Crystalline representations and Wach modules in the relative case II

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Abstract. We study relative Wach modules, generalising our previous works on this subject. Our main result shows a categorical equivalence between relative Wach modules and lattices inside relative crystalline representations. Using this result, we deduce a purity statement for relative crystalline representations and provide a criteria for checking the crystallinity of relative p-adic representations. Furthermore, we interpret relative Wach modules as modules with q-connections and show that for a crystalline representation, its associated Wach module together with the Nygaard filtration is the canonical q-deformation (after inverting p) of the filtered (φ, ∂)-module associated to the representation.

1. Introduction

The study of arithmetic Wach modules and their relationship to crystalline representations is of classical nature, having been taken up in the works of Fontaine [Fon90], Wach [Wac96; Wac97], Colmez [Col99] and Berger [Ber04]. More precisely, in op. cit. the authors studied the situation of an absolutely unramified extension of $\mathbb{Q}_p$ with perfect residue field. In [Abh21] we defined a similar concept in the relative case, i.e. for certain étale algebras over a formal torus (see §1.4 for precise setup) and showed that such objects give rise to crystalline representations of the fundamental group of the generic fiber. On the other hand, in [Abh23a], we generalised the theory of Wach modules and their relationship to crystalline representations, to the imperfect residue field case. In this article, we combine these two generalisations of the classical theory, to discuss the equivalence between Wach modules and crystalline representations in its most natural generality. In addition, we provide some applications of the preceding result and also show that Wach modules are q-deformations of lattices inside the filtered (φ, ∂)-module attached to crystalline representations.

Before providing further motivations for our results, let us remark that recent developments in the theory of prismatic F-crystals [BS23; DLMS22; GR22] provide a new approach to the classification of lattices inside crystalline representations. These exciting new developments have motivated us in seeking the results of the current paper. However, instead of using the tools from the prismatic theory, we employ techniques from the classical theory of (φ, Γ)-modules to obtain our results due to the very nature of the objects studied in this article, i.e. relative Wach modules. Additionally, our proof enables us to provide interesting applications as well, for example, using [Abh23a, Theorem 1.5] and Theorem 1.5, we provide a new criteria for checking the crystallinity of a p-adic representation in the relative case (see Theorem 1.7 and Corollary 1.8). We refer the reader to §1.1.2 for precise statements of these results, to §1.1.3 for a sketch of our proof strategy and to §1.3 for more details on relation of our results to the prismatic theory.

Our motivation for studying relative Wach modules is twofold, largely stemming from geometry. In [Abh23b], for smooth (p-adic formal) schemes, we defined the notion of crystalline syntomic complex with coefficients in global relative Fontaine-Laffaille modules. Moreover, [Abh23b, Theorem 1.15] showed that such a complex is naturally comparable to the complex of p-adic nearby cycles of the associated crystalline $\mathbb{Z}_p$-local system on the (rigid analytic) generic fiber of the (formal) scheme. The work in loc. cit. was motivated by the results of [FM87], [Tsu96], [Tsu99] and [CN17], and the proof of [Abh23b, Theorem 1.15] follows via careful computations in the local setting in which relative Wach modules play a pivotal role (see [Abh23b, Corollary 1.12]). To generalise these results beyond the Fontaine-Laffaille case, it is therefore necessary to understand the relationship between crystalline representations of the fundamental group and general relative Wach modules (see Theorem 1.5).

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On the other hand, in [BMS19], for smooth p-adic formal schemes, the authors defined a prismatic syntomic complex and compared it to the complex of p-adic nearby cycles integrally. In the same vein, comparison results beyond the smooth case, have also been obtained in [AMMN22] and [BM23], where the latter uses the theory of prismatic cohomology from [BS22]. The aforementioned results were obtained in the case of constant coefficients and it is natural to ask if [BMS19, Theorem 10.1] could be generalised to non-constant coefficients, i.e. prismatic F-crystals. In our approach to resolving this question, results pertaining to Wach modules from the current paper will play a critical role.

Another motivation for considering Wach modules is to construct a deformation of crystalline cohomology, i.e. the functor $\mathbf{D}_{\text{cris}}$ from classical p-adic Hodge theory, to better capture mixed characteristic information. In [Fon90, §B.2.3] Fontaine expressed similar expectations which were verified by Berger in [Ber04, Théorème III.4.4] and generalised to finer integral conjectures in [Sch17, §6]. Some conjectures of [Sch17] were resolved by the introduction of prismatic cohomology [BS22]. Furthermore, it is also worth mentioning that the proof of the result comparing prismatic syntomic complex to p-adic nearby cycles, i.e. [BMS19, Theorem 10.1], relies on a local computation of prismatic cohomology using the q-de Rham complex, i.e. a q-deformation of the usual de Rham complex. Additionally, the importance of q-de Rham cohomology in computation of prismatic cohomology has also been emphasised in [BL22, §3].

In this paper, we interpret Wach modules as q-de Rham complexes (see Theorem 1.9). Moreover, we show that such an object is the q-deformation of a lattice inside the filtered (φ, ∂)-module attached to a crystalline representation. In a subsequent work [Abh24], we show that in our setting, a relative Wach module can be regarded as the evaluation of a prismatic F-crystal over a covering (by a suitable q-de Rham prism) of the final object of a certain prismatic topos. Hence, from these apparent tight connections between Wach modules and prismatic F-crystals and p-adic crystalline representations, we expect these objects to play a pivotal role in the study of p-adic nearby cycles of crystalline $\mathbb{Z}_p$-local systems (for smooth formal schemes) and its comparison to prismatic syntomic complex with coefficients.

In summary, within the overarching program sketched above, this paper realises two of our goals (see Theorem 1.5 and Theorem 1.9). Additionally, we provide interesting applications of our results to purity statements in p-adic Hodge theory (see Theorem 1.7 and Corollary 1.8).

1.1. Crystalline representations and Wach modules. Let p be a fixed prime number and κ a perfect field of characteristic p; set $O_F := W(\kappa)$ to be the ring of p-typical Witt vectors with coefficients in κ and $F := O_F[1/p]$. Let $d \in \mathbb{N}$ and take $X_1, X_2, \ldots, X_d$ to be some indeterminates. We set $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ to be the p-adic completion of Laurent polynomial ring $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$. Let R denote the p-adic completion of an étale algebra over $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ with non-empty and geometrically integral special fiber. Denote by G_R the étale fundamental group of $R[1/p]$ and by Γ_R the Galois group of $R_\infty[1/p]$ over $R[1/p]$, where R_∞ is obtained from R by adjoining to it all p-power roots of unity and all p-power roots of X_i, for each $1 \leq i \leq d$. Then we have $\Gamma_R \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$ (see §2 for precise definitions). Set $O_L := (R_{(p)})^\wedge$ as a complete discrete valuation ring with uniformiser p, residue field a finite étale extension of $\kappa(X_1, \ldots, X_d)$ and set $L := O_L[1/p]$. Let G_L denote the absolute Galois group of L such that we have a continuous homomorphism $G_L \to G_R$; let Γ_L denote the Galois group of L_∞ over L, where L_∞ is obtained from L by adjoining to it all p-power roots of unity and all p-power roots of X_i, for each $1 \leq i \leq d$. The continuous homomorphism $G_L \to G_R$ induces a continuous isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$. In this setting, we have the theory of crystalline representations of G_R from [Bri08] and the theory of étale (φ, Γ)-modules from [And06; AB08].

1.1.1. Relative Wach modules. Set $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots)$ in $R_\infty^\flat$ (the tilt of R_∞) and its Teichmüller lift $[\varepsilon]$ in $A_{\text{inf}}(R_\infty) := W(R_\infty^\flat)$, the ring of p-typical Witt vectors with coefficients in $R_\infty^\flat$. Additionally, set $\mu := [\varepsilon] - 1$ and $[p]_q := \varphi(\mu)/\mu$, as elements of $A_{\text{inf}}(R_\infty)$. Moreover, for $1 \leq i \leq d$, fix $X_i^\flat := (X_i, X_i^{1/p}, \ldots)$ in $R_\infty^\flat$ and their Teichmüller lifts $[X_i^\flat]$ in $A_{\text{inf}}(R_\infty)$. Let A_R^+ denote the (p, μ)-adic completion of the unique extension of the (p, μ)-adic completion of $O_F[\![\mu]\!][X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]$ along the p-adically completed étale map $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle \to R$ (see §1.4 and §2.2). The ring A_R^+ is equipped with a Frobenius endomorphism φ and a continuous action of Γ_R; set A_L^+ to be the (p, μ)-adic completion of the localisation $(A_R^+)_{(p,\mu)}$ equipped with an induced Frobenius endomorphism φ and a continuous action of $\Gamma_L \xrightarrow{\sim} \Gamma_R$. With this setup, we define the following:

Definition 1.1. A *Wach module* over A_R^+ with weights in the interval $[a,b]$, for some $a,b \in \mathbb{Z}$ with $b \geq a$, is a finitely generated A_R^+-module N satisfying the following assumptions:

(1) The sequences $\{p,\mu\}$ and $\{\mu,p\}$ are regular on N.

(2) N is equipped with a semilinear action of Γ_R such that the induced action of Γ_R on $N/\mu N$ is trivial.

(3) N admits a Frobenius-semilinear operator $\varphi : N[1/\mu] \rightarrow N[1/\varphi(\mu)]$ compatible with the action of Γ_R on each side, and such that $\varphi(\mu^b N) \subset \mu^b N$ and the cokernel of the A_R^+-linear map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \rightarrow \mu^b N$ is killed by $[p]_q^{b-a}$.

Denote by $(\varphi,\Gamma_R)\text{-Mod}_{A_R^+}^{[p]_q}$ the category of Wach modules over A_R^+, with morphisms between objects being A_R^+-linear, φ-equivariant (after inverting μ) and Γ_R-equivariant morphisms.

Remark 1.2. The condition (1) in Definition 1.1 is new and relaxes finite projectivity assumption of relative Wach modules in [Abh21, Definition 4.8]. Moreover, condition (1) above is equivalent to the vanishing of local cohomology of N with respect to the ideal $(p,\mu) \subset A_R^+$ in degree 1 (see Lemma 3.3 and Remark 3.4), in particular, it is equivalent to having $\{p,[p]_q\}$ and $\{[p]_q,p\}$ as regular sequences on N (see Lemma 3.6). Furthermore, one can also show that the $A_R^+[1/p]$-module $N[1/p]$ is finite projective (see Proposition A.1, where we use some ideas from [BMS18; DLMS22]), the $A_R^+[1/\mu]$-module $N[1/\mu]$ is finite projective (see Proposition 3.11) and $N = N[1/p] \cap N[1/\mu] \subset N[1/p, 1/\mu]$ (see Lemma 3.5).

Remark 1.3. In Definition 1.1, note that in contrast to the definition of Wach modules in the arithmetic case (see [Ber04, Definition III.4.1]), we have dropped the assumption on the continuity of the action of Γ_R on N. However, in Lemma 3.7 we show that the condition (2) in Definition 1.1, i.e. triviality of the action of Γ_R on $N/\mu N$, automatically implies that the action of Γ_R on N is continuous.

Remark 1.4. Definition 1.1 may be adapted to the case of a field, i.e. over the ring $A_F^+ = O_F[\![\mu]\!]$ (resp. A_L^+). In such cases, from the assumptions of Definition 1.1 it follows that a Wach module over A_F^+ (resp. A_L^+) is necessarily finite free. Indeed, if N is a Wach module over A_F^+ (resp. A_L^+), in the sense of Definition 1.1, then one first observes that N is torsion-free since $N \subset N[1/p]$ and the latter is finite free over $A_F^+[1/p]$ (resp. $A_L^+[1/p]$) by [Abh23a, Lemma 2.14]. Then using [Fon90, §B.1.2.4 Proposition] (resp. Lemma 3.5 and [Abh23a, Remark 2.15]) it follows that N is finite free. In particular, Definition 1.1 is equivalent to [Ber04, Definition III.4.1] over A_F^+ (resp. [Abh23a, Definition 1.3] over A_L^+).

Set $A_R := A_R^+[1/\mu]^\wedge$ as the p-adic completion, equipped with the induced Frobenius endomorphism φ and the induced continuous action of Γ_R, and similarly, set $A_L := A_L^+[1/\mu]^\wedge$ equipped with the induced Frobenius endomorphism φ and the induced continuous action of Γ_L. Let T be a finite free $\mathbb{Z}_p$-representation of G_R and note that one can functorially attach to T a finite projective étale (φ,Γ_R)-module $\mathbf{D}_R(T)$ over A_R of rank $= \text{rk}_{\mathbb{Z}_p} T$, equipped with a semilinear and continuous action of Γ_R and a Frobenius-semilinear operator φ commuting with the action of Γ_R. In fact, the preceding functor induces a categorical equivalence between the category of finite free $\mathbb{Z}_p$-representations of G_R and the category of finite projective étale (φ,Γ_R)-modules over A_R (see [And06, Theorem 7.11]). Additionally, the category of Wach modules over A_R^+ fully faithfully embeds into the latter category, i.e. the category of étale (φ,Γ_R)-modules over A_R (see Proposition 3.15).

1.1.2. Main results. Let $\text{Rep}_{\mathbb{Z}_p}^{\text{cris}}(G_R)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_R. For T a $\mathbb{Z}_p$-lattice inside a p-adic crystalline representation of G_R, we construct a Wach module $\mathbf{N}_R(T)$ over A_R^+, functorial in T, and contained in $\mathbf{D}_R(T)$ (see Theorem 4.1). Our first main result is as follows:

Theorem 1.5 (Corollary 4.3)**.** *The Wach module functor induces an equivalence of categories*

$$\begin{aligned}\text{Rep}_{\mathbb{Z}_p}^{\text{cris}}(G_R) &\xrightarrow{\sim} (\varphi,\Gamma_R)\text{-Mod}_{A_R^+}^{[p]_q}\\ T &\longmapsto \mathbf{N}_R(T),\end{aligned}$$

with a quasi-inverse given as $N \mapsto \mathbf{T}_R(N) := \big(W(\overline{R}^\flat[1/p^\flat]) \otimes_{A_R^+} N\big)^{\varphi=1}$.

