan correspondence (and similarly for the wedge powers). One may also obtain $L_{S/R}$ by left Kan extending the functor $\Omega^1_{-/R}$ on polynomial R-algebras to all simplicial commutative R-algebras. Restricting to commutative R-algebras one obtains the cotangent complex (upto isomorphism) described in Definition A.1.

Definition A.3. Let $a, b \in \mathbb{Z} \cup \{\pm\infty\}$.

(1) An object M in $D(R)$ has *p-complete Tor amplitude* in $[a, b]$ if $M \otimes^{\mathbb{L}}_R R/pR$ in $D(R/pR)$ has Tor amplitude in $[a, b]$.

(2) An object M in $D(R)$ is *p-completely (faithfully) flat* if $M \otimes^{\mathbb{L}}_R R/pR$ in $D(R/pR)$ is concentrated in degree 0 and it is a (faithfully) flat R/pR-module.

Definition A.4 (Quasisyntomic site, [BMS19, Definition 4.10])**.** We define the following:

(1) A ring R is called *quasisyntomic* if it is p-complete, has bounded p^∞-torsion and $L_{R/\mathbb{Z}_p}$ in $D(R)$ has p-complete Tor amplitude in $[-1, 0]$. Denote by Qsyn the category of quasisyntomic rings.

(2) A map $R \to S$ of p-complete rings with bounded p^∞-torsion is said to be *quasyntomic* (resp. *quasisyntomic cover*) if S is p-completely flat (resp. p-completely faithfully flat) over R and $L_{S/R}$ in $D(S)$ has p-complete Tor amplitude in $[-1, 0]$. Endow $\mathrm{Qsyn}^{\mathrm{op}}$ with the structure of a site via the quasisyntomic covers (see [BMS19, Lemma 4.17]).

Definition A.5 (Perfectoid rings, [BMS18, Definition 3.5])**.** A ring R is called *perfectoid* if it is p-adically complete and there is some π in R such that $\pi^p = pu$ for some unit u in $R^\times$, the ring R/p is semiperfect, i.e. the absolute Frobenius map on R/p is surjective, and the kernel of the map $\theta : A_{\mathrm{inf}}(R) \to R$ is principal.

Definition A.6 (Quasiregular semiperfectoid rings, [BMS19, Definition 4.20])**.** A ring S is called *quasiregular semiperfectoid* if S is quasisyntomic in the sense of Definition A.4 (1), there exists a map $R \to S$ with R perfectoid in the sense of Definition A.5 and S/pS is semiperfect, i.e. the Frobenius on S/pS is surjective. Denote by Qrsp the category of quasiregular semiperfectoid rings and endow $\mathrm{Qrsp}^{\mathrm{op}}$ with the topology generated by quasisyntomic covers.

Definition A.7. Let $(X, \mathcal{O})$ denote a ringed topos. An $\mathcal{O}$-module $\mathcal{E}$ is called a *vector bundle* on $(X, \mathcal{O})$ if there exists a cover $\{U_i\}$ of X and finite projective $\mathcal{O}(U_i)$-modules $\mathcal{P}_i$ such that $\mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{P}_i \otimes_{\mathcal{O}(U_i)} \mathcal{O}_{U_i}$, for each i. Denote by $\mathrm{Vect}(X, \mathcal{O})$ the category of all vector bundles on $(X, \mathcal{O})$.

Definition A.8 (Koszul complex, [BMS18, Definition 7.1])**.** Let M be an abelian group and for $i = 1, \ldots, d$, let $f_i : M \to M$ denote d commuting endomorphisms of M. The Koszul complex $K_M(f_1, \ldots, f_d)$ is defined to be the following complex:

$$M \xrightarrow{(f_1,\ldots,f_d)} \bigoplus_{1 \leq i \leq d} M \longrightarrow \bigoplus_{1 \leq i_1 < i_2 \leq d} M \longrightarrow \cdots \longrightarrow \bigoplus_{1 \leq i_1 < \cdots < i_k \leq d} M \longrightarrow \cdots,$$

where the differential from M at the index $i_1 < \cdots < i_k$ to M at the index $j_1 < \cdots < j_{k+1}$ is nonzero if and only if the set $\{i_1, \ldots, i_k\}$ is contained in the set $\{j_1, \ldots, j_{k+1}\}$, and in that case the differential is given as $(-1)^n f_{j_n}$, where n is the unique integer in $\{1, \ldots, k+1\}$ such that j_n does not belong to the set $\{i_1, \ldots, i_k\}$.

We note the following fact from the proof of [BMS18, Corollary 12.5]:

Lemma A.9. *For $1 \leq i \leq d$, if f_i are d commuting endomorphisms of an abelian group M and h_i are d automorphisms of M commuting with each other and f_i's, then we have a natural quasi-isomorphism of complexes*

$$K_M(f_1h_1, \ldots, f_dh_d) \xrightarrow{\sim} K_M(f_1, \ldots, f_d).$$

A.2. Modules with $\mathbb{Z}_p^\times$-action. In this subsection, we will consider objects admitting a continuous action of $\Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$. From (1.6), recall that $\Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$, via the p-adic cyclotomic character, fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1,$$

where, for $p \geq 3$, we have $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$ and for $p = 2$, we have $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$. Moreover, for $p \geq 3$, we have that $\omega : \Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$ and the projection map $\Gamma_F \to \Gamma_{\mathrm{tor}}$, admits a section $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \to \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$, where the second map is given as $a \mapsto [a]$, the Teichmüller lift of a. Finally, for $p = 2$, we have $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \{\pm 1\}$, as groups.