Remark 1.6. In Theorem 1.5, we do not expect the functor $\mathbf{N}_R$ to be an exact equivalence. However, note that after inverting p, the Wach module functor induces an exact equivalence between $\otimes$-categories: $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{B_R^+}$, via $V \mapsto \mathbf{N}_R(V)$, where $B_R^+ = A_R^+[1/p]$, and an exact $\otimes$-compatible quasi-inverse functor given as $M \mapsto \mathbf{V}_R(M) := (W(\overline{R}^\flat[1/p^\flat]) \otimes_{A_R^+} M)^{\varphi=1}$ (see Corollary 4.4).

As an application of Theorem 1.5, we obtain the following purity statement:

Theorem 1.7 (Theorem 4.5)**.** *Let V be a p-adic representation of G_R. Then V is crystalline as a representation of G_R if and only if it is crystalline as a representation of G_L.*

For a p-adic representation V of G_R, let $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ denote the associated filtered (φ, ∂)-module over $R[1/p]$ (see [Bri08, §8.2]). We show the following criterion for checking the crystallinity of V:

Corollary 1.8 (Theorem 4.5 & Corollary 4.6)**.** *Let V be a p-adic representation of G_R. Then V is crystalline if and only if $\mathrm{rk}_{R[1/p]}\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$. Moreover, under these equivalent conditions, we have a natural isomorphism $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ of filtered (φ, ∂)-modules over L.*

Important inputs for the proof of Corollary 1.8 are Theorem 1.7 and a careful study of the period rings for the localisation of $\overline{R}$ at its minimal primes above $(p) \subset R$ (see §2.1).

1.1.3. Strategy for the proof of Theorem 1.5. The proof of Theorem 1.5 crucially uses analogous results obtained in the imperfect residue field case (see [Abh23a, Theorem 1.5]). Starting with a Wach module N over A_R^+, we use ideas from [Abh21, Theorem 4.25 & Proposition 4.28], the observation that $A_L^+ \otimes_{A_R^+} N$ is a Wach module over A_L^+ and [Abh23a, Lemma 3.6 & Theorem 3.12] to establish that $\mathbf{T}_R(N)$ is a $\mathbb{Z}_p$-representation of G_R such that $\mathbf{T}_R(N)[1/p]$ is crystalline (see Theorem 3.34). Conversely, starting with a $\mathbb{Z}_p$-lattice T inside a p-adic crystalline representation of G_R, we observe that $T[1/p]$ is a p-adic crystalline representation of G_L, and we use [Abh23a, Theorem 4.1] to obtain a unique Wach module $\mathbf{N}_L(T)$ over A_L^+. Moreover, note that from the theory of (φ, Γ)-modules we have an étale (φ, Γ_R)-module $\mathbf{D}_R(T)$ over A_R (see [And06]).

We set $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T) \subset \mathbf{D}_L(T)$ as an A_R^+-module, where $\mathbf{D}_L(T)$ is the (φ, Γ_L)-module over A_L, associated to T. Then, using the compatible Frobenius-semilinear endomorphism φ and the continuous action of $\Gamma_L \xrightarrow{\sim} \Gamma_R$ on $\mathbf{N}_L(T)$ and $\mathbf{D}_R(T)$, we equip the A_R^+-module $\mathbf{N}_R(T)$ with a natural (φ, Γ_R)-action. Let us remark that the definition of $\mathbf{N}_R(T)$ is parallel to the Breuil-Kisin setting studied in [DLMS22] and we employ some (modified) ideas from op. cit. to show that $\mathbf{N}_R(T)$ has "good" properties as a module over A_R^+. However, there are two key differences: first, op. cit. uses [BT08] as an important ingredient but our constructions use [Abh23a] instead; next, note that relative Breuil-Kisin modules admit a prismatic descent datum whereas Wach modules admit an action of Γ_R. Equipping $\mathbf{N}_R(T)$ with a natural action of Γ_R is non-trivial and we resolve it by using the theory of Wach modules in the imperfect residue field case from [Abh23a] and the theory of étale (φ, Γ)-modules from [And06] as important inputs. Finally, we utilise the properties of $\mathbf{N}_L(T)$ and $\mathbf{D}_R(T)$ to show that $\mathbf{N}_R(T)$ is the unique Wach module associated to T.

1.2. Wach modules as q-deformations. In §5 we recall the definition of a q-connection axiomatically, following [MT20]. Moreover, we show that a Wach module N over A_R^+ can also be seen as a φ-module equipped with a q-connection. More precisely, let $D := O_F[\![\mu]\!]$, and let $\{\gamma_1, \ldots, \gamma_d\}$ be topological generators of the geometric part of Γ_R, i.e. Γ'_R (see §2). Then in Proposition 5.3 we show that the q-connection defined as

$$\nabla_q : N \longrightarrow N \otimes_{A_R^+} \Omega^1_{A_R^+/D}, \qquad x \longmapsto \textstyle\sum_{i=1}^d \frac{\gamma_i(x)-x}{\mu} d\log([X_i^\flat]),$$

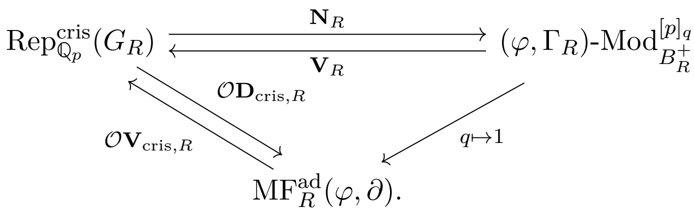
describes (N, ∇_q) as a φ-module with $(p, [p]_q)$-adically quasi-nilpotent D-linear flat q-connection over A_R^+. We equip N with the Nygaard filtration as in Definition 3.24. Then, it follows that $N/\mu N$ is a φ-module over R equipped with a p-adically quasi-nilpotent flat connection and we further equip it with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $N[1/p]/\mu N[1/p] = (N/\mu N)[1/p]$ with induced structures, in particular, it is a filtered (φ, ∂)-module over $R[1/p]$.

Theorem 1.9 (Theorem 5.6)**.** *Let N be a Wach module over A_R^+ and $V := \mathbf{T}_R(N)[1/p]$, the associated crystalline representation from Theorem 1.5. Then we have a natural isomorphism $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ of filtered (φ, ∂)-modules over $R[1/p]$.*

Note that $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ denotes the filtered (φ, ∂)-module over $R[1/p]$ associated to V (see [Bri08, §8.2]). Our proof of the theorem follows from computations done for the proof of Theorem 3.34 (building upon ideas developed in [Abh21, Theorem 4.25 & Proposition 4.28] and [Abh23a, Theorem 1.7]).

Finally, let us summarise the relationship between various categories considered in Theorem 1.5 and Theorem 1.9. Recall that $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R)$ is the category of p-adic crystalline representations of G_R, and let $\mathrm{MF}_R(\varphi, \partial)$ denote the category of filtered (φ, ∂)-modules over $R[1/p]$. From [Bri08, §8.2] we have a $\otimes$-compatible functor $\mathcal{O}\mathbf{D}_{\mathrm{cris},R} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R) \to \mathrm{MF}_R(\varphi, \partial)$, and let $\mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial)$ denote its essential image. Then, from [Bri08, Théorème 8.5.1], we have an exact equivalence of $\otimes$-categories $\mathcal{O}\mathbf{D}_{\mathrm{cris},R} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R) \xrightarrow{\sim} \mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial)$, with an exact $\otimes$-compatible quasi-inverse $\mathcal{O}\mathbf{V}_{\mathrm{cris},R}$ (see §2.6). So, Remark 1.6 and Theorem 1.9 can be summarised as follows:

Corollary 1.10 (Corollary 5.10)**.** *Functors in the following diagram induce exact equivalence of $\otimes$-categories*



1.3. Relation to previous works. Our first main result, Theorem 1.5, is a generalisation of arithmetic Wach modules from [Wac96; Col99; Ber04] and [Abh23a, Theorem 1.5]. That said, the methods of op. cit. do not directly apply to our current situtation. In fact, the proof of Theorem 1.5 uses crucial inputs of results and ideas from [Abh21] and [Abh23a].

Recent developments in the theory of prismatic F-crystals in [BS23; DLMS22; GR22] would suggest that there is a categorical equivalence between the category of Wach modules over A_R^+ and (completed/analytic) prismatic F-crystals on the absolute prismatic site $(\mathrm{Spf}\ R)_{\Delta}$. From that perspective, Theorem 1.5 could be seen as an analogue of [DLMS22, Theorem 1.2 & Proposition 1.4]. In our constructions, for a lattice T inside a crystalline representation of G_R, the definition of $\mathbf{N}_R(T)$ is parallel to the Breuil-Kisin case studied in op. cit. and we employ some (modified) ideas from op. cit. to show that $\mathbf{N}_R(T)$ has "good" properties as a module over A_R^+. However, there are two key differences: first, op. cit. uses [BT08] as an important ingredient but our constructions use [Abh23a] instead; next, note that Wach modules admit a natural action of Γ_R whereas relative Breuil-Kisin modules admit a prismatic descent datum. Equipping $\mathbf{N}_R(T)$ with a natural action of Γ_R is non-trivial and we resolve it by using the theory of Wach modules in the imperfect residue field case from [Abh23a] and the theory of étale (φ, Γ)-modules from [And06] as important inputs. Furthermore, as our base ring R is absolutely unramified (at p), the action of Γ_R is rich enough to establish the categorical equivalence claimed in Theorem 1.5.

In the current paper, we provide two applications of Theorem 1.5. The first application, i.e. Theorem 1.7 establishes a certain purity statement for crystalline representations. Our result is similar to the purity statement for Hodge-Tate representations in [Tsu11, Theorem 9.1] and rigidity of de Rham local systems in [LZ17, Theorem 1.3]. It should be noted that the purity result in Theorem 1.7 can also be obtained by combining [LZ17, Theorem 1.3] and some unpublished works of Tsuji. Moreover, the result of loc. cit. works for general ramified (at p) small base. A similar statement has been obtained in [Moo22, Theorem 1.4] using the results of [DLMS22].

The second application of Theorem 1.5 is given in Corollary 1.8. Our result provides a new criterion for checking the crystallinity of a p-adic representation of G_R. Note that the analogous statement for de Rham representations is true from the results of [LZ17]. However, our result in the crystalline case is entirely new and uses Theorem 1.7 as an important input. At this point, it is worth mentioning that for general ramified (at p) small base, a statement analogous to Corollary 1.8 appears to be true. In particular, we expect that one can deduce the statement using [LZ17, Theorem 1.3], the unpublished results of Tsuji mentioned above and employing arguments similar to our proof of Theorem 4.5.

For our second main result, Theorem 1.9, the motivation for interpreting a Wach module as a q-de Rham complex and as the q-deformation of crystalline cohomology, i.e. $\mathcal{O}\mathbf{D}_{\text{cris}}$, comes from [Fon90, §B.2.3], [Ber04, Théorème III.4.4] and [Sch17, §6]. In particular, we provide a direct generalisation of [Ber04, Théorème III.4.4], as well as verify expectations put forth in [Abh21, Remark 4.48] and [Abh23a, Remark 1.8] (see Remark 5.9 for the latter).

1.4. Setup and notations. In this section we will describe our setup and fix some notations, which are essentially the same as in [Abh21, §1.4]. We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers.

Let p be a fixed prime number, κ a perfect field of characteristic p, $O_F := W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ. Then O_F is a complete discrete valuation ring with uniformiser p and set $F := O_F[1/p]$ to be the fraction field of O_F. Let $\overline{F}$ denote a fixed algebraic closure of F so that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of κ. Furthermore, denote the absolute Galois group of F to be $G_F := \mathrm{Gal}(\overline{F}/F)$.

Notation. Let Λ be an I-adically complete algebra for a finitely generated ideal $I \subset \Lambda$. Let $Z := (Z_1, \ldots, Z_s)$ denote a set of indeterminates and $\mathbf{k} := (k_1, \ldots, k_s) \in \mathbb{N}^s$ be a multi-index, then we write $Z^{\mathbf{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$. For $\mathbf{k} \to +\infty$ we will mean that $\sum k_i \to +\infty$. Define

$$\Lambda\langle Z\rangle := \Big\{ \sum_{\mathbf{k} \in \mathbb{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}, \text{ where } a_{\mathbf{k}} \in \Lambda \text{ and } a_{\mathbf{k}} \to 0 \ I\text{-adically as } \mathbf{k} \to +\infty \Big\}.$$

We fix $d \in \mathbb{N}$ and let $X := (X_1, X_2, \ldots, X_d)$ be some indeterminates. Let R be the p-adic completion of an étale algebra over $R^\square := O_F\langle X, X^{-1}\rangle$, with non-empty geometrically integral special fiber. We fix an algebraic closure $\overline{\mathrm{Frac}(R)}$ of $\mathrm{Frac}(R)$ containing $\overline{F}$. Let $\overline{R}$ denote the union of finite R-subalgebras $S \subset \overline{\mathrm{Frac}(R)}$, such that $S[1/p]$ is étale over $R[1/p]$. Let $\overline{\eta}$ denote the fixed geometric point of the generic fiber $\mathrm{Spec}\, R[1/p]$ (defined by $\overline{\mathrm{Frac}(R)}$), and let $G_R := \pi_1^{\text{ét}}(\mathrm{Spec}\, R[1/p], \overline{\eta})$ denote the étale fundamental group. We can write this étale fundamental group as the Galois group (of the fraction field of $\overline{R}[1/p]$ over the fraction field of $R[1/p]$), i.e. $G_R = \pi_1^{\text{ét}}(\mathrm{Spec}\,(R[1/p]), \overline{\eta}) = \mathrm{Gal}(\overline{R}[1/p]/R[1/p])$. For $k \in \mathbb{N}$, let Ω^k_R denote the p-adic completion of module of k-differentials of R relative to $\mathbb{Z}$. Then, we have $\Omega^1_R = \oplus_{i=1}^d R\, d\log X_i$, and $\Omega^k_R = \wedge^k_R \Omega^1_R$.

Let φ denote an endomorphism of $R^\square$ which extends the natural Frobenius on O_F by setting $\varphi(X_i) = X_i^p$, for all $1 \leq i \leq d$. The morphism $\varphi : R^\square \to R^\square$ is flat by [Bri08, Lemma 7.1.5], and it is faithfully flat since $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ for any maximal ideal $\mathfrak{m} \subset R^\square$. Moreover, using Nakayama Lemma and the fact that the absolute Frobenius on $R^\square/p$ is evidently of degree p^d, it easily follows that φ on $R^\square$ is finite of degree p^d. Recall that the O_F-algebra R is given as the p-adic completion of an étale algebra $R^\square$, therefore, the Frobenius endomorphism φ on $R^\square$ admits a unique extension $\varphi : R \to R$ such that the induced map $\varphi : R/p \to R/p$ is the absolute Frobenius $x \mapsto x^p$ (see [CN17, Proposition 2.1]). Similar to above, again note that the endomorphism $\varphi : R \to R$ is faithfully flat and finite of degree p^d.