A.2.1. The action of $\mathbb{F}_p^\times$. In this subsubsection, we will assume that $p \geq 3$ and recall the definition of $\mathbb{F}_p^\times$-decomposition from [Iwa59, Section 3] (Δ-decomposition in loc. cit.). For $0 \leq i \leq p-2$, let ϵ_i denote the following element in the group ring $\mathbb{Z}_p[\![\Gamma_F]\!]$:

$$\epsilon_i = \tfrac{1}{p-1} \sum_{a \in \mathbb{F}_p} \omega(a)^{-i} a.$$

Then it is easy to check that we have $\epsilon_i^2 = \epsilon_i$, $\epsilon_i\epsilon_j = 0$ for $i \neq j$, $\sum_{i=0}^{p-2} \epsilon_i = 1$.

Now, let M be a compact $\mathbb{Z}_p$-module admitting a continuous action of Γ_F. Set $M_i := \epsilon_i(M)$ for $0 \leq i \leq p-2$. Then we have a canonical decomposition of M as follows:

$$M = \bigoplus_{i=0}^{p-1} M_i. \tag{A.1}$$

Here, each M_i can also be characterised as the submodule of all x in M such that $ax = \omega(a)^i x$ for all a in $\mathbb{F}_p^\times$. We will refer to the decomposition of M in (A.1) as the $\mathbb{F}_p^\times$-decomposition of M. Moreover, since the action of $\mathbb{F}_p^\times$ and Γ_0 commute with each other, therefore, we see that each M_i admits a continuous action of Γ_0.

A.2.2. The action of $\{\pm 1\}$. In this subsubsection, we will assume that $p = 2$ and recall the following construction: Let M be a compact $\mathbb{Z}_p$-module admitting a continuous action of Γ_F and let σ be a generator of $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \{\pm 1\}$. Then we set $M_+ := \{x \in M \text{ such that } \sigma(x) = x\}$ and $M_- := \{x \in M \text{ such that } \sigma(x) = -x\}$. Using these notations, we have a canonical injective map of $\mathbb{Z}_p$-modules

$$M_+ \oplus M_- \longrightarrow M, \tag{A.2}$$

sending $(x, y) \mapsto x + y$. Note that the map (A.2) need not be surjective. If $M = A$ is a $\mathbb{Z}_p$-algebra, then it is easy to verify that A_+ is a $\mathbb{Z}_p$-algebra as well, A_- is an A_+-module and the map in (A.2) is A_+-linear.

Let us consider the ring $\mathbb{Z}_p[\![q-1]\!]$ and equip it with a $\mathbb{Z}_p$-linear action of Γ_{tor} given as $\sigma(q) = q^{-1}$. Let M be a topological $\mathbb{Z}_p[\![q-1]\!]$-module admitting an action of Γ_{tor} such that the induced action of Γ_{tor} is trivial on $M/(q-1)M$, in particular, $\mathbb{Z}_p[\![q-1]\!]$ satisfies these conditions. Then we note that the operator $\nabla_\sigma := \frac{\sigma-1}{q-1}$ is well-defined on M and we claim the following:

Lemma A.10. *The $\mathbb{Z}_p[\![q-1]\!]$-module M admits an $\mathbb{Z}_p[\![q-1]\!]^{\nabla_\sigma=0}$-linear decomposition $M = M^{\nabla_\sigma=0} \oplus M^{\nabla_\sigma=1}$.*

Proof. Let us first note that $\nabla_\sigma^2 = \nabla_\sigma$. Indeed,

$$\begin{aligned}\nabla_\sigma^2 = \tfrac{\sigma-1}{q-1} \circ \tfrac{\sigma-1}{q-1} &= \tfrac{1}{q-1}\big(\tfrac{\sigma(\sigma-1)}{\sigma(q)-1} - \tfrac{\sigma-1}{q-1}\big) = \tfrac{1}{q-1}\big(\tfrac{1-\sigma}{q^{-1}-1} - \tfrac{\sigma-1}{q-1}\big)\\ &= \tfrac{1}{q-1}\big(\tfrac{q(\sigma-1)}{q-1} - \tfrac{\sigma-1}{q-1}\big) = \tfrac{\sigma-1}{q-1} = \nabla_\sigma.\end{aligned}$$

As ∇_σ is an idempotent operator on M, it follows that M admits a decomposition $M = \nabla_\sigma(M) \oplus (\nabla_\sigma - 1)M = M^{\nabla_\sigma=1} \oplus M^{\nabla_\sigma=0}$. It is easy to see that the decomposition is $\mathbb{Z}_p[\![q-1]\!]^{\nabla_\sigma=0}$-linear. ∎

Let M and N be two $\mathbb{Z}_p[\![q-1]\!]$-modules admitting actions of Γ_{tor} such that the induced action of Γ_{tor} is trivial modulo $(q-1)$ and let $M \twoheadrightarrow N$ be a $\mathbb{Z}_p[\![q-1]\!]$-linear map compatible with the action of Γ_{tor}. Then we claim the following:

Lemma A.11. *The $\mathbb{Z}_p[\![q-1]\!]^{\Gamma_{\mathrm{tor}}}$-linear map $M^{\Gamma_{\mathrm{tor}}} \to N^{\Gamma_{\mathrm{tor}}}$ is surjective.*

Proof. Note that we hve $M^{\Gamma_{\mathrm{tor}}} = M^{\nabla_\sigma=0}$ and similarly for N. We get the claim by using the decomposition in Lemma A.10. ∎

B. δ-rings and divided power algebras

The content of this section has been adapted from some notes of Takeshi Tsuji on prismatic envelopes; we are thankful to him for sharing his computations.