Let $O_L := (R_{(p)})^\wedge$, where $^\wedge$ denotes the p-adic completion. Let $\overline{L}$ denote a fixed algebraic closure of L with ring of integers $O_{\overline{L}}$ such that we have an embedding $\overline{R} \to O_{\overline{L}}$. Then we get a continuous homomorphism $G_L := \mathrm{Gal}(\overline{L}/L) \to G_R$, inducing an isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$. The Frobenius on R extends to a unique Frobenius endomorphism $\varphi : O_L \to O_L$, lifting the absolute Frobenius on O_L/pO_L (see [CN17, Proposition 2.1]). Similar to above, φ on O_L is faithfully flat and finite of degree p^d.

Let S be a commutative ring with $\pi := p^{1/p} \in S$ such that S is π-adically complete and π-torsion free, for example, $S = O_{F_\infty}, O_{L_\infty}, O_{\overline{F}}, O_{\overline{L}}, R_\infty, \overline{R}$. Then the tilt of S is defined as $S^\flat := \lim_\varphi S/p$ and the tilt of $S[1/p]$ is defined as $S[1/p]^\flat := S^\flat[1/p^\flat]$, where $p^\flat := (1, p^{1/p}, \ldots) \in S^\flat$ (see [Fon77, Chapitre V, §1.4] and [BMS18, §3]). Finally, consider a $\mathbb{Z}_p$-algebra A equipped with a lift of the absolute Frobenius on A/p, i.e. an endomorphism $\varphi : A \to A$ such that φ modulo p is the absolute Frobenius. Then for any A-module M we write $\varphi^*(M) := A \otimes_{\varphi, A} M$.

Outline of the paper. This article consists of four main sections. In §2 we collect relevant results in relative p-adic Hodge theory. In §2.1 we consider localisations of $\overline{R}$ at minimal primes above $(p) \subset R$ and study their properties. Then in §2.2, §2.3 & §2.4 we define relative period rings and study their localisations at primes of $\overline{R}$ above $(p) \subset R$. In §2.5 we quickly recall important rings from the theory of relative (φ, Γ)-modules and in §2.6 we recall the relation between (φ, Γ)-module theory and p-adic representations, as well as, definition and properties of crystalline representations. The aim of §3 is to define and study properties of a Wach module in the relative case and the associated representation of G_R. In §3.1 we first note some technical lemmas and then in §3.2 we define relative Wach modules, study its properties and relate these objects to étale (φ, Γ)-modules (see Proposition 3.15). Furthermore, in §3.3, we functorially attach a $\mathbb{Z}_p$-representation of G_R to a relative Wach module and in §3.4 we show that such representations are closely related to finite $[p]_q$-height representations studied in [Abh21]. In §3.5 we study the Nygaard filtration on relative Wach modules. Finally, in §3.6 we show that the $\mathbb{Z}_p$-representation of G_R associated to a relative Wach module, as in §3.3, is a lattice inside a p-adic crystalline representation of G_R (see Theorem 3.34). In §4 we prove our first main result, i.e. Theorem 1.5. Before proving the theorem, we draw some important conclusions from the statement, in particular, in §4.1 we prove Theorem 1.7 and Corollary 1.8. Finally, in §4.2 we construct the promised relative Wach module and prove Theorem 1.5. In §5, we state and prove our second main result, i.e. Theorem 1.9. In §5.1, we recall the formalism on q-connections. Then in §5.2, we show that a Wach module can be interpreted as a φ-module equipped with a q-connection (see Proposition 5.3). Finally, using the computations done in the proof of Theorem 3.34, we prove Theorem 1.9.

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2. Period rings and p-adic representations

We will use the setup and notations from §1.4. Recall that R is the p-adic completion of an étale algebra over $O_F\langle X_1^{\pm 1}, \ldots, X_d^{\pm 1}\rangle$ and $O_L := (R_{(p)})^\wedge$. Set $R_\infty := \cup_{i=1}^d R[\mu_{p^\infty}, X_i^{1/p^\infty}]$ and recall that $\overline{R}$ is the union of finite R-subalgebras S in a fixed algebraic closure $\overline{\mathrm{Frac}(R)} \supset \overline{F}$, such that $S[1/p]$ is étale over $R[1/p]$. We have (see [Abh21, §2 & §3]),

$$\begin{aligned}
&G_R := \mathrm{Gal}(\overline{R}[1/p]/R[1/p]),\ H_R = \mathrm{Gal}(\overline{R}[1/p]/R_\infty[1/p]),\\
&\Gamma_R := G_R/H_R = \mathrm{Gal}(R_\infty[1/p]/R[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times,\\
&\Gamma_R' := \mathrm{Gal}(R_\infty[1/p]/R(\mu_{p^\infty})[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d,\ \mathrm{Gal}(R(\mu_{p^\infty})[1/p]/R[1/p]) = \Gamma_R/\Gamma_R' \xrightarrow{\sim} \mathbb{Z}_p^\times.
\end{aligned}$$

We fixed $\overline{L}$ as an algebraic closure of $L := O_L[1/p]$ with ring of integers $O_{\overline{L}}$ and an embedding $\overline{R} \to O_{\overline{L}}$. So, we have a continuous homomorphism of groups $G_L := \mathrm{Gal}(\overline{L}/L) \to G_R$, which induces an isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$. For $1 \leq i \leq d$, we fix $X_i^\flat := (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots)$ in $R_\infty^\flat$ and take $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ in Γ_R such that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ_R' satisfying $\gamma_j(X_i^\flat) = \varepsilon X_i^\flat$ if $i = j$ and $X_i^\flat$ otherwise, and γ_0 is a lift of a topological generator of Γ_R/Γ_R'.

2.1. Localisation.

Let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$. The set $\mathcal{S}$ is equipped with a transitive action of G_R (see [Mat89, Theorem 9.3]). For each prime $\mathfrak{p} \in \mathcal{S}$, set $G_R(\mathfrak{p}) := \{g \in G_R \text{ such that } g(\mathfrak{p}) = \mathfrak{p}\}$, i.e. the decomposition group of G at $\mathfrak{p}$. Recall that $O_L = (R_{(p)})^\wedge$ and $L = O_L[1/p]$. For each $\mathfrak{p} \in \mathcal{S}$, let $\overline{L}(\mathfrak{p})$ denote an algebraic closure of L with ring of integers $O_{\overline{L}(\mathfrak{p})}$ containing $(\overline{R})_\mathfrak{p}$. Set $\widehat{G}_R(\mathfrak{p}) := \mathrm{Gal}(\overline{L}(\mathfrak{p})/L)$ so that we have a natural homomorphism $\widehat{G}_R(\mathfrak{p}) \to G_R$ which factors as $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p}) \subset G_R$ (see [Bri08, Lemme 3.3.1]). Note that for each $\mathfrak{p} \in \mathcal{S}$, we have a natural embedding $\overline{R} \subset O_{\overline{L}(\mathfrak{p})}$ and hence we have a (non-canonical) isomorphism of Galois groups $\widehat{G}_R(\mathfrak{p}) \xrightarrow{\sim} G_L$.

Now, for each $\mathfrak{p} \in \mathcal{S}$, let $\mathbb{C}_\mathfrak{p}^+$ denote the p-adic completion of $O_{\overline{L}(\mathfrak{p})}$ and let $\mathbb{C}_\mathfrak{p} := \mathrm{Frac}(\mathbb{C}_\mathfrak{p}^+)$. Then $\mathbb{C}_\mathfrak{p}$ is an algebraically closed valuation field equipped with a continuous action of $\widehat{G}_R(\mathfrak{p})$ and $(\mathbb{C}_\mathfrak{p}^+)^{\widehat{G}_R(\mathfrak{p})} = O_L$ (see [Hyo86, Theorem 1]). Furthermore, let $\mathbb{C}^+(\mathfrak{p})$ denote the p-adic completion of $(\overline{R})_\mathfrak{p}$ and let $\mathbb{C}(\mathfrak{p}) := \mathbb{C}^+(\mathfrak{p})[1/p]$ equipped with a continuous action of $G_R(\mathfrak{p})$.

Lemma 2.1. *For each $\mathfrak{p} \in \mathcal{S}$, we have $(\overline{R})_\mathfrak{p} \subset \mathbb{C}^+(\mathfrak{p})$ and $(\overline{R})_\mathfrak{p} \cap p\mathbb{C}^+(\mathfrak{p}) = p(\overline{R})_\mathfrak{p}$. Moreover, $(\overline{R})_\mathfrak{p} \cap pO_{\overline{L}(\mathfrak{p})} = p(\overline{R})_\mathfrak{p}$.*

Proof. The proof is similar to [Bri08, Proposition 2.0.3]. Let $\mathfrak{p} \in \mathcal{S}$ and $x \in (\overline{R})_{\mathfrak{p}}$. Then there exists a finite normal R-subalgebra $S \subset \overline{R}$ such that $S[1/p]$ is étale over $R[1/p]$ and $\mathfrak{q} := \mathfrak{p} \cap S$ is a height 1 prime ideal of S with $p \in \mathfrak{q}$ (since $\overline{R}$ is integral over S) and $x \in S_{\mathfrak{q}}$. Moreover, $S_{\mathfrak{q}}$ is a 1-dimensional normal noetherian domain, in particular, a discrete valuation ring. Now if the image of x is zero in $\mathbb{C}^+(\mathfrak{p})$, then we have that $x \in p^n(\overline{R})_{\mathfrak{p}} \cap S_{\mathfrak{q}} = p^n S_{\mathfrak{q}}$, for each $n \in \mathbb{N}$, since $S_{\mathfrak{q}}$ is normal. So x must be zero since $S_{\mathfrak{q}}$ is p-adically separated. This shows the first claim. For the second claim, let $x = py$ for some $y \in \mathbb{C}^+(\mathfrak{p})$. We have that $y \in S_{\mathfrak{q}}[1/p]$ and we need to show that $y \in S_{\mathfrak{q}}$. Let $\widehat{S}_{\mathfrak{q}}$ denote the completion of $S_{\mathfrak{q}}$ for the valuation (say $v_{\mathfrak{q}}$) described above. Then $\widehat{S}_{\mathfrak{q}}[1/p]$ is a finite separable extension of L and $\widehat{S}_{\mathfrak{q}}$ embeds into $\mathbb{C}_{\mathfrak{p}}^+$. Moreover, the image of $\mathbb{C}^+(\mathfrak{p})$ in $\mathbb{C}_{\mathfrak{p}}$ is contained in $\mathbb{C}_{\mathfrak{p}}^+$, therefore $v_{\mathfrak{q}}(y) \geq 0$, i.e. $y \in S_{\mathfrak{q}}[1/p] \cap \widehat{S}_{\mathfrak{q}} = S_{\mathfrak{q}}$, as desired. Finally, let $x = pz$ for some $z \in O_{\overline{L(\mathfrak{p})}}$. Then similar to above, we have $z \in S_{\mathfrak{q}}[1/p]$ and $v_{\mathfrak{q}}(z) \geq 0$, so $z \in S_{\mathfrak{q}}$. This shows the third claim. ∎

All rings discussed above are p-torsion free, so from Lemma 2.1, it easily follows that the inclusion $\mathbb{C}^+(\mathfrak{p}) \subset \mathbb{C}_{\mathfrak{p}}^+$ is compatible with respective actions of $\widehat{G}_R(\mathfrak{p})$, where the action of $\widehat{G}_R(\mathfrak{p})$ on the left-hand term factors through $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p})$. In particular, we get that $\mathbb{C}^+(\mathfrak{p})^{G_R(\mathfrak{p})} = O_L$ (see [Bri08, p. 24]). Now, note that we have natural injective maps $\overline{R} \to (\overline{R})_{\mathfrak{p}} \to O_{\overline{L(\mathfrak{p})}}$. Upon passing to p-adic completions and setting $\mathbb{C}^+(\overline{R}) := \widehat{\overline{R}}$, we obtain natural maps $\mathbb{C}^+(\overline{R}) \to \mathbb{C}^+(\mathfrak{p}) \to \mathbb{C}_{\mathfrak{p}}^+$, where the first map need not be injective. However, recall that $\overline{R}$ is a direct limit of finite and normal R-algebras, therefore the natural map $\overline{R}/p^n \to \oplus_{\mathfrak{p}\in\mathcal{S}}(\overline{R})_{\mathfrak{p}}/p^n$ is injective. Passing to the limit over n, we obtain injective maps

$$\mathbb{C}^+(\overline{R}) \longrightarrow \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p}) \longrightarrow \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}_{\mathfrak{p}}^+. \tag{2.1}$$

Note that in (2.1) the leftmost term admits a natural action of G_R, the middle term admits a natural action of $\prod_{\mathfrak{p}\in\mathcal{S}} G_R(\mathfrak{p})$ and the rightmost term admits a natural action of $\prod_{\mathfrak{p}\in\mathcal{S}} \widehat{G}_R(\mathfrak{p})$. The two homomorphisms in (2.1) are compatible with these respective actions. Moreover, from [Bri08, Remarque 3.3.2] the middle term of (2.1) can be equipped with an action of G_R and the left homomorphism in (2.1) is equivariant with respect to this action of G_R.

Remark 2.2. Note that $\mathbb{C}^+(\mathfrak{p})$ is an O_L-algebra for each $\mathfrak{p} \in \mathcal{S}$, so the maps in (2.1) extend to injective maps $O_L \otimes_R \mathbb{C}^+(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}_{\mathfrak{p}}^+$ (see [Bri08, Proposition 3.3.3]).

Lemma 2.3. *The O_L-algebra $\mathbb{C}^+(\mathfrak{p})$ is perfectoid in the sense of [BMS18, Definition 3.5].*

Proof. Note that we have $\pi := p^{1/p} \in \overline{R} \subset (\overline{R})_{\mathfrak{p}} \subset \mathbb{C}^+(\mathfrak{p})$ and $\pi^p = p$ divides p. Moreover, it is clear that $\mathbb{C}^+(\mathfrak{p})$ is π-adically complete. Now, consider the following commutative diagram:

$$\begin{array}{ccccc} \mathbb{C}^+(\mathfrak{p})/\pi^p & \twoheadrightarrow & \mathbb{C}^+(\mathfrak{p})/\pi & \xrightarrow[\varphi]{} & \mathbb{C}^+(\mathfrak{p})/\pi^p \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}_{\mathfrak{p}}^+/\pi^p & \twoheadrightarrow & \mathbb{C}_{\mathfrak{p}}^+/\pi & \xrightarrow[\varphi]{\sim} & \mathbb{C}_{\mathfrak{p}}^+/\pi^p, \end{array}$$

where the left and right vertical arrows are injective by Lemma 2.1 and the middle vertical arrow is also injective by an argument similar to the proof of Lemma 2.1. So it follows that the top right horizontal arrow is injective as well. Then, using [BMS18, Lemma 3.9 and Lemma 3.10], we are left to show that $\varphi : \mathbb{C}^+(\mathfrak{p})/p = (\overline{R})_{\mathfrak{p}}/p \to (\overline{R})_{\mathfrak{p}}/p = \mathbb{C}^+(\mathfrak{p})/p$ is surjective. So let $x \in (\overline{R})_{\mathfrak{p}}/p$ and take a lift $y \in (\overline{R})_{\mathfrak{p}}$. Then there exists an $a \in \overline{R} \setminus \mathfrak{p}$ such that $ay \in \overline{R}$. Now, from [Bri08, Proposition 2.0.1], there exists $z, w \in \overline{R}$ such that $ay = z^p + pw$. Moreover, there exists $b \in \overline{R} \setminus \mathfrak{p}$ and $c \in \overline{R}$ such that $a = b^p + pc$. Then we can write $b^p y + pcy = z^p + pw$, or equivalently, $y = (z/b)^p + p(cy + w)/b^p$ with $(z/b)^p \in (\overline{R})_{\mathfrak{p}}$ and $p(cy+w)/b^p \in p(\overline{R})_{\mathfrak{p}}$. Hence, $x = (z/b)^p \mod p(\overline{R})_{\mathfrak{p}}$, proving that $\varphi : (\overline{R})_{\mathfrak{p}}/p \to (\overline{R})_{\mathfrak{p}}/p$ is surjective. ∎

2.2. The period ring A_{inf}. In this subsection we will study the relative version of Fonatine's infinitesimal period ring A_{inf} to be used in the sequel (see [Abh21, §2 and §3] for details). Let $A_{\text{inf}}(R_\infty) := W(R_\infty^\flat)$ and $A_{\text{inf}}(\overline{R}) := W(\overline{R}^\flat)$ admitting the Frobenius on Witt vectors and continuous G_R-action (for the weak topology). Moreover, we have $A_{\text{inf}}(R_\infty) = A_{\text{inf}}(\overline{R})^{H_R}$ (see [And06, Proposition 7.2]). Let $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots)$, $\overline{\mu} := \varepsilon - 1 \in O_{F_\infty}^\flat$ and set $\mu := [\varepsilon] - 1, \xi := \mu/\varphi^{-1}(\mu) \in A_{\text{inf}}(O_{F_\infty})$. Let χ denote the p-adic cyclotomic character, then for $g \in G_R$, we have $g(1+\mu) = (1+\mu)^{\chi(g)}$. Additionally, we have a G_R-equivariant surjection $\theta : A_{\inf}(\overline{R}) \to \mathbb{C}^+(\overline{R})$ and $\operatorname{Ker} \theta = \xi A_{\inf}(\overline{R})$. The map θ further induces a Γ_R-equivariant surjection $\theta : A_{\inf}(R_\infty) \to \widehat{R}_\infty$.

Let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$ and for each prime $\mathfrak{p} \in \mathcal{S}$ let $\mathbb{C}_\mathfrak{p}$ denote the valuation field described in §2.1 and $\mathbb{C}_\mathfrak{p}^+$ its ring of integers. Moreover, from Lemma 2.3, we have that $\mathbb{C}^+(\mathfrak{p})$ is a perfectoid algebra. So we set $A_{\inf}(\mathbb{C}_\mathfrak{p}^+) := W(\mathbb{C}_\mathfrak{p}^{+,\flat})$ (resp. $A_{\inf}(\mathbb{C}^+(\mathfrak{p})) := W(\mathbb{C}^+(\mathfrak{p})^\flat)$) admitting the Frobenius on Witt vectors and continuous $\widehat{G}_R(\mathfrak{p})$-action (resp. $G_R(\mathfrak{p})$-action). Similar to above, we have a $\widehat{G}_R(\mathfrak{p})$-equivariant surjection $\theta : A_{\inf}(\mathbb{C}_\mathfrak{p}^+) \to \mathbb{C}_\mathfrak{p}^+$ with $\operatorname{Ker} \theta = \xi A_{\inf}(\mathbb{C}_\mathfrak{p}^+)$ (resp. a $G_R(\mathfrak{p})$-equivariant surjection $\theta : A_{\inf}(\mathbb{C}^+(\mathfrak{p})) \to \mathbb{C}^+(\mathfrak{p})$ with $\operatorname{Ker} \theta = \xi A_{\inf}(\mathbb{C}^+(\mathfrak{p}))$).

Lemma 2.4. *For each $\mathfrak{p} \in \mathcal{S}$ we have $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant embeddings $A_{\inf}(\mathbb{C}^+(\mathfrak{p})) \to A_{\inf}(\mathbb{C}_\mathfrak{p}^+)$ and $W(\mathbb{C}(\mathfrak{p})^\flat) \to W(\mathbb{C}_\mathfrak{p}^\flat)$, where the action of $\widehat{G}_R(\mathfrak{p})$ on left-hand terms factor through $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p})$. Moreover, we have a $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant identification $A_{\inf}(\mathbb{C}^+(\mathfrak{p})) = A_{\inf}(\mathbb{C}_\mathfrak{p}^+) \cap W(\mathbb{C}(\mathfrak{p})^\flat)$ as subrings of $W(\mathbb{C}_\mathfrak{p}^\flat)$.*

Proof. From the discussion before (2.1), we have a $\widehat{G}_R(\mathfrak{p})$-equivariant injective map $\mathbb{C}^+(\mathfrak{p}) \to \mathbb{C}_\mathfrak{p}^+$. By applying the tilting functor, we further obtain a $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant commutative diagram of rings

$$\begin{array}{ccc} \mathbb{C}^+(\mathfrak{p})^\flat & \longrightarrow & \mathbb{C}_\mathfrak{p}^{+,\flat} \\ \downarrow & & \downarrow \\ \mathbb{C}(\mathfrak{p})^\flat & \longrightarrow & \mathbb{C}_\mathfrak{p}^\flat, \end{array} \tag{2.2}$$

where the vertical arrows are injective. Note that the natural map $\mathbb{C}^+(\mathfrak{p})/p = (\overline{R})_\mathfrak{p}/p \to O_{\overline{L(\mathfrak{p})}}/p = \mathbb{C}_\mathfrak{p}^+/p$ is injective, so by left exactness of $\lim_\varphi$, we obtain that in (2.2) the top horizontal arrow is injective. Moreover, note that $\mathbb{C}(\mathfrak{p})^\flat = \lim_{x \mapsto x^p} \mathbb{C}(\mathfrak{p})$ as a multiplicative monoid, and similarly for $\mathbb{C}_\mathfrak{p}^\flat$. Therefore, again by left exactness of lim, it follows that the bottom horizontal arrow in (2.2) is injective. Now, since $\mathbb{C}_\mathfrak{p}^\flat$ is a valuation field, let $\upsilon_\mathfrak{p}^\flat$ denote the normalised valuation on it such that $\upsilon_\mathfrak{p}^\flat(p^\flat) = 1$. Then we have that $x \in \mathbb{C}_\mathfrak{p}^{+,\flat}$ if and only if $\upsilon_\mathfrak{p}^\flat(x) \geq 0$. Moreover, we have $\mathbb{C}(\mathfrak{p})^\flat = \mathbb{C}^+(\mathfrak{p})^\flat[1/p^\flat]$ and $\mathbb{C}_\mathfrak{p}^\flat = \mathbb{C}_\mathfrak{p}^{+,\flat}[1/p^\flat]$. From (2.2) and injectivity of its arrows, it now follows that for $x \in \mathbb{C}(\mathfrak{p})^\flat$ we have $x \in \mathbb{C}^+(\mathfrak{p})^\flat$ is and only if $\upsilon_\mathfrak{p}^\flat(x) \geq 0$. In particular,

$$\mathbb{C}^+(\mathfrak{p})^\flat = \mathbb{C}(\mathfrak{p})^\flat \cap \mathbb{C}_\mathfrak{p}^{+,\flat} \subset \mathbb{C}_\mathfrak{p}^\flat. \tag{2.3}$$

Furthermore, recall that the p-typical Witt vector functor is left exact since it is right adjoint to the forgetful functor from the category of δ-rings to the category of rings (see [Joy85]). Therefore, all maps in the following natural $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant commutative diagram are injective

$$\begin{array}{ccc} A_{\inf}(\mathbb{C}^+(\mathfrak{p})) & \longrightarrow & A_{\inf}(\mathbb{C}_\mathfrak{p}^+) \\ \downarrow & & \downarrow \\ W(\mathbb{C}(\mathfrak{p})^\flat) & \longrightarrow & W(\mathbb{C}_\mathfrak{p}^\flat). \end{array}$$

Hence, from (2.3) it follows that $A_{\inf}(\mathbb{C}^+(\mathfrak{p})) = W(\mathbb{C}(\mathfrak{p})^\flat) \cap A_{\inf}(\mathbb{C}_\mathfrak{p}^+) \subset W(\mathbb{C}_\mathfrak{p}^\flat)$. ■

Remark 2.5. From Lemma 2.4, the discussion preceding it (see the map θ) and the fact that $\mathbb{C}^+(\mathfrak{p})$ is a subring of $\mathbb{C}_\mathfrak{p}^+$, it easily follows that $\xi A_{\inf}(\mathbb{C}^+(\mathfrak{p})) = A_{\inf}(\mathbb{C}^+(\mathfrak{p})) \cap \xi A_{\inf}(\mathbb{C}_\mathfrak{p}^+) \subset A_{\inf}(\mathbb{C}_\mathfrak{p}^+)$.

Remark 2.6. By functoriality of the tilting construction and Witt vector construction, we note that the action of G_R on $\prod_{\mathfrak{p} \in \mathcal{S}} \mathbb{C}^+(\mathfrak{p})$ described after (2.1) (see [Bri08, Remarque 3.3.2]), extends to respective natural actions of G_R on $\prod_{\mathfrak{p} \in \mathcal{S}} A_{\inf}(\mathbb{C}^+(\mathfrak{p}))$ and $\prod_{\mathfrak{p} \in \mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)$.

Lemma 2.7. *In the notations described above, we have (φ, G_R)-equivariant embeddings $A_{\inf}(\overline{R}) \to \prod_{\mathfrak{p} \in \mathcal{S}} A_{\inf}(\mathbb{C}^+(\mathfrak{p}))$ and $W(\mathbb{C}(\overline{R})^\flat) \to \prod_{\mathfrak{p} \in \mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)$, where right-hand terms are equipped with a G_R-action as described in Remark 2.6. Moreover, we have a (φ, G_R)-equivariant identification $A_{\inf}(\overline{R}) = W(\mathbb{C}(\overline{R})^\flat) \cap \prod_{\mathfrak{p} \in \mathcal{S}} A_{\inf}(\mathbb{C}^+(\mathfrak{p}))$ as subrings of $\prod_{\mathfrak{p} \in \mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)$.*

Proof. From (2.1) recall that we have injective maps $\mathbb{C}^+(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}}(\overline{R})^{\wedge}_{\mathfrak{p}}$. By applying the tilting functor, we further obtain a (φ, G_R)-equivariant commutative diagram:

$$\begin{array}{ccccc} \mathbb{C}^+(\overline{R})^\flat & \longrightarrow & & & \prod_{\mathfrak{p}\in\mathcal{S}}\mathbb{C}^+(\mathfrak{p})^\flat \\ \downarrow & & & & \downarrow \\ \mathbb{C}(\overline{R})^\flat & \longrightarrow & \Big(\prod_{\mathfrak{p}\in\mathcal{S}}\mathbb{C}^+(\mathfrak{p})^\flat\Big)\big[\tfrac{1}{p^\flat}\big] & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}}\mathbb{C}(\mathfrak{p})^\flat, \end{array} \tag{2.4}$$

where the bottom right horizontal arrow and vertical arrows are injective. From the injectivity of $\overline{R}/p \to \prod_{\mathfrak{p}\in\mathcal{S}}\mathbb{C}^+(\mathfrak{p})/p$ and left exactness of $\lim_\varphi$, we obtain that in (2.4) the top horizontal arrow is injective and since we have $\mathbb{C}(\overline{R})^\flat = \mathbb{C}^+(\overline{R})^\flat[1/p^\flat]$, it also follows that the bottom left horizontal arrow is injective. Now let $\upsilon_{\mathfrak{p}}^\flat$ denote the valuation on $\mathbb{C}_{\mathfrak{p}}^\flat$ intriduced in the proof of Lemma 2.4. Then under the composition of left vertical and bottom horizontal arrows of (2.4), it follows that for any $x \in \mathbb{C}(\overline{R})^\flat$ we have that x belongs to $\mathbb{C}^+(\overline{R})^\flat$ if and only if $\upsilon_{\mathfrak{p}}^\flat(x) \geq 0$ for each $\mathfrak{p} \in \mathcal{S}$. In particular,

$$\mathbb{C}^+(\overline{R})^\flat = \mathbb{C}(\overline{R})^\flat \cap \prod_{\mathfrak{p}\in\mathcal{S}}\mathbb{C}^+(\mathfrak{p})^\flat \subset \prod_{\mathfrak{p}\in\mathcal{S}}\mathbb{C}(\mathfrak{p})^\flat. \tag{2.5}$$

Furthermore, recall that the p-typical Witt vector functor is left exact since it is right adjoint to the forgetful functor from the category of δ-rings to the category of rings (see [Joy85]). Therefore, all maps in the following natural (φ, G_R)-equivariant commutative diagram are injective

$$\begin{array}{ccc} A_{\mathrm{inf}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \\ \downarrow & & \downarrow \\ W(\mathbb{C}(\overline{R})^\flat) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat). \end{array}$$

Then from (2.5) we obtain $A_{\mathrm{inf}}(\overline{R}) = W(\mathbb{C}(\overline{R})^\flat) \cap \prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ as subrings of $\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)$. ■

2.3. de Rham period rings. In this subsection we will recall the de Rham period rings (see [Abh21, §2.1]). Note that the Γ_R-equivariant map $\theta : A_{\mathrm{inf}}(R_\infty) \twoheadrightarrow \widehat{R}_\infty$ described in §2.2 extends to a surjective map $\theta : A_{\mathrm{inf}}(R_\infty)[1/p] \twoheadrightarrow \widehat{R}_\infty[1/p]$. We set $B_{\mathrm{dR}}^+(R_\infty) := \lim_n (A_{\mathrm{inf}}(R_\infty)[1/p])/\xi^n$. Let $t := \log(1+\mu) \in B_{\mathrm{dR}}^+(R_\infty)$, then $B_{\mathrm{dR}}^+(R_\infty)$ is t-torsion free and we set $B_{\mathrm{dR}}(R_\infty) := B_{\mathrm{dR}}^+(R_\infty)[1/t]$. Furthermore, one can define period rings $\mathcal{O}B_{\mathrm{dR}}^+(R_\infty)$ and $\mathcal{O}B_{\mathrm{dR}}(R_\infty)$. These rings are equipped with a Γ_R-action, an appropriate extension of the map θ and a decreasing filtration. Rings with a prefix "$\mathcal{O}$" are further equipped with an integrable connection satisfying Griffiths transversality with respect to the filtration. One can define variations of these rings over $\overline{R}$ as well.