In this section, we will describe certain prismatic envelopes explicitly. We begin this section by fixing some terminology. Let p be a fixed prime and A a commutative ring. A *δ-ring* is a pair (A, δ) where A is a commutative ring and $\delta : A \to A$ is a map of sets with $\delta(0) = \delta(1) = 0$ and satisfying:

$$\begin{aligned}\delta(x+y) &= \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p},\\ \delta(xy) &= x^p\delta(y) + y^p\delta(x) + p\delta(x)\delta(y).\end{aligned} \tag{B.1}$$

Given a δ-ring (A, δ) define an endomorphism $\varphi : A \to A$ by the formula $\varphi(x) = x^p + p\delta(x)$, for any x in A. This determines a *lifting of (the absolute) Forbenius* on A/pA. Then the product formula in (B.1) can be rewritten as

$$\delta(xy) = \varphi(x)\delta(y) + \delta(x)y^p. \tag{B.2}$$

If A is p-torsion-free then any lift $\varphi : A \to A$ of the absolute Frobenius on A/pA determines a unique δ-structure on A given as $\delta(x) = (\varphi(x) - x)/p$ for all x in A.

A *δ-homomorphism* $f : A \to B$ between δ-rings is a homomorphism of the underlying rings compatible with the respective δ-structures, i.e. $f \circ \delta(x) = \delta \circ f(x)$, for all x in A. For a δ-ring A, a *δ-algebra* over A is a δ-ring B equipped with a δ-homomorphism $A \to B$. A *δ-ideal* of a δ-ring A is an ideal I of the underlying ring, stable under δ, i.e. $\delta(I) \subset I$. If I is a δ-ideal of a δ-ring A, then the quotient ring A/I is equipped with a unique δ-structure over A. Now, let S be a subset of a δ-ring A. Then, we see that the ideal of A generated by $\{\delta^n(S)\}_{n\geq 1}$ is stable under δ and it is the smallest δ-ideal of A containing S. (Note that for any ideal J of A, the map $J \to A/J$, given by $x \mapsto \delta(x)$ mod J, is a φ-semilinear homomorphism of A-modules by (B.1) and (B.2).) We write this ideal as $(S)_\delta$ and call it the *δ-ideal generated by S*. If I is an ideal of A, then we write I_δ for $(I)_\delta$. If the ideal I is generated by a subset S of I, then we have $I_\delta = (S)_\delta$ because the latter contains I. This implies that if B is a δ-algebra over a δ-ring A and I a δ-ideal of A, then the ideal IB of B generated by I is a δ-ideal of the δ-ring B.

A *δ-subring* of a δ-ring A is a subring $A' \subset A$, stable under the δ-structure on A, i.e. $\delta(A') \subset A'$. Similarly, a *δ-subalgebra* of a δ-algebra B over a δ-ring A is an A-subalgebra $B' \subset B$, stable under the δ-structure on B, i.e. $\delta(B') \subset B'$. Let S be a subset of a δ-algebra B over a δ-ring A and note that $\delta(A[\delta^k(S), 1 \leq k \leq n]) \subset A[\delta^k(S), 1 \leq k \leq n+1]$, for each $n \in \mathbb{N}_{\geq 1}$. (Note that for any

A-subalgebra B' of B, the subset $\{x \text{ in } B' \text{ such that } \delta(x) \text{ in } B\}$ of B' is an A-subalgebra by (B.1) and (B.2)). This implies that the A-subalgebra of B generated by $\{\delta^n(S)\}_{n\geq 1}$ is stable under δ and it is the smallest δ-subalgebra of B over A containing S; we denote this δ-subalgebra by $A[S]_\delta$ and call it the *δ-subalgebra of B over A generated by S*.

Assumption B.1. Let A be a δ-ring equipped with an element q such that $\varphi(q) = q^p$. Set $\mu := q-1$ and assume that $\varphi^n(\mu)$ is a nonzerodivisor on A, for each $n \in \mathbb{N}$. Moreover, assume that A and $A/\mu A$ are p-torsion free.

In the rest of this section we will consider a δ-ring A satisfying Assumption B.1. Set $[p]_q = \frac{q^p-1}{q-1} = 1+q+\cdots+q^{p-1}$. Then, by Assumption B.1, we see that for each $n \in \mathbb{N}$, the element $\varphi^n([p]_q)$ is a nonzerodivisor on A. Before proceeding further, we note a simple lemma on regular sequences.

Lemma B.2. *Let R be a ring.*

(1) *If a sequence $\{x, y\}$ is regular on R, then x is regular on R/yR.*

(2) *Let $x, y_1, \ldots, y_d$ be elements of R. If the sequence $\{x, y_1, \ldots, y_d\}$ is regular on R, then x is regular on $R/\sum_{i=1}^d y_i R$.*

Proof. The claim in (1) is obtained by applying the snake lemma to the map induced by multiplication by y on the exact sequence $0 \to R \xrightarrow{x} R \to R/xR \to 0$. We will prove the claim in (2) by induction on d, where the case $d = 1$ is clear from (1). So let $d \geq 2$ and assume that x is regular on $R' = R/\sum_{i=1}^{d-1} y_i R$. Now, as y_d is regular on R' by assumption, therefore, the claim in (1) implies that x is regular on $R'/y_d R' = R/\sum_{i=1}^d y_i R$. ∎

Remark B.3. By using Lemma B.2 (1), we see that for the ring A as in Assumption B.1, the quotient A/pA is μ-torsion free. Moreover, since $\varphi^n(\mu) = \mu^{p^n} \bmod pA$, for each $n \in \mathbb{N}$, and A is p-torsion free, therefore, by using Lemma B.2 (1), we get that $A/\varphi^n(\mu)A$ is p-torsion free.

Let $\overline{A}$ denote the p-torsion free algebra $A/\mu A$ and note that image of $\varphi^n([p]_q)$ in $\overline{A}$ is p, for every $n \in \mathbb{N}$. Moreover, the lifting of Frobenius on A induces a lifting of Frobenius on $\overline{A}$. Furthermore, as $A/\mu A$ is p-torsion free, we see that the image of $\delta([p]_q)$ in A is $\delta(p)$ in A/pA, which can be computed as $(\varphi(p)-p^p)/p = 1-p^{p-1}$. Hence, it follows that $\delta([p]_q)$ is a unit modulo any power of the ideal $(p, \mu) \subset A$.