Next, let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$ as in §2.1. Similar to above, for each $\mathfrak{p} \in \mathcal{S}$, we set $B_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+) := \lim_n (A_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)[1/p])/(\mathrm{Ker}\ \theta)^n$ and $B_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+) := B_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)[1/t]$ equipped with a $\widehat{G}_R(\mathfrak{p})$-action (resp. $B_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p})) := \lim_n (A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))[1/p])/(\mathrm{Ker}\ \theta)^n$ as well as $B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) := B_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))[1/t]$ equipped with a $G_R(\mathfrak{p})$-action), an appropriate extension of the map θ and a decreasing filtration.

Lemma 2.8. *The $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to A_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$ of Lemma 2.4 extends to a $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to B_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+)$.*

Proof. Note that by definition, the $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to A_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$ induces a $\widehat{G}_R(\mathfrak{p})$-equivariant map $B_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p})) \to B_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$. Then from Remark 2.5 and the fact that lim is a left exact functor on the category of abelian groups, we get that the map $B_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p})) \to B_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ is injective. The claim now follows since $B_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ and $B_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))$ are t-torsion free (see [Bri08, Proposition 5.1.4]). ■

Moreover, for each $\mathfrak{p} \in \mathcal{S}$ we have big period rings $\mathcal{O}B_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ and $\mathcal{O}B_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+)$ equipped with an L-linear $\widehat{G}_R(\mathfrak{p})$-action (resp. $\mathcal{O}B_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))$ and $\mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ equipped with an L-linear $G_R(\mathfrak{p})$-action), an appropriate extension of the map θ, a decreasing filtration and a connection. From [Bri08, §5.2 &

§5.3], in particular, from the alternative description of $\mathcal{O}B_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ (resp. $\mathcal{O}B_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))$) as a power series ring over $B_{\mathrm{dR}}^+(\mathbb{C}_{\mathfrak{p}}^+)$ (resp. $B_{\mathrm{dR}}^+(\mathbb{C}^+(\mathfrak{p}))$) and using Lemma 2.8, the following is obvious:

Lemma 2.9. *The $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to B_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+)$ of Lemma 2.8 extends to an L-linear $\widehat{G}_R(\mathfrak{p})$-equivariant embedding $\mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+)$ compatible with respective filtrations and connections.*

Remark 2.10. Recall that product is an exact functor on the category of abelian groups. So the natural embeddings $A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, for each $\mathfrak{p} \in \mathcal{S}$, extend to embeddings $\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$. By an argument similar to [Bri08, Remarque 3.3.2] the products $\prod_{\mathfrak{p}\in\mathcal{S}} B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ and $\prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ can respectively be equipped with an action of G_R, extending the G_R-action on $\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ (see Remark 2.6), in particular, the embeddings $\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ are G_R-equivariant.

Lemma 2.11. *In the notations described above, we have an $R[1/p]$-linear G_R-equivariant embedding $\mathcal{O}B_{\mathrm{dR}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, where the right-hand term is equipped with a G_R-action as described in Remark 2.10. Moreover, for each $\mathfrak{p} \in \mathcal{S}$, the induced natural map $\mathcal{O}B_{\mathrm{dR}}(\overline{R}) \to \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ is compatible with respective filtrations and connections.*

Proof. Note that from Lemma 2.7 and Remark 2.10, we have G_R-equivariant injective maps $A_{\mathrm{inf}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$. Then from the definition of $B_{\mathrm{dR}}(\overline{R})$ and $\mathcal{O}B_{\mathrm{dR}}(\overline{R})$, the preceding maps naturally induce an $R[1/p]$-linear and G_R-equivariant commutative diagram:

$$\begin{array}{ccccc} B_{\mathrm{dR}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) & \longrightarrow & B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}B_{\mathrm{dR}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) & \longrightarrow & \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})), \end{array} \tag{2.6}$$

where the vertical maps are injective, with the leftmost and rightmost vertical arrows being compatible with respective filtrations and connections for each $\mathfrak{p} \in \mathcal{S}$. We need to show that the top left and bottom left horizontal arrows are injective. But first, let us note that from the explicit description of filtration on B_{dR} and $\mathcal{O}B_{\mathrm{dR}}$ in [Bri08, §5.2], it easily follows that compositions of horizontal arrows in (2.6) are compatible with respective filtrations and connections, i.e. for each $k \in \mathbb{Z}$, the respective images of $\mathrm{Fil}^k B_{\mathrm{dR}}(\overline{R})$ and $\mathrm{Fil}^k \mathcal{O}B_{\mathrm{dR}}(\overline{R})$ are contained in $\mathrm{Fil}^k B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ and $\mathrm{Fil}^k \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, under the composition of horizontal arrows. Similarly, from the explicit description of connection on $\mathcal{O}B_{\mathrm{dR}}$ in [Bri08, §5.3], it easily follows that the composition of bottom horizontal arrows in (2.6) is further compatible with respective connections, for each $\mathfrak{p} \in \mathcal{S}$. Note that the injectivity of the top left horizontal arrow in (2.6) will follow from the injectvity of the the lower horizontal arrow, which we show next (our argument will be similar to [Bri08, Proposition 6.2.6]). Note that the filtration on $\mathcal{O}B_{\mathrm{dR}}(\overline{R})$ and $\mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, for each $\mathfrak{p} \in \mathcal{S}$, is separated. Therefore, it is enough to show that the induced map on grading of the filtration is injective. From [Bri08, Proposition 5.2.7] recall that $\mathrm{gr}^\bullet \mathcal{O}B_{\mathrm{dR}}(\overline{R}) \xrightarrow{\sim} \mathbb{C}^+(\overline{R})[z_1, \ldots, z_d, t^{\pm 1}]$, where z_i denotes the image of $(X_i - [X_i^\flat])/t$ in $\mathrm{gr}^0 \mathcal{O}B_{\mathrm{dR}}(\overline{R}) \xrightarrow{\sim} \mathbb{C}^+(\overline{R})[z_1, \ldots, z_d]$. Similarly, we have $\mathrm{gr}^\bullet \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \xrightarrow{\sim} \mathbb{C}^+(\mathfrak{p})[z_1, \ldots, z_d, t^{\pm 1}]$, for each $\mathfrak{p} \in \mathcal{S}$. The claim now follows from injectivity of the natural map $\mathbb{C}^+(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathbb{C}^+(\mathfrak{p})$ (see (2.1)). This concludes our proof. ■

Remark 2.12. The G_R-equivariant embedding $\mathcal{O}B_{\mathrm{dR}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ of Lemma 2.11 admits a natural L-linear and G_R-equivariant extension to an embedding $L \otimes_{R[1/p]} \mathcal{O}B_{\mathrm{dR}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$. Indeed, this follows from an argument similar to Lemma 2.11 or directly from [Bri08, Proposition 6.2.6]. Furthermore, from Lemma 2.11, it also follows that for each $\mathfrak{p} \in \mathcal{S}$, the induced natural map $L \otimes_{R[1/p]} \mathcal{O}B_{\mathrm{dR}}(\overline{R}) \to \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ is compatible with respective filtrations and connections, where the left-hand term is equipped with filtration on $\mathcal{O}B_{\mathrm{dR}}(\overline{R})$ and tensor product connection.

2.4. Crystalline period rings.

In this subsection we will recall crystalline period rings (see [Abh21, §2.2]). We set $A_{\mathrm{cris}}(R_\infty) := A_{\mathrm{inf}}(R_\infty)\langle \xi^k/k!, k \in \mathbb{N}\rangle$ and we have $t = \log(1+\mu) \in A_{\mathrm{cris}}(O_{F_\infty})$ and $A_{\mathrm{cris}}(R_\infty)$ is p-torsion free and t-torsion free. So, we set $B_{\mathrm{cris}}^+(R_\infty) := A_{\mathrm{cris}}(R_\infty)[1/p]$ and $B_{\mathrm{cris}}(R_\infty) :=$

$B_{\mathrm{cris}}^+(R_\infty)[1/t]$. Furthermore, one can define period rings $\mathcal{O}A_{\mathrm{cris}}(R_\infty)$, $\mathcal{O}B_{\mathrm{cris}}^+(R_\infty)$ and $\mathcal{O}B_{\mathrm{cris}}(R_\infty)$. These rings are equipped with a continuous action of Γ_R, a Frobenius endomorphism φ and a natural extension of the map θ. Rings with a subscript "cris" are equipped with a natural decreasing filtration and rings with a prefix "$\mathcal{O}$" are additionally equipped with an integrable connection satisfying Griffiths transversality with respect to the filtration. Moreover, we have G_R-equivariant and filtration compatible natural embeddings $B_{\mathrm{cris}}(R_\infty) \subset B_{\mathrm{dR}}(R_\infty)$ and $\mathcal{O}B_{\mathrm{cris}}(R_\infty) \subset \mathcal{O}B_{\mathrm{dR}}(R_\infty)$. One can define variations of these rings over $\overline{R}$ as well. From [MT20, Corollary 4.34] we have a (φ, Γ_R)-equivariant isomorphism $\mathcal{O}A_{\mathrm{cris}}(R_\infty) \xrightarrow{\sim} \mathcal{O}A_{\mathrm{cris}}(\overline{R})^{H_R}$.

As in §2.1, let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$. Similar to above, for each $\mathfrak{p} \in \mathcal{S}$, we have rings $A_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$, $B_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$, $\mathcal{O}A_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$ and $\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$ equipped with a $\widehat{G}_R(\mathfrak{p})$-action (resp. $A_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$, $B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$, $\mathcal{O}A_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ and $\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ equipped with a $G_R(\mathfrak{p})$-action), an appropriate extension of the map θ, a Frobenius endomorphism φ, a decreasing filtration and a connection (for rings with prefix "$\mathcal{O}$"). Then, we have the following:

Lemma 2.13. *The $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant embedding $A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to A_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$ of Lemma 2.4 extends to $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant and filtration compatible embeddings $B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to B_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$ and $\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$, where the latter is L-linear and also compatible with respective connections.*

Proof. By definition, the $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant embedding $A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to A_{\mathrm{inf}}(\mathbb{C}_{\mathfrak{p}}^+)$ naturally extends to $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant maps $A_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to A_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$ and $\mathcal{O}A_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}A_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+)$, where the latter is O_L-linear and compatible with respective connections. Now consider the following $\widehat{G}_R(\mathfrak{p})$-equivariant commutative diagram

$$\begin{array}{ccccc} A_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) & \longrightarrow & \mathcal{O}A_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) & \longrightarrow & \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})) \\ \downarrow & & \downarrow & & \downarrow \\ A_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+) & \longrightarrow & \mathcal{O}A_{\mathrm{cris}}(\mathbb{C}_{\mathfrak{p}}^+) & \longrightarrow & \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}_{\mathfrak{p}}^+), \end{array}$$

where all horizontal arrows are injective and compatible with respective filtrations and the right vertical arrow is injective and compatible with respective filtrations and connections. Therefore, it follows that the left and middle vertical arrows are injective and compatible with respective filtrations and connections. Finally, the claims for B_{cris} and $\mathcal{O}B_{\mathrm{cris}}$ follow by inverting t in the left and middle columns of the diagram. ∎

Remark 2.14. From Remark 2.10 it is easy to see that we have injective maps $\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$, where the first two maps are compatible with respective Frobenii. By an argument similar to [Bri08, Remarque 3.3.2] the products $\prod_{\mathfrak{p}\in\mathcal{S}} B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ and $\prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ are stable under the G_R-action on $\prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ (see Remark 2.10) and we equip them with the induced action. Then it follows that the injective maps $\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p}))$ are G_R-equivariant as well.

Lemma 2.15. *In the notations described above, we have an $R[1/p]$-linear (φ, G_R)-equivariant embedding $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$, where the right-hand term is equipped with a G_R-action as described in Remark 2.14. Moreover, for each $\mathfrak{p} \in \mathcal{S}$, the induced natural map $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$ is compatible with respective Frobenii, filtrations and connections.*

Proof. From Lemma 2.7 and Remark 2.14, note that we have (φ, G_R)-equivariant injective maps $A_{\mathrm{inf}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$. Then from the definition of $\mathcal{O}B_{\mathrm{cris}}$, the preceding maps naturally induce an $R[1/p]$-linear and (φ, G_R)-equivariant map $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$. The claim on injectivity of the latter map follows in a manner similar to [Bri08, Proposition 6.2.6]. Indeed, consider the following natural diagram

$$\begin{array}{ccc} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \\ \downarrow & & \downarrow \\ \mathcal{O}B_{\mathrm{dR}}(\overline{R}) & \longrightarrow & \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{dR}}(\mathbb{C}^+(\mathfrak{p})), \end{array}$$

where the left and right vertical arrows are natural inclusions and the bottom arrow is injective from Lemma 2.11. The diagram commutes since the top and bottom horizontal arrows are defined using the embedding $A_{\text{inf}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} A_{\text{inf}}(\mathbb{C}^+(\mathfrak{p}))$ of Lemma 2.7. In particular, it follows that the top horizontal arrow is injective, proving the first claim. Finally, for each $\mathfrak{p} \in \mathcal{S}$, the induced natural map $\mathcal{O}B_{\text{cris}}(\overline{R}) \to \mathcal{O}B_{\text{cris}}(\mathbb{C}^+(\mathfrak{p}))$ is tautologically compatible with respective Frobenii and the claims on filtrations and connections follow from the corresponding claims on $\mathcal{O}B_{\text{dR}}$ in Lemma 2.11. Hence, the lemma is proved. ∎

Remark 2.16. The (φ, G_R)-equivariant embedding $\mathcal{O}B_{\text{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\text{cris}}(\mathbb{C}^+(\mathfrak{p}))$ of Lemma 2.15 admits a natural L-linear and (φ, G_R)-equivariant extension to an embedding $L \otimes_{R[1/p]} \mathcal{O}B_{\text{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\text{cris}}(\mathbb{C}^+(\mathfrak{p}))$. Indeed, this follows from an argument similar to Lemma 2.15 or directly from [Bri08, Proposition 6.2.6]. Furthermore, from Lemma 2.15 it also follows that the induced natural map $L \otimes_{R[1/p]} \mathcal{O}B_{\text{cris}}(\overline{R}) \to \mathcal{O}B_{\text{cris}}(\mathbb{C}^+(\mathfrak{p}))$ is compatible with respective Frobenii, filtrations and connections, where the left-hand term is equipped with filtration on $\mathcal{O}B_{\text{cris}}(\overline{R})$ and tensor product Frobenius and connection.

2.5. Rings of (φ, Γ)-modules. Let us fix Teichmüller lifts $[X_i^\flat] \in A_{\text{inf}}(R_\infty)$, for $1 \leq i \leq d$, and let $A_\square^+$ denote the (p, μ)-adic completion of $O_F[\mu, [X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]$. By defininition, there exists a natural embedding $A_\square^+ \subset A_{\text{inf}}(R_\infty)$ and its image is stable under the Witt vector Frobenius endomorphism φ and the Γ_R-action on $A_{\text{inf}}(R_\infty)$ (see [Abh21, §3]); we equip $A_\square^+$ with induced structures. Furthermore, note that we have an embedding $\iota : R^\square \to A_\square^+$ defined by the map $X_i \mapsto [X_i^\flat]$ and it is easy to see that ι extends to an isomorphism of rings $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+$ (enough to check modulo μ since both source and target are μ-adically complete and μtorsion-free). We extend the Frobenius endomorphism on $R^\square$ to a Frobenius endomorphism φ on $R^\square[\![\mu]\!]$ by setting $\varphi(\mu) = (1+\mu)^p - 1$. Then the Frobenius on $R^\square[\![\mu]\!]$ is finite and faithfully flat of degree p^{d+1} . Moreover, by the preceding discussion, it also follows that the embedding ι and the isomorphism $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+$ are Frobenius-equivariant.

Let A_R^+ denote the (p, μ)-adic completion of the unique extension of the embedding $A_\square^+ \to A_{\text{inf}}(R_\infty)$ along the p-adically completed étale map $R^\square \to R$ (see [Abh21, §3.3.2] and [CN17, Proposition 2.1]). Then there exists a natural embedding $A_R^+ \subset A_{\text{inf}}(R_\infty)$ and its image is stable under the Witt vector Frobenius and Γ_R-action on $A_{\text{inf}}(R_\infty)$; we equip A_R^+ with induced structures. Furthermore, the embedding $\iota : R^\square \to A_\square^+ \subset A_R^+$ and the isomorphism $R^\square[\![\mu]\!] \xrightarrow{\sim} A_\square^+ \subset A_R^+$ naturally extend to a unique embedding $\iota : R \to A_R^+$ and an isomorphism of rings $R[\![\mu]\!] \xrightarrow{\sim} A_R^+$. We extend the Frobenius endomorphism on R to a Frobenius endomorphism φ on $R[\![\mu]\!]$ by setting $\varphi(\mu) = (1+\mu)^p - 1$. Then the Froebnius on $R[\![\mu]\!]$ is finite and faithfully flat of degree p^{d+1} . Moreover, by the preceding disucssion, it is easy to see that the embedding ι and the isomorphism $R[\![\mu]\!] \xrightarrow{\sim} A_R^+$ are Frobenius-equivariant. In particular, the induced Frobenius endomorphism φ on A_R^+ is finite and faithfully flat of degree p^{d+1} and we have $\varphi^*(A_R^+) := A_R^+ \otimes_{\varphi, A_R^+} A_R^+ \xrightarrow{\sim} \oplus_\alpha \varphi(A_R^+) u_\alpha$, where $u_\alpha := (1+\mu)^{\alpha_0}[X_1^\flat]^{\alpha_1} \cdots [X_d^\flat]^{\alpha_d}$ for $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$.

Set $A_R := A_R^+[1/\mu]^\wedge$ as the p-adic completion and note that the Frobenius endomorphism φ and the continuous action of Γ_R on A_R^+ naturally extend to A_R. Similar to above, the induced Frobenius endomorphism φ on A_R is finite and faithfully flat of degree p^{d+1} and $\varphi^*(A_R) := A_R \otimes_{\varphi, A_R} A_R \xrightarrow{\sim} \oplus_\alpha \varphi(A_R) u_\alpha = (\oplus_\alpha \varphi(A_R^+) u_\alpha) \otimes_{\varphi(A_R^+)} \varphi(A_R) \xleftarrow{\sim} A_R^+ \otimes_{\varphi, A_R^+} A_R$.

Recall that $\mathbb{C}(\overline{R}) = \mathbb{C}^+(\overline{R})[1/p]$ and we set $\tilde{A} := W(\mathbb{C}(\overline{R})^\flat)$ and $\tilde{B} := \tilde{A}[1/p]$, equipped with the Frobenius on Witt vectors and a continuous (for the weak topology) action of G_R. Moreover, the natural Frobenius and Γ_R-equivariant embedding $A_R^+ \subset A_{\text{inf}}(R_\infty)$ extends to a Frobenius and Γ_R-equivariant embedding $A_R \subset \tilde{A}^{H_R}$ and we set $B_R := A_R[1/p]$ equipped with induced Frobenius and Γ_R-action. Take A to be the p-adic completion of the maximal unramified extension of A_R inside $\tilde{A}$ and set $B := A[1/p] \subset \tilde{B}$. The rings A and B are stable under the action of G_R and Frobenius endomorphism on $\tilde{B}$ and we equip A and B with induced structures. Moreover, we have $A_R = A^{H_R}$ and $B_R = B^{H_R}$. Next, let us set $A^+ := A_{\text{inf}}(\overline{R}) \cap A \subset \tilde{A}$ and $B^+ := A^+[1/p] \subset B$ and note that these rings are stable under the Frobenius and G_R-action on B. Furthemore, we have $A_R^+ = (A^+)^{H_R}$ and $B_R^+ = (B^+)^{H_R}$.

Also note that by identifying the groups $\Gamma_L \xrightarrow{\sim} \Gamma_R$, we have a (φ, Γ_L)-equivariant isomorphism $A_L^+ \xrightarrow{\sim} ((A_R^+)_{(p,\mu)})^\wedge$, where $^\wedge$ denotes the (p, μ)-adic completion. The preceding isomorphism extends to an isomorphism $A_L \xrightarrow{\sim} ((A_R)_{(p)})^\wedge$, where $^\wedge$ denotes the p-adic completion. It is easy to see that we have

$A_R^+ = A_L^+ \cap A_R$ as subrings of A_L and $B_R^+ := A_R^+[1/p] = B_L^+ \cap B_R$ as subrings of B_L.

2.6. p-adic representations. Let T be a finite free $\mathbb{Z}_p$-representation of G_R. By the theory of étale (φ, Γ)-modules (see [Fon90] and [And06]), one can functorially associate to T a finite projective étale (φ, Γ_R)-module $\mathbf{D}_R(T) := (A \otimes_{\mathbb{Z}_p} T)^{H_R}$ over A_R of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$. Moreover, $\tilde{D}_R(T) := (\tilde{A} \otimes_{\mathbb{Z}_p} T)^{H_R} \xrightarrow{\sim} \tilde{A}^{H_R} \otimes_{A_R} \mathbf{D}_R(T)$ and we have a natural (φ, Γ_R)-equivariant isomorphism

$$A \otimes_{A_R} \mathbf{D}_R(T) \xrightarrow{\sim} A \otimes_{\mathbb{Z}_p} T. \tag{2.7}$$

These constructions are functorial in $\mathbb{Z}_p$-representations and induce an exact equivalence of $\otimes$-categories (see [And06, Theorem 7.11])

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{\text{ét}}, \tag{2.8}$$

with an exact $\otimes$-compatible quasi-inverse given as $\mathbf{T}_R(D) := (A \otimes_{A_R} D)^{\varphi=1} = (\tilde{A} \otimes_{A_R} D)^{\varphi=1}$. Similar statements are also true for p-adic representations of G_R. Furthermore, let $\mathbf{D}_R^+(T) := (A^+ \otimes_{\mathbb{Z}_p} T)^{H_R}$ be the (φ, Γ_R)-module over A_R^+ associated to T and for $V := T[1/p]$ let $\mathbf{D}_R^+(V) := \mathbf{D}_R^+(T)[1/p]$ be the (φ, Γ_R)-module over B_R^+ associated to V.

Let V be a p-adic representation of G_R. From p-adic Hodge theory of G_R (see [Bri08]), one can attach to V a filtered (φ, ∂)-module over $R[1/p]$ of rank $\leq \dim_{\mathbb{Q}_p} V$ given by the functor

$$\begin{aligned} \mathcal{O}\mathbf{D}_{\mathrm{cris},R} : \mathrm{Rep}_{\mathbb{Q}_p}(G_R) &\longrightarrow \mathrm{MF}_R(\varphi, \partial) \\ V &\longmapsto (\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V)^{G_R}. \end{aligned}$$

The representation V is said to be crystalline if the natural map $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \to \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ is an isomorphism, in particular, if V is crystalline then $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$. Restricting $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}$ to the category of crystalline representations of G_R and writing $\mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial)$ for the essential image of restricted functor, we have an exact equivalence of $\otimes$-categories (see [Bri08, Théorème 8.5.1])

$$\mathcal{O}\mathbf{D}_{\mathrm{cris},R} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R) \xrightarrow{\sim} \mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial), \tag{2.9}$$

with an exact $\otimes$-compatible quasi-inverse given as $\mathcal{O}\mathbf{V}_{\mathrm{cris},R}(D) := (\mathrm{Fil}^0(\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} D))^{\partial=0, \varphi=1}$. Furthermore, we have a continuous homomorphism $G_L \to G_R$, i.e. V is also a p-adic representation of G_L. Base changing the isomorphism $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$ along $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \to \mathcal{O}B_{\mathrm{cris}}(O_{\overline{L}})$, we obtain a G_L-equivariant isomorphism $\mathcal{O}B_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}B_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$, i.e. V is a crystalline representation of G_L. Taking G_L-invariants in the preceding isomorphism we further obtain a natural isomorphism $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ compatible with respective Frobenii, filtrations and connections.

3. Relative Wach modules

In this section we will describe relative Wach modules and finite $[p]_q$-height representations of G_R and relate them to crystalline representations. We start by noting some technical lemmas.

3.1. Some technical results. In $A_{\mathrm{inf}}(O_{F_\infty})$, let us fix $q := [\varepsilon]$, $\mu := [\varepsilon] - 1 = q - 1$ and $[p]_q := \varphi(\mu)/\mu$.

Definition 3.1. Let N be a finitely generated A_R^+-module. The sequence $\{p, \mu\}$ in A_R^+ is said to be *N-regular* if N is p-torsion free and N/pN is μ-torsion free. Similarly, $\{\mu, p\}$ is N-regular if N is μ-torsion free and $N/\mu N$ is p-torsion free. The sequence $\{p, \mu\}$ in A_R^+ is said to be *strictly N-regular* if both $\{p, \mu\}$ and $\{\mu, p\}$ are N-regular.

Remark 3.2. In Definition 3.1 note that the sequence $\{p, \mu\}$ is strictly N-regular if and only if N is a-torsion free for every nonzero element a in the ideal $(p, \mu) \subset A_R^+$ and $N/\mu N$ is p-torsion free. Indeed, the "only if" direction is obvious and for the converse one needs to check that N/pN is μ-torsion free. So let $x \in N$ such that $\mu x = py$ for some $y \in N$; we claim that $x \in pN$. Reducing the preceding equality modulo p and using $(N/\mu N)[p] = 0$, we get that $y = \mu z$ for some $z \in \mathbb{N}$. From μ-torsion freeness of N, it follows that $x = pz$, as claimed.

Lemma 3.3. *Let N be a finitely generated A_R^+-module and consider the complex,*

$$\mathscr{C}^\bullet : N \xrightarrow{(p,\mu)} N \oplus N \xrightarrow{(\mu,-p)} N,$$

where the first map is given by $x \mapsto (px, \mu x)$ and the second map is given by $(x, y) \mapsto \mu x - py$. Then the sequence $\{p, \mu\}$ is strictly N-regular if and only if $H^1(\mathscr{C}^\bullet) = 0$. Moreover, under these equivalent conditions $H^0(\mathscr{C}^\bullet) = 0$.