Assumption B.4. Let B be a δ-algebra over A. Assume that B and $B/\mu B$ are p-torsion free and $\varphi^n(\mu)$ is a nonzerodivisor on B, for each $n \in \mathbb{N}$. Let $Y_0, Y_1, \ldots, Y_d$ be elements of B such that the sequence $\{Y_1, \ldots, Y_d\}$ is regular on B and the sequence $\{Y_0, Y_1, \ldots, Y_d\}$ is regular on $B/[p]_qB$ and $B/(p, \mu)B$.

Remark B.5. Similar to the case of A as in Remark B.3, by using Lemma B.2 (1), we see that B/pB is μ-torsion free and $B/\varphi^n(\mu)B$ is p-torsion free, for each $n \in \mathbb{N}$.

Let I denote the set of natural numbers $\{0, 1, \ldots, d\}$. Let $C_0 := B[Y_{0,0}, \ldots, Y_{d,0}]$ denote a polynomial ring over B in $d+1$ variables $Y_{0,0}, \ldots Y_{d,0}$, and let $D_0 := C_0/([p]_qY_{i,0} - Y_i, i \in I)$. Moreover, set E_0 to be the B-subalgebra of $B[1/[p]_q]$ generated by $y_i := Y_i/[p]_q$, for $i \in I$. Then the surjective homomorphism of B-algebras $C_0 \to E_0$ via $Y_{i,0} \mapsto y_i$ for $i \in I$, induces a surjective homomorphism of B-algebras,

$$D_0 \longrightarrow E_0. \tag{B.3}$$

Set $\overline{B} := B/\mu B$, $\overline{C}_0 := C_0/\mu C_0$, $\overline{D}_0 := D_0/\mu D_0$ and $\overline{E}_0 := E_0/\mu E_0$, and write $\overline{Y}_{i,0}$ (resp. $\overline{y}_i$) for the image of $Y_{i,0}$ (resp. y_i) in $\overline{C}_0$ (resp. $\overline{E}_0$). We claim the following:

Lemma B.6. *With notations as above, we have the following:*

(1) *The homomorphism (B.3) is an isomorphism.*

(2) *The algebra $\overline{E}_0$ is p-torsion free.*

Proof. Note that we have $D_0[1/[p]_q] \xrightarrow{\sim} B[1/[p]_q][Y_{i,0}, i \in I]/(Y_{i,0}-y_i, i \in I) \xrightarrow{\sim} B[1/[p]_q]$. Therefore, to show (1), it suffices to show that D_0 is $[p]_q$-torsion free. Now, note that by Assumption B.4, the element $[p]_q$ is regular on C_0 and the sequence $\{[p]_qY_{0,0}-Y_0, \ldots, [p]_qY_{d,0}-Y_d\}$ is regular on $C_0/[p]_qC_0$. Hence, from Lemma B.2 (2), it follows that $[p]_q$ is regular on D_0. For (2), note that using claim (1), we are reduced to showing that $\overline{D}_0$ is p-torsion free. Moreover, as $\overline{C}_0$ is p-torsion free, by using Lemma B.2 (2), we see that it is enough to show that the sequence $\{[p]_qY_{0,0}-Y_0, \ldots, [p]_qY_{d,0}-Y_d\}$ is regular on $\overline{C}_0/p\overline{C}_0$. Now note that $\overline{C}_0/p\overline{C}_0 = C_0/(p, \mu)C_0$ and since $[p]_q$ is in the ideal $(p, \mu) \subset A$, therefore, by Assumption B.4, we get that the sequence $\{[p]_qY_{0,0}-Y_0, \ldots, [p]_qY_{d,0}-Y_d\}$ is regular on $\overline{C}_0/p\overline{C}_0$. This concludes our proof. ∎

The lifting of Frobenius on B naturally extends to $B[1/p, 1/\varphi^n([p]_q), n \in \mathbb{N}]$ and its E_0-subalgebra $E_0[1/p, 1/\varphi^n([p]_q), n \in \mathbb{N}]$ is stable under φ. Let E be the δ-subalgebra of $E_0[1/p, 1/\varphi^n([p]_q), n \in \mathbb{N}]$ generated over B by y_i, for all $i \in I$. Note that we have $E_0 \subset E$. Now, recall that $\varphi^n([p]_q) = p \bmod \mu B$, for each $n \in \mathbb{N}$, therefore, we get that $E_0[1/p, 1/\varphi^n([p]_q), n \geq 1]/\mu \xrightarrow{\sim} \overline{E}_0[1/p]$. Via the preceding isomorphism, the lifting of Frobenius on $E_0[1/p, 1/\varphi^n([p]_q), n \geq 1]$ induces a lifting of Frobenius on $\overline{E}_0[1/p]$. Define $\overline{E}$ to be δ-subalgebra of $\overline{E}_0[1/p]$ generated over $\overline{B}$ by $\overline{y}_i$, for all $i \in I$. Note that we have $\overline{E}_0 \subset \overline{E}$ by Lemma B.6 (2) and the natural ring homomorphism $E_0[1/p, 1/\varphi^n([p]_q), n \geq 1] \to \overline{E}_0[1/p]$ induces the following surjective ring homomorphism compatible with φ,

$$E \longrightarrow \overline{E}. \tag{B.4}$$

We will study the δ-rings E and $\overline{E}$ by comparing them to the δ-rings obtained by universally adjoining $Y_i/[p]_q$, for all $i \in I$, to the δ-rings B and $\overline{B}$, respectively. Let C denote a polynomial ring over B in variables $Y_{i,n}$, for $i \in I$ and $n \in \mathbb{N}$. Equip C with a lifting of Frobenius φ compatible with that on A and defined on the variables as $\varphi(Y_{i,n}) = Y_{i,n}^p + pY_{i,n+1}$, for each $i \in I$ and $n \in \mathbb{N}$. Since, C is p-torsion free, therefore, for δ-structure associated to φ, we have $\delta(Y_{i,n}) = Y_{i,n+1}$, for each $i \in I$ and $n \in \mathbb{N}$. Set $\overline{C} := C/\mu C$ equipped with a lifting of Frobenius induced by that of C. Denote by $\overline{Y}_i$ (resp. $\overline{Y}_{i,n}$) the image of Y_i (resp. $Y_{i,n}$) in $\overline{C}$. We define D (resp. $\overline{D}$) to be the quotient of C (resp. $\overline{C}$) by the δ-ideal generated by $[p]_qY_{i,0} - Y_i$ (resp. $[p]_q\overline{Y}_{i,0} - \overline{Y}_i$), for all $i \in I$. Then we have a φ-compatible (in particular, δ-compatible) surjective homomorphism of B-algebras $C \to E$ (resp. surjective homomorphism of $\overline{B}$-algebras $\overline{C} \to \overline{E}$) defined by sending $Y_{i,n}$ to $\delta^n(y_i)$ (resp. by sending $\overline{Y}_{i,n}$ to $\delta^n(\overline{y}_i)$), for each $i \in I$ and $n \in \mathbb{N}$. The preceding ring homomorphisms induce surjective horizontal arrows in the following diagram:

$$\begin{tikzcd} D = C/([p]_qY_{i,0} - Y_i)_\delta \arrow[r] \arrow[d] & E, \arrow[d, "(B.4)"] \\ \overline{D} = \overline{C}/(p\overline{Y}_{i,0} - \overline{Y}_i)_\delta \arrow[r] & \overline{E}, \end{tikzcd} \tag{B.5}$$

where the left vertical arrow is induced by the isomorphism $D/\mu D \xrightarrow{\sim} \overline{D}$ and the diagram commutes by definition. In particular, all arrows in the diagram (B.5) are surjective.

For each $i \in I$ and $n \in \mathbb{N}$, let $C_n^{(i)}$ be the B-subalgebra of C generated by $Y_{i,m}$ for $0 \leq m \leq n$. Set $C_{-1}^{(i)} := B$ and note that we have $\delta(C_{n-1}^{(i)}) \subset C_n^{(i)}$, for each $n \in \mathbb{N}$.

Lemma B.7. *For each $i \in I$ and $n \in \mathbb{N}_{\geq 1}$, the element $\delta^n([p]_qY_{i,0} - Y_i)$ in $C_n^{(i)}$ can be written as*

$$\delta^n([p]_qY_{i,0} - Y_i) = \varphi^n([p]_q)Y_{i,n} + c_0Y_{i,n-1}^p + \textstyle\sum_{k=0}^{p-1} c_kY_{i,n-1}^k,$$

where $c_0 = \varphi^{n-1}(\delta([p]_q))\sum_{m=0}^{n-1} p^{m(p-1)}$ is in A and c_k is in $C_{n-2}^{(i)}$, for each $0 \leq k \leq p-1$.

Proof. We will prove the claim by induction on n. For $n = 1$, note that we have

$$\begin{aligned} \delta([p]_qY_{i,0} - Y_i) &= \delta([p]_qY_{i,0}) + \delta(-Y_i) - \textstyle\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}([p]_qY_{i,0})^k(-Y_i)^{p-k} \\ &= \varphi([p]_q)Y_{i,1} + \delta([p]_q)Y_{i,0}^p + \delta(-Y_i) - \textstyle\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}[p]_q^k(-Y_i)^{p-k}Y_{i,0}^k, \end{aligned}$$

which is of the required form because the elemenets $\delta(-Y_i)$, $[p]_q$ and Y_i belong to $C_{-1}^{(i)} = B$. Now, let $n \geq 1$ and assume that the claim holds for $\delta^n([p]_q Y_{i,0} - Y_i)$, with c_0 in A and c_k in $C_{n-2}^{(i)}$. Set $b = c_0 Y_{i,n-1}^p + \sum_{k=0}^{p-1} c_k Y_{i,n-1}^k$ in $C_{n-1}^{(i)}$ and note that we have

$$\begin{aligned}\delta^{n+1}([p]_q Y_{i,0} - Y_i) &= \delta(\varphi^n([p]_q)Y_{i,n} + b)\\ &= \delta(\varphi^n([p]_q)Y_{i,n}) + \delta(b) - \textstyle\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}(\varphi^n([p]_q)Y_{i,n})^k b^{p-k}\\ &= \varphi^{n+1}([p]_q)Y_{i,n+1} + \delta(\varphi^n([p]_q))Y_{i,n}^P + \delta(b) - \textstyle\sum_{k=1}^{p-1} \frac{1}{p}\binom{p}{k}\varphi^n([p]_q)^k b^{p-k} Y_{i,n}^k.\end{aligned}$$

As the elements $\varphi^n([p]_q)$ and b are in $C_{n-1}^{(i)}$, therefore, it is enough to show that $\delta(b)$ is of the form $p^{p-1}c_0 + \sum_{k=0}^{p-1} d_k Y_{i,n}^k$, for some d_k in $C_{n-1}^{(i)}$. From (B.1), note that for any x and y in $C_{n-1}^{(i)}$, we have that $\delta(x+y) = \delta(x) + \delta(y) \bmod C_{n-1}^{(i)}$ and from (B.2), for x in $C_{n-2}^{(i)}$ and y in $C_{n-1}^{(i)}$, we have that $\delta(xy) = \varphi(x)\delta(y) \bmod C_{n-1}^{(i)}$. Therefore, we can write

$$\delta(b) = \varphi(c_0)\delta(Y_{i,n-1}^p) + \textstyle\sum_{k=0}^{p-1} \varphi(c_k)\delta(Y_{i,n-1}^k) \bmod C_{n-1}^{(i)}.$$