Proof. If $\{p, \mu\}$ is strictly N-regular then $(N/p)[\mu] = (N/\mu)[p] = 0$. Therefore, we must have $H^0(\mathscr{C}^\bullet) = H^1(\mathscr{C}^\bullet) = 0$. For the converse, consider the following diagram

$$\begin{array}{ccccccccccc}
 & & N[p,\mu] & \longrightarrow & N[p] & \xrightarrow{\mu} & N[p] & \longrightarrow & (N/\mu)[p] & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N[\mu] & \longrightarrow & N & \xrightarrow{\mu} & N & \longrightarrow & N/\mu & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle p} & & \\
0 & \longrightarrow & N[\mu] & \longrightarrow & N & \xrightarrow{\mu} & N & \longrightarrow & N/\mu & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & (N/p)[\mu] & \longrightarrow & N/p & \xrightarrow{\mu} & N/p & \longrightarrow & N/(p,\mu). & &
\end{array} \tag{3.1}$$

Since $H^1(\mathscr{C}^\bullet) = 0$, we get that the top right and bottom left corners of (3.1) are zero, i.e. $(N/\mu)[p] = (N/p)[\mu] = 0$. Now let $x \in N[\mu]$, then from the surjectivity of the leftmost vertical arrow from second to third row it follows that there exists $x_1 \in N[\mu]$ such that $x = px_1$. Proceeding by induction it is easy to see that $x \in p^n N[\mu] \subset p^n N$ for all $n \in \mathbb{N}$. But since N is finitely generated over A_R^+, which is (p, μ)-adically complete, it follows that N is p-adically separated, i.e. $x = 0$, in particular, $N[\mu] = 0$. A similar argument shows that $N[p] = 0$, in particular, $N[p, \mu] = 0$. This proves both claims in the lemma. ∎

Remark 3.4. The complex $\mathscr{C}^\bullet$ in Lemma 3.3 computes local cohomology of N with respect to the ideal $(p, \mu) \subset A_R^+$ (see [Wei94, Theorem 4.6.8]). So, if we set $Z := V(p, \mu) \subset \mathrm{Spec}\,(A_R^+) =: X$ as a closed subset, then one also says that $\mathscr{C}^\bullet$ computes $H_Z^i(X, N)$, i.e. cohomology of X with compact support along Z (see [Wei94, Generalization 4.6.2]).

Lemma 3.5. *Let N be a finitely generated A_R^+-module such that $\{p, \mu\}$ is strictly N-regular. Then we have $N = N[1/p] \cap N[1/\mu] \subset N[1/p, 1/\mu]$ as A_R^+-modules. Moreover, $N = N[1/p] \cap N[1/\mu]^\wedge \subset N[1/\mu]^\wedge[1/p]$, where $^\wedge$ denotes the p-adic completion.*

Proof. Note that from definitions we have $(N/p)[\mu] = (N/\mu)[p] = 0$ and $(N[1/\mu])/p = (N/p)[1/\mu]$. So it follows that $N/p^n N \subset (N/p^n)[1/\mu]$, for all $n \in \mathbb{N}$, and therefore, $N \cap p^n N[1/\mu] = p^n N$. Hence, $N[1/p] \cap N[1/\mu] = N$. Furthermore, since $(N[1/\mu]^\wedge)/p^n = (N[1/\mu])/p^n = (N/p^n)[1/\mu]$, therefore, similar to above we also get that $N \cap p^n N[1/\mu]^\wedge = p^n N$, for all $n \in N$. Hence, $N[1/p] \cap N[1/\mu]^\wedge = N$. ∎

Lemma 3.6. *Let N be a finitely generated A_R^+-module. Then the sequence $\{p, \mu\}$ is strictly N-regular if and only if the sequence $\{p, [p]_q\}$ is strictly N-regular.*

Proof. Let us first assume that the sequence $\{p, \mu\}$ is strictly N-regular. Note that we have $[p]_q = \mu^{p-1} \bmod pA_R^+$, therefore, it follows that N/p is $[p]_q$-torsion free, in particular, the sequence $\{p, [p]_q\}$ is regular on N. Moreover, as $[p]_q$ is an element of the ideal $(p, \mu) \subset A_R^+$, from Remark 3.2 we have that N is $[p]_q$-torsion free. Now considering a diagram similar to (3.1) with μ replaced by $[p]_q$ and using that N is p-torsion free and N/p is $[p]_q$-torsion free, it follows that $N/[p]_q$ is p-torsion free, i.e. the sequence $\{p, [p]_q\}$ is strictly N-regular. Conversely, assume that the sequence $\{p, [p]_q\}$ is strictly N-regular. Then, again as we have $[p]_q = \mu^{p-1} \bmod pA_R^+$, so from [Sta23, Tag 07DV], it follows that $\{p, \mu\}$ is a regular sequence on N. Next, let us note that μ^{p-1} is an element of the ideal $(p, [p]_q) \subset A_R^+$, so it follows that N is μ^{p-1}-torsion free, therefore, μ-torsion free. Now considering the diagram (3.1) and using that N is p-torsion free and N/p is μ-torsion free, it follows that N/μ is p-torsion free, i.e. the sequence $\{p, \mu\}$ is strictly N-regular. Hence, the lemma is proved. ∎

Finally, let us note an important observation for the action of Γ_R on A_R^+-modules. Note that the action of Γ_R is continuous on A_R^+ for the (p, μ)-adic topology and the induced action of Γ_R on $A_R^+/\mu \xrightarrow{\sim} R$ is trivial. More generally, we claim the following:

Lemma 3.7. *Let N be a finitely generated A_R^+-module equipped with a semilinear action of Γ_R such that the induced action of Γ_R on $N/\mu N$ is trivial. Then the action of Γ_R on N is continuous for the (p, μ)-adic topology.*

Proof. Recall that from §2 we have $\Gamma_R \xrightarrow{\sim} \Gamma'_R \rtimes \Gamma_F \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$. Moreover, we fixed $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of Γ'_R and γ_0 in Γ_R to be a lift of a topological generator of Γ_R/Γ'_R. Additionally, we may assume that $\chi(\gamma_0) = 1+pa$, for $p \geq 3$, and $\chi(\gamma_0) = 1+4a$, for $p = 2$, where χ is the p-adic cyclotomic character and a is a unit in $\mathbb{Z}_p$. To show that the action of Γ_R is continuous on N, for the (p, μ)-adic topology, we need to show that for any x in N, any $n \geq 1$ and for each γ_i, there exists an $m \in \mathbb{N}$ such that $\gamma_i^{p^m}(x) = x \bmod (p, \mu)^n$. As the action of Γ_R is trivial on $N/\mu N$, let us note that for each $0 \leq i \leq d$, the operators $\nabla_{q,i} := \frac{\gamma_i - 1}{\mu} : N \to N$ are well-defined (see §5.2 for more on such operators). Moreover, note that for any a in A_R^+, x in N and $0 \leq i \leq d$, we have $(\gamma_i - 1)(a \otimes x) = (\gamma_i - 1)a \otimes x + \gamma_i(a) \otimes (\gamma_i - 1)(x)$, and therefore, $\nabla_{q,i}(a \otimes x) = \nabla_{q,i}(a) \otimes x + \gamma_i(a) \otimes \nabla_{q,i}(x)$. Now for $1 \leq i \leq d$, note that $\nabla_{q,i}(\mu) = \mu$, so by setting $m = n$, we get that

$$\gamma_i^{p^n}(x) = (1 + \mu\nabla_{q,i}(x))^{p^n} = x + \textstyle\sum_{k=1}^{p^n} \binom{p^n}{k} \mu^k \nabla_{q,i}^k(x),$$

where the summation in the third term is easily seen to be an element of $(p, \mu)^n N$. Next, let $i = 0$ and using the action of γ_0 on μ, it is easy to see that $\nabla_{q,0}(\mu) = (1 + \mu)((1 + \mu)^{pa} - 1)/\mu$ is an element of $(p, \mu)A_F^+$. Then an easy induction on $k \geq 1$ shows that for any x in N, we must have that $(\mu\nabla_{q,0})^k(x)$ is an element of $(p, \mu)^k N$. In particular, by setting $m = n$, it follows that we have

$$\gamma_0^{p^n}(x) = (1 + \mu\nabla_{q,0}(x))^{p^n} = x + \textstyle\sum_{k=1}^{p^n} \binom{p^n}{k} (\mu\nabla_{q,0})^k(x),$$

where the summation in the third term is again an element of $(p, \mu)^n N$. Hence, we conclude that the action of Γ_R is continuous on N. ■

3.2. Wach modules over A_R^+.

We start with the definition of Wach modules.

Definition 3.8. A *Wach module* over A_R^+ with weights in the interval $[a, b]$, for some $a, b \in \mathbb{Z}$ with $b \geq a$, is a finitely generated A_R^+-module N satisfying the following assumptions:

(1) The sequences $\{p, \mu\}$ and $\{\mu, p\}$ are regular on N.

(2) N is equipped with a semilinear action of Γ_R such that the induced action of Γ_R on $N/\mu N$ is trivial.

(3) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ compatible with respective actions of Γ_R such that $\varphi(\mu^b N) \subset \mu^b N$, and the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is injective and its cokernel is killed by $[p]_q^{b-a}$.

We define the $[p]_q$-*height* of N to be the largest value of $-a$ for $a \in \mathbb{Z}$ as above. The module N is said to be *effective* if we can take $b = 0$ and $a \leq 0$. A Wach module over B_R^+ is a finitely generated module M equipped with a semilinear action of Γ_R and a Frobenius-semilinear operator $\varphi : M[1/\mu] \to M[1/\varphi(\mu)]$ compatible with respective actions of Γ_R and such that there exists a Γ_R-stable and φ-stable (after inverting μ) A_R^+-submodule $N \subset M$ and equipped with induced (φ, Γ_R)-action N is a Wach module over A_R^+ and $N[1/p] = M$. Denote by $(\varphi, \Gamma_R)\text{-Mod}_{A_R^+}^{[p]_q}$, the category of Wach modules over A_R^+ with morphisms between objects being A_R^+-linear φ-equivariant (after inverting μ) and Γ_R-equivariant morphisms.

Remark 3.9. In Definition 3.8, note that from the triviality of the action of Γ_R on $N/\mu N$ and Lemma 3.7, it follows that the action of Γ_R on N is continuous.

Next, we note some structural properties of Wach modules.

Lemma 3.10. *Let N be a finitely generated A_R^+-module. Then (3) of Definition 3.8 is equivalent to giving an A_R^+-linear and Γ_R-equivariant isomorphism $\varphi_N : (\varphi^* N)[1/[p]_q] = (A_R^+ \otimes_{\varphi, A_R^+} N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$.*

Proof. Suppose N satisfes condition (3) of Definition 3.8. Then, the map $1 \otimes \varphi : \varphi^*(\mu^b N) \to \mu^b N$ induces an isomorphism $1 \otimes \varphi : (\mu^b \varphi^* N)[1/[p]_q] \xrightarrow{\sim} (\mu^b N)[1/[p]_q]$. Hence, we obtain an isomorphism

$$\varphi_N : (\varphi^* N)[1/[p]_q] \xrightarrow[\sim]{\mu^b} (\mu^b \varphi^* N)[1/[p]_q] \xrightarrow[\sim]{1\otimes\varphi} (\mu^b N)[1/[p]_q] \xleftarrow[\sim]{\mu^b} N[1/[p]_q].$$

Since, $1 \otimes \varphi$ commutes with the action of Γ_R, we deduce that φ_N is Γ_R-equivariant.

Conversely, suppose that we have an A_R^+-linear Γ_R-equivariant isomorphism $\varphi_N : (\varphi^* N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Then note that for some $a, b \in \mathbb{Z}$ with $b \geq a$ we can write $[p]_q^b \varphi_N(\varphi^* N) \subset N \subset [p]_q^a \varphi_N(\varphi^* N)$. So we get an A_R^+-semilinear and Γ_R-equivariant map as the composition $\varphi : \mu^b N \xrightarrow{\text{can}} \varphi^*(\mu^b N) \xrightarrow{\varphi_N} \mu^b N$. This extends to an A_R^+-semilinear Γ_R-equivariant map $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ and we have

$$\varphi_N(\varphi^*(\mu^b N)) = \mu^b [p]_q^b \varphi_N(\varphi^* N) \subset \mu^b N \subset [p]_q^{a-b} \varphi_N(\varphi^*(\mu^b N))$$

Then it easily follows that $1 \otimes \varphi = \varphi_N : \varphi^*(\mu^b N) \to \mu^b N$ is injective, its cokernel is killed by $[p]_q^{b-a}$ and it commutes with the action of Γ_R. Hence, N satisfies condition (3) of Definition 3.8. ∎

Proposition 3.11. *Let N be a Wach module over A_R^+. Then $N[1/p]$ is finite projective over $A_R^+[1/p]$ and $N[1/\mu]$ is finite projective over $A_R^+[1/\mu]$.*

Proof. For $r \in \mathbb{N}$ large enough, note that the Wach module $\mu^r N(-r)$ is always effective. So without loss of generality, we may assume that N is effective. Then the first claim follows from Lemma 3.10 and Proposition A.1. For the second claim, note that N is p-torsion free, so $A_R \otimes_{A_R^+} N$ is a p-torsion free étale (φ, Γ_R)-module over A_R, and therefore, finite projective by [And06, Lemma 7.10]. Since $A_R^+[1/\mu]$ is noetherian, we have $N[1/\mu]^\wedge \xrightarrow{\sim} A_R \otimes_{A_R^+[1/\mu]} N[1/\mu] = A_R \otimes_{A_R^+} N$, where $^\wedge$ denotes the p-adic completion. Moreover, the natural map $\operatorname{Spec}(A_R^+[1/\mu]^\wedge) \cup \operatorname{Spec}(A_R^+[1/\mu, 1/p]) \to \operatorname{Spec}(A_R^+[1/\mu])$ is a flat cover. Therefore, by faithfully flat descent it follows that $N[1/\mu]$ is finite projective over $A_R^+[1/\mu]$. ∎

Remark 3.12. Note that the map $\operatorname{Spec}(A_R^+[1/[p]_q]_p^\wedge) \cup \operatorname{Spec}(A_R^+[1/[p]_q, 1/p]) \to \operatorname{Spec}(A_R^+[1/[p]_q])$ is a flat cover and $A_R^+[1/\mu]_p^\wedge = A_R^+[1/[p]_q]_p^\wedge$. Now for a Wach module N over A_R^+, we have that the $A_R^+[1/p]$-module $N[1/p]$ is finite projective and the $A_R^+[1/\mu]$-module $N[1/\mu]$ is finite projective (see Proposition 3.11). Therefore, by faithfully flat descent, we get that the $A_R^+[1/[p]_q]$-module $N[1/[p]_q]$ is finite projective. Moreover, from Lemma 3.6 we also have that the sequence $\{p, [p]_q\}$ is strictly N-regular and equivalent to condition (1) in Definition 3.8.

Remark 3.13. Note that for a Wach module N over A_R^+, we have that N is p-torsion free, in particular, N is contained in $N[1/p]$. As $N[1/p]$ is finite projective over $A_R^+[1/p]$ by Proposition 3.11, therefore, we obtain that N is a torsion free A_R^+-module.