Now, we have that $\delta(1) = 0$ on A because A is p-torsion free, and by using Lemma B.8, for each $1 \leq k \leq p$, we have

$$\delta(Y_{i,n-1}^k) = \textstyle\sum_{j=1}^{k} \binom{k}{j} p^{j-1} Y_{i,n-1}^{p(k-j)} Y_{i,n}^j.$$

Note that $Y_{i,n-1}$ is in $C_{n-1}^{(i)}$ and $Y_{i,n}^p$ appears only when $k = p$, in which case the coefficient of $Y_{i,n}^p$ is p^{p-1}. This allows us to conclude. ■

The following fact was used above:

Lemma B.8. *Let R be a δ-ring and x any element of R. Then for each $n \in \mathbb{N}_{\geq 1}$, we have*

$$\delta(x^n) = \textstyle\sum_{j=1}^{n} \binom{n}{j} p^{j-1} x^{p(n-j)} \delta(x)^j.$$

Proof. We will prove the claim by induction on n. The case $n = 1$ is obvious, so assume that the claim holds for some $n \geq 1$. Then, we have that

$$\begin{aligned}\delta(x^{n+1}) &= \delta(x^n)x^p + x^{np}\delta(x) + p\delta(x)\delta(x^n)\\ &= \textstyle\sum_{j=1}^{n} \binom{n}{j} p^{j-1} x^{p(n+1-j)}\delta(x)^j + x^{np}\delta(x) + \sum_{j=1}^{n} \binom{n}{j} p^j x^{p(n-j)} \delta(x)^{j+1}.\end{aligned}$$

In the final term of the second line of the displayed equation above, by replacing j with $k-1$, for $2 \leq k \leq n+1$, and expanding, it easily follows that the expression thus obtained is the sum $\sum_{j=1}^{n+1} \binom{n+1}{j} p^{j-1} x^{p(n+1-j)} \delta(x)^j$. ■

Let $\overline{C}_n$ be the $\overline{B}$-subalgebra of $\overline{C}$ generated by $\overline{Y}_{i,m}$, for all $i \in I$ and $0 \leq m \leq n$. Set $\overline{C}_{-1} := \overline{B}$ and note that we have $\delta(\overline{C}_{n-1}) \subset \overline{C}_n$, for each $n \in \mathbb{N}$. Define $\overline{D}_n$ to be the quotient of $\overline{C}_n$ by the ideal generated by $\delta^m(p\overline{Y}_{i,0} - \overline{Y}_i)$, for all $i \in I$ and $0 \leq m \leq n$. Note that the $\overline{B}$-algebras $\overline{C}_0$ and $\overline{D}_0$ coincide with those defined before Lemma B.6. For each $i \in I$ and $n \in \mathbb{N}_{\geq 1}$, let us set

$$X_n^{(i)} := \delta^n(p\overline{Y}_{i,0} - \overline{Y}_i) - p\overline{Y}_{i,n} \in \overline{C}, \tag{B.6}$$

and by Lemma B.7 note that it is contained in $\overline{u}_n \overline{Y}_{i,n-1}^p + \sum_{k=0}^{p-1} \overline{Y}_{i,n-1}^k \overline{C}_{n-2} \subset \overline{C}_{n-1}$, where $\overline{u}_n = \varphi^{n-1}(\delta(p)) \sum_{m=0}^{n-1} p^{m(p-1)}$ is in $\overline{A}$. In particular, we see that for each $n \in \mathbb{N}$, we have

$$\overline{D}_{n+1} = \overline{D}_n[\overline{Y}_{i,n+1}, i \in I]/(p\overline{Y}_{i,n+1} + X_{n+1}^{(i)}, i \in I). \tag{B.7}$$

The inclusion maps $\overline{C}_n \hookrightarrow \overline{C}_m \hookrightarrow \overline{C}$, for $0 \leq n < m$, induce maps $\overline{D}_n \to \overline{D}_m \to \overline{D}$ and an isomorphism

$$\operatorname*{colim}_n \overline{D}_n \xrightarrow{\sim} \overline{D}. \tag{B.8}$$

Lemma B.9. *With notations as above, we have the following:*

(1) *The homomorphism $\overline{D}_n \to \overline{D}_{n+1}$ induces an isomorphism $\overline{D}_n[1/p] \xrightarrow{\sim} \overline{D}_{n+1}[1/p]$, for each $n \in \mathbb{N}$.*

(2) *Let S_n denote the image of $\overline{D}_n/p\overline{D}_n$ in $\overline{D}_{n+1}/p\overline{D}_{n+1}$, for each $n \in \mathbb{N}$. Then, we have $S_n = (\overline{D}_n/p\overline{D}_n)/(X_{n+1}^{(i)}, i \in I)$ and $\overline{D}_{n+1}/p\overline{D}_{n+1} = S_n[\overline{Y}_{i,n+1}, i \in I]$, for each $n \in \mathbb{N}$.*

(3) *For each $n \in \mathbb{N}$, the algebra $\overline{D}_n$ is p-torsion free.*

Proof. The claim in (1) follows immediately from (B.7) and the fact that $X_{n+1}^{(i)}$ is in $\overline{C}_n$. For (2), note that by taking the reduction modulo p of (B.8), we get that

$$\overline{D}_{n+1}/p\overline{D}_{n+1} = (\overline{D}_n/p\overline{D}_n)/(X_{n+1}^{(i)}, i \in I)[\overline{Y}_{i,n+1}, i \in I].$$

To prove (3), let us first recall that $\overline{D}_0$ is p-torsion free by Lemma B.6. So, for each $n \in \mathbb{N}$, by Lemma B.2 (2) and (B.7), it suffices to show that the sequence $\{X_{n+1}^{(i)}, i \in I\}$, where $X_{n+1}^{(i)}$ is in $\overline{C}_n$, is regular on $(\overline{D}_n/p\overline{D}_n)[\overline{Y}_{i,n+1}, i \in I]$, i.e. it is regular on $\overline{D}_n/p\overline{D}_n$. For $n \geq 1$, the claim follows because we have $\overline{D}_n/p\overline{D}_n = S_{n-1}[\overline{Y}_{i,n}, i \in I]$ and $\overline{u}_{n+1}$ modulo p is a unit in $\overline{A}/p\overline{A}$ (see the discussions after (B.6) and before Assumption B.4). For $n = 0$, it suffices to show that the sequence $\{\{[p]_q Y_{i,0} - Y_i, i \in I\}, \{X_1^{(i)}, i \in I\}\}$ is regular on $\overline{C}_0/p\overline{C}_0$. As $[p]_q$ is contained in the ideal $(p, \mu) \subset B$, we are reduced to showing that the sequence $\{\{-Y_i, i \in I\}, \{X_1^{(i)}, i \in I\}\}$ is regular on $\overline{C}_0/p\overline{C}_0 = B/(p,\mu)[\overline{Y}_{i,0}, i \in I]$. This is obvious since $\overline{u}_1$ modulo p is a unit in $\overline{A}/p\overline{A}$ (see the discussions after (B.6) and before Assumption B.4). Hence, the lemma is proved. ■

Now, note that reducing the top arrow in (B.5) modulo μ, we obtain the following commutative diagram with surjective arrows:

$$\begin{tikzcd} D/\mu D \arrow[r] \arrow[d, "\wr"] & E/\mu E, \arrow[d, "(B.4)"] \\ \overline{D} \arrow[r] & \overline{E}. \end{tikzcd} \tag{B.9}$$

Proposition B.10. *All arrows in the diagram (B.9) are isomorphisms.*

Proof. By definition, the left vertical arrow in (B.9) is an isomorphism and all arrows are surjective. Hence, it suffices to show that the lower horizontal arrow is injective. Note that we have the following commutative diagram:

$$\begin{tikzcd} \overline{D}_0[1/p] \arrow[r] \arrow[d, "\wr"', "(B.3)"] & \overline{D}[1/p] \arrow[d] \\ \overline{E}_0[1/p] \arrow[r] & \overline{E}[1/p]. \end{tikzcd}$$

The top horizontal arrow is a bijection by (B.8) and Lemma B.9 (1), the left vertical arrow is a bijection by Lemma B.6 and the bottom horizontal arrow is a bijection by the definition of $\overline{E}$. Therefore, it follows that the right vertical arrow is a bijection as well, i.e. $\overline{D}[1/p] \xrightarrow{\sim} \overline{E}[1/p]$. Finally, from (B.8) and Lemma B.9 (3), note that $\overline{D}$ is p-torsion free. Hence, it follows that the map $\overline{D} \to \overline{E}$ in (B.9) is injective. This completes our proof. ■

References

[Abh21] Abhinandan. "Crystalline representations and Wach modules in the relative case". In: arXiv:2103.17097 (Mar. 2021). To appear in Annales de l'Institut Fourier (cit. on pp. 1, 2, 5, 6, 12, 23, 45).

[Abh23a] Abhinandan. "Crystalline representations and Wach modules in the imperfect residue field case". In: arXiv:2308.11191 (Aug. 2023) (cit. on pp. 1, 5, 6).

[Abh23b] Abhinandan. "Crystalline representations and Wach modules in the relative case II". In: arXiv:2309.16446 (Sept. 2023) (cit. on pp. 1, 2, 5, 6, 7, 24, 42, 43, 51, 61, 62, 64).

[Abh23c] Abhinandan. "Syntomic complex and p-adic nearby cycles". In: arXiv:2308.10736 (Aug. 2023) (cit. on p. 2).

[Abh24] Abhinandan. "Crystalline part of the Galois cohomology of crystalline representations". In: arXiv:2405.17012 (May 2024) (cit. on p. 5).

[And06] Fabrizio Andreatta. "Generalized ring of norms and generalized (ϕ, Γ)-modules". In: *Ann. Sci. École Norm. Sup. (4)* 39.4 (2006), pp. 599–647. ISSN: 0012-9593 (cit. on pp. 5, 12).

[AL23] Johannes Anschütz and Arthur-César Le Bras. "Prismatic Dieudonné theory". In: *Forum Math. Pi* 11 (2023), Paper No. e2, 92. ISSN: 2050-5086 (cit. on p. 1).

[AMMN22] Benjamin Antieau, Akhil Mathew, Matthew Morrow, and Thomas Nikolaus. "On the Beilinson fiber square". In: *Duke Math. J.* 171.18 (2022), pp. 3707–3806. ISSN: 0012-7094,1547-7398 (cit. on p. 2).

[Ber04] Laurent Berger. "Limites de représentations cristallines". In: *Compos. Math.* 140.6 (2004), pp. 1473–1498. ISSN: 0010-437X (cit. on pp. 1, 5).

[Ber74] Pierre Berthelot. *Cohomologie cristalline des schémas de caractéristique* $p > 0$. Springer-Verlag, Berlin-New York,, 1974, p. 604 (cit. on pp. 1, 47, 49, 54, 64).

[BL22] Bhargav Bhatt and Jacob Lurie. "Absolute prismatic cohomology". In: arXiv:2201.06120 (Jan. 2022) (cit. on p. 13).

[BM23] Bhargav Bhatt and Akhil Mathew. "Syntomic complexes and p-adic étale Tate twists". In: *Forum Math. Pi* 11 (2023), Paper No. e1, 26. ISSN: 2050-5086 (cit. on p. 2).

[BMS18] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 128 (2018), pp. 219–397. ISSN: 0073-8301 (cit. on pp. 1, 45, 78, 79).

[BMS19] Bhargav Bhatt, Matthew Morrow, and Peter Scholze. "Topological Hochschild homology and integral p-adic Hodge theory". In: *Publ. Math. Inst. Hautes Études Sci.* 129 (2019), pp. 199–310. ISSN: 0073-8301,1618-1913 (cit. on pp. 1, 2, 9, 15, 78).

[BS22] Bhargav Bhatt and Peter Scholze. "Prisms and prismatic cohomology". In: *Ann. of Math. (2)* 196.3 (2022), pp. 1135–1275. ISSN: 0003-486X,1939-8980 (cit. on pp. 1, 2, 7, 8, 9, 14, 15, 16, 17, 18, 22, 28, 29, 32, 37, 39).

[BS23] Bhargav Bhatt and Peter Scholze. "Prismatic F-crystals and crystalline Galois representations". In: *Camb. J. Math.* 11.2 (2023), pp. 507–562. ISSN: 2168-0930,2168-0949 (cit. on pp. 1, 2, 5, 6, 7, 8, 9, 11).

[Bri08] Olivier Brinon. "Représentations p-adiques cristallines et de de Rham dans le cas relatif". In: *Mém. Soc. Math. Fr. (N.S.)* 112 (2008), pp. vi+159. ISSN: 0249-633X (cit. on pp. 5, 6, 23, 27).

[Col99] Pierre Colmez. "Représentations cristallines et représentations de hauteur finie". In: *J. Reine Angew. Math.* 514 (1999), pp. 119–143. ISSN: 0075-4102 (cit. on pp. 1, 5).

[CN17] Pierre Colmez and Wiesława Nizioł. "Syntomic complexes and p-adic nearby cycles". In: *Invent. Math.* 208.1 (2017), pp. 1–108. ISSN: 0020-9910 (cit. on pp. 2, 7).

[DL21] Heng Du and Tong Liu. "A prismatic approach to $(\varphi, \hat{G})$-modules and F-crystals". In: arXiv:2107.12240 (July 2021) (cit. on pp. 1, 6).

[DLMS24] Heng Du, Tong Liu, Yong Suk Moon, and Koji Shimizu. "Completed prismatic F-crystals and crystalline Z_p-local systems". In: *Compos. Math.* 160.5 (2024), pp. 1101–1166. ISSN: 0010-437X,1570-5846 (cit. on pp. 1, 2, 5, 6, 7, 8, 10, 11, 61).

[Fon77] Jean-Marc Fontaine. *Groupes p-divisibles sur les corps locaux.* Société Mathématique de France, Paris,, 1977, pp. i+262 (cit. on p. 7).

[Fon90] Jean-Marc Fontaine. "Représentations p-adiques des corps locaux. I". In: *The Grothendieck Festschrift, Vol. II.* Vol. 87. Progr. Math. Birkhäuser Boston, Boston, MA, 1990, pp. 249–309 (cit. on pp. 1, 5, 51).

[Fon94] Jean-Marc Fontaine. "Le corps des périodes p-adiques". In: *Astérisque* 223 (1994). With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988), pp. 59–111. ISSN: 0303-1179 (cit. on pp. 13, 14, 15, 36).

[GLQ22] Michel Gros, Bernard Le Stum, and Adolfo Quirós. "Twisted differential operators of negative level and prismatic crystals". In: *Tunis. J. Math.* 4.1 (2022), pp. 19–53. ISSN: 2576-7658,2576-7666 (cit. on p. 1).

[GLQ23] Michel Gros, Bernard Le Stum, and Adolfo Quirós. "Absolute calculus and prismatic crystals on cyclotomic rings". In: arXiv:2310.13790 (Oct. 2023) (cit. on p. 1).

[GR24] Haoyang Guo and Emanuel Reinecke. "A prismatic approach to crystalline local systems". In: *Invent. Math.* 236.1 (2024), pp. 17–164. ISSN: 0020-9910,1432-1297 (cit. on pp. 1, 2, 5, 6, 7, 8, 10, 11).

[Iwa59] Kenkichi Iwasawa. "On the theory of cyclotomic fields". In: *Ann. of Math. (2)* 70 (1959), pp. 530–561. ISSN: 0003-486X (cit. on pp. 7, 79).

[MW21] Yu Min and Yupeng Wang. "Relative (φ, Γ)-modules and prismatic F-crystals". In: arXiv:2110.06076 (Oct. 2021) (cit. on pp. 1, 6).

[MT20] Matthew Morrow and Takeshi Tsuji. "Generalised representations as q-connections in integral p-adic Hodge theory". In: arXiv:2010.04059 (Oct. 2020) (cit. on pp. 1, 2, 6, 24, 25, 43, 45, 47, 64, 65, 75, 77).

[Sta23] The Stacks project authors. *The Stacks project.* `https://stacks.math.columbia.edu`. 2023 (cit. on pp. 16, 17, 44, 62).

[Tia23] Yichao Tian. "Finiteness and duality for the cohomology of prismatic crystals". In: *J. Reine Angew. Math.* 800 (2023), pp. 217–257. ISSN: 0075-4102,1435-5345 (cit. on p. 1).

[Tsu24] Takeshi Tsuji. "Prismatic crystals and q-Higgs fields". In: arXiv:2403.11676 (Mar. 2024) (cit. on p. 1).

[Wac96] Nathalie Wach. "Représentations p-adiques potentiellement cristallines". In: *Bull. Soc. Math. France* 124.3 (1996), pp. 375–400. ISSN: 0037-9484 (cit. on pp. 1, 5).

[Wac97] Nathalie Wach. "Représentations cristallines de torsion". In: *Compositio Math.* 108.2 (1997), pp. 185–240. ISSN: 0010-437X (cit. on p. 1).

[Wu21] Zhiyou Wu. "Galois representations, (φ, Γ)-modules and prismatic F-crystals". In: *Doc. Math.* 26 (2021), pp. 1771–1798. ISSN: 1431-0635,1431-0643 (cit. on pp. 1, 6).

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