Lemma 3.14. *Let N be a Wach module over A_R^+, then we have $N = (A_L^+ \otimes_{A_R^+} N) \cap (A_R \otimes_{A_R^+} N) \subset A_L \otimes_{A_R^+} N$ as A_R^+-modules.*

Proof. Let $N_R := N$, $N_L := A_L^+ \otimes_{A_R^+} N$ and $D_R := A_R \otimes_{A_R^+} N$. Note that $N_R[1/p]$ is finite projective over B_R^+, with $N_L[1/p] = B_L^+ \otimes_{B_R^+} N_R[1/p]$ and $D_R[1/p] = B_R \otimes_{B_R^+} N_R[1/p]$, therefore $N_L[1/p] \cap D_R[1/p] = (B_L^+ \cap B_R) \otimes_{B_R^+} N_R[1/p] = N_R[1/p]$. Moreover, we have $N_L \cap D_R \subset N_L[1/p] \cap D_R[1/p] = N_R[1/p]$, and using Lemma 3.5 we see that $N_L \cap D_R = N_L \cap D_R \cap N_R[1/p] = N_R$. ∎

From the proof of Proposition 3.11, it is clear that extending scalars along $A_R^+ \to A_R$ induces a functor $(\varphi, \Gamma_R)\text{-Mod}_{A_R^+}^{[p]_q} \to (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{\text{ét}}$, and we make the following claim:

Proposition 3.15. *The following natural functor is fully faithful*

$$\begin{aligned}(\varphi, \Gamma_R)\text{-Mod}_{A_R^+}^{[p]_q} &\longrightarrow (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{\text{ét}}\\ N &\longmapsto A_R \otimes_{A_R^+} N.\end{aligned}$$

Proof. Let N, N' be two Wach modules over A_R^+. Write $N_R := N$, $N_L := A_L^+ \otimes_{A_R^+} N$, $D_L := A_R \otimes_{A_R^+} N$ and similarly for N'. We need to show that for Wach modules N_R and N'_R, we have

$$\mathrm{Hom}_{(\varphi,\Gamma_R)\text{-}\mathrm{Mod}_{A_R^+}^{[p]_q}}(N_R, N'_R) \xrightarrow{\sim} \mathrm{Hom}_{(\varphi,\Gamma_R)\text{-}\mathrm{Mod}_{A_R}^{\text{ét}}}(D_R, D'_R) \tag{3.2}$$

Note that $A_R^+ \to A_R = A_R^+[1/\mu]^\wedge$ is injective, in particular, the map in (3.2) is injective. To check that (3.2) is surjective, take an A_R-linear and (φ, Γ_R)-equivariant map $f : D_R \to D'_R$. We need to show that $f(N_R) \subset N'_R$. Base changing f along $A_R \to A_L$ and using the isomorphism $\Gamma_L \xrightarrow{\sim} \Gamma_R$ induces an A_L-linear and (φ, Γ_L)-equivariant map $f : D_L \to D'_L$. Then from [Abh23a, Proposition 3.3] we have $f(N_L) \subset N'_L$. Finally, using Lemma 3.14, we get that inside D'_L we have $f(N_R) = f(N_L \cap D_R) = f(N_L) \cap f(D_R) \subset N'_L \cap D'_R = N'_R$, concluding the proof. ∎

Analogous to above, one can define categories $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{B_R^+}^{[p]_q}$ and $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{B_R}^{\text{ét}}$ and a functor from the former to latter by extending scalars along $B_R^+ \to B_R$. Then passing to the associated isogeny categories and using Proposition 3.15, we get the following:

Corollary 3.16. *The natural functor* $(\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{B_R^+}^{[p]_q} \to (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{B_R}^{\text{ét}}$ *is fully faithful.*

3.3. G_R-representations attached to Wach modules.

Composing the functor in Proposition 3.15 with the equivalence in (2.8), we obtain a fully faithful functor,

$$\begin{aligned}\mathbf{T}_R : (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{A_R^+}^{[p]_q} &\longrightarrow \mathrm{Rep}_{\mathbb{Z}_p}(G_R)\\ N &\longmapsto (A \otimes_{A_R^+} N)^{\varphi=1} \xrightarrow{\sim} (W(\mathbb{C}(\overline{R})^\flat) \otimes_{A_R^+} N)^{\varphi=1}.\end{aligned} \tag{3.3}$$

Proposition 3.17. *Let N be a Wach module over A_R^+ and $T := \mathbf{T}_R(N)$, the associated finite free $\mathbb{Z}_p$-representation of G_R. Then we have a natural G_R-equivariant comparison isomorphism*

$$A_{\mathrm{inf}}(\overline{R})[1/\mu] \otimes_{A_R^+} N \xrightarrow{\sim} A_{\mathrm{inf}}(\overline{R})[1/\mu] \otimes_{\mathbb{Z}_p} T. \tag{3.4}$$

Additionally, (3.4) is compatible with Frobenius after base change along $A_{\mathrm{inf}}(\overline{R})[1/\mu] \to W(\mathbb{C}(\overline{R})^\flat)$.

Proof. Note that for $T = \mathbf{T}_R(N)$, from the equivalence in (2.8), we have $\mathbf{D}_R(T) \xrightarrow{\sim} A_R \otimes_{A_R^+} N$ as étale (φ, Γ_R)-modules over A_R. Then extending scalars of the isomorphism in (2.7) along $A \to W(\mathbb{C}(\overline{R})^\flat)$ gives (φ, G_R)-equivariant isomorphism,

$$W(\mathbb{C}(\overline{R})^\flat) \otimes_{A_R^+} N \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T. \tag{3.5}$$

Nowa, for $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have $\mathbf{T}_R(\mu^r N(-r)) = T(-r)$ (the twist $(-r)$ denotes the Tate twist on which Γ_R acts via the cyclotomic character). Therefore, we see that it is enough to show the claim for effective Wach modules (see Definition 3.8), in particular, in the rest of the proof we will assume that N is effective.

Let $\mathcal{S}$ denote the set of minimal primes of $\overline{R}$ above $pR \subset R$. From §2.1, recall that for each $\mathfrak{p} \in \mathcal{S}$, we have $\overline{L}(\mathfrak{p}) \subset \mathbb{C}_\mathfrak{p}$, an algebraic closure of L containing $(\overline{R})_\mathfrak{p}$, and we have $\widehat{G}_R(\mathfrak{p}) = \mathrm{Gal}(\overline{L}(\mathfrak{p})/L)$. Moreover, we have an isomorphism of groups $\Gamma_L \xrightarrow{\sim} \Gamma_R$ and for each prime $\mathfrak{p} \in \mathcal{S}$, let $A_L^+(\mathfrak{p})$ denote the base ring for Wach modules in the imperfect residue field case (see [Abh23a, §2.1.2]). To avoid confusion, let us write $N_R := N$ and $N_L(\mathfrak{p}) := A_L^+(\mathfrak{p}) \otimes_{A_R^+} N$, in particular, $N_L(\mathfrak{p})$ is a Wach module over $A_L^+(\mathfrak{p})$ finite free of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$. From [Abh23a, Lemma 3.6] note that we have $\widehat{G}_R(\mathfrak{p})$-equivariant inclusions for each $\mathfrak{p} \in \mathcal{S}$,

$$\mu^s A_{\mathrm{inf}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{\mathbb{Z}_p} T \subset A_{\mathrm{inf}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{A_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \subset A_{\mathrm{inf}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{\mathbb{Z}_p} T. \tag{3.6}$$

Now, note that the $(\varphi, G_R(\mathfrak{p}))$-equivariant composition $A_R^+ \to W(\mathbb{C}(\overline{R})^\flat) \to W(\mathbb{C}(\mathfrak{p})^\flat)$ naturally factors as the $(\varphi, G_R(\mathfrak{p}))$-equivariant maps $A_R^+ \to A_L^+(\mathfrak{p}) \to W(\mathbb{C}(\mathfrak{p})^\flat)$. So, by base changing the (φ, G_R)-equivariant isomorphism in (3.5) along the $(\varphi, G_R(\mathfrak{p}))$-equivariant map $W(\mathbb{C}(\overline{R})^\flat) \to W(\mathbb{C}(\mathfrak{p})^\flat)$, we obtain a natural $(\varphi, G_R(\mathfrak{p}))$-equivariant isomorphism,

$$W(\mathbb{C}(\mathfrak{p})^\flat) \otimes_{A_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \xrightarrow{\sim} W(\mathbb{C}(\mathfrak{p})^\flat) \otimes_{\mathbb{Z}_p} T. \tag{3.7}$$

All terms in (3.6) and (3.7) admit $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant embedding into $W(\mathbb{C}_\mathfrak{p}^\flat) \otimes_{A_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \xrightarrow{\sim} W(\mathbb{C}_\mathfrak{p}^\flat) \otimes_{\mathbb{Z}_p} T$, where the action of $\widehat{G}_R(\mathfrak{p})$ on (3.7) factors through $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p})$. Therefore, taking the intersection of (3.6) with (3.7) inside $W(\mathbb{C}_\mathfrak{p}^\flat) \otimes_{A_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \xrightarrow{\sim} W(\mathbb{C}_\mathfrak{p}^\flat) \otimes_{\mathbb{Z}_p} T$ and using Lemma 2.4, for each $\mathfrak{p} \in \mathcal{S}$, we obtain the following $(\varphi, G_R(\mathfrak{p}))$-equivariant inclusions:

$$\mu^s A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T \subset A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{A_L^+(\mathfrak{p})} N_L(\mathfrak{p}) \subset A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T, \tag{3.8}$$

where the middle term can be written as $A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{A_L^+(\mathfrak{p})} N_L(\mathfrak{p}) = A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{A_R^+} N_R$.

Now, from Remark 2.6, recall that $\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$ is equipped with an action of G_R and from Lemma 2.7 we have a (φ, G_R)-equivariant embedding $A_{\mathrm{inf}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))$. Then, we can equip $\prod_{\mathfrak{p}\in\mathcal{S}}(A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T) = (\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))) \otimes_{\mathbb{Z}_p} T$ with the diagonal action of (φ, G_R) and similarly for $\prod_{\mathfrak{p}\in\mathcal{S}}(A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{A_L^+(\mathfrak{p})} N_L(\mathfrak{p})) = \prod_{\mathfrak{p}\in\mathcal{S}}(A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{A_R^+} N_R) = (\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))) \otimes_{A_R^+} N_R$, where the second equality follows from the fact that product is an exact functor on the category of A_R^+-modules and N_R is finitely presented over the noetherian ring A_R^+ (see [Sta23, Tag 059K]). So, taking the product of (3.8) over all $\mathfrak{p} \in \mathcal{S}$, we obtain the following (φ, G_R)-equivariant inclusions:

$$\mu^s \prod_{\mathfrak{p}\in\mathcal{S}} (A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T) \subset \prod_{\mathfrak{p}\in\mathcal{S}} (A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{A_R^+} N_R) \subset \prod_{\mathfrak{p}\in\mathcal{S}} (A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T). \tag{3.9}$$

Inverting μ in (3.9) and from the discussion above we get a G_R-equivariant isomorphism

$$\Big(\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{A_R^+[1/\mu]} N_R[\tfrac{1}{\mu}] \xrightarrow{\sim} \Big(\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T. \tag{3.10}$$

Furthermore, the (φ, G_R)-equivariant isomorphism in (3.5) can be written as

$$W(\mathbb{C}(\overline{R})^\flat) \otimes_{A_R^+[1/\mu]} N_R[\tfrac{1}{\mu}] \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T. \tag{3.11}$$

Using Lemma 2.7, all terms in (3.10) and (3.11) admit an embedding into $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{A_R^+} N_R \xrightarrow{\sim} \big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbb{Z}_p} T$ compatible with respective actions of φ and G_R. Note that $N_R[1/\mu]$ is finite projective over $A_R^+[1/\mu]$ (see Proposition 3.11), so the intersection of the left-hand terms in (3.10) and (3.11), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{A_R^+} N_R$, gives

$$\begin{aligned}\Big(W(\mathbb{C}(\overline{R})^\flat) \otimes_{A_R^+[1/\mu]} N_R[\tfrac{1}{\mu}]\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{A_R^+[1/\mu]} N_R[\tfrac{1}{\mu}]\Big) \\ = A_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{A_R^+[1/\mu]} N_R[\tfrac{1}{\mu}] = A_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{A_R^+} N_R,\end{aligned}$$

where the first equality follows from Lemma 2.7. Similarly, the intersection of the right-hand terms in (3.10) and (3.11), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big) \otimes_{\mathbb{Z}_p} T$, gives

$$\Big(W(\mathbb{C}(\overline{R})^\flat) \otimes_{\mathbb{Z}_p} T\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} A_{\mathrm{inf}}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T\Big) = A_{\mathrm{inf}}(\overline{R})[\tfrac{1}{\mu}] \otimes_{\mathbb{Z}_p} T,$$

where the equality again follows from Lemma 2.7. Since (3.10) and (3.11) are isomorphisms, we obtain the natural G_R-equivariant isomorphism claimed in (3.4) as,

$$A_{\mathrm{inf}}(\overline{R})[1/\mu] \otimes_{A_R^+} N_R \xrightarrow{\sim} A_{\mathrm{inf}}(\overline{R})[1/\mu] \otimes_{\mathbb{Z}_p} T.$$

From the proof, it also follows that the isomorphism above is compatible with Frobenius after base change along $A_{\mathrm{inf}}(\overline{R}) \to W(\mathbb{C}(\overline{R})^\flat)$. ∎

Corollary 3.18. *Let N be a Wach module over A_R^+ and let $T := \mathbf{T}_R(N)$ denote the associated finite free $\mathbb{Z}_p$-representation of G_R. Then we have a natural (φ, G_R)-equivariant comparison isomorphism*

$$A^+[1/\mu] \otimes_{A_R^+} N \xrightarrow{\sim} A^+[1/\mu] \otimes_{\mathbb{Z}_p} T.$$

Additionally, the isomorphism above is compatible with Frobenius after base change along $A^+[1/\mu] \to A$.

Proof. Since $N[1/\mu]$ is finite projective over $A_R^+[1/\mu]$, taking the intersection of the isomorphism in Proposition 3.17 with the isomorphism in (2.7), inside $\tilde{A} \otimes_{\mathbb{Z}_p} T$, we obtain a G_R-equivariant isomorphism $A^+[1/\mu] \otimes_{A_R^+[1/\mu]} N[1/\mu] \xrightarrow{\sim} A^+[1/\mu] \otimes_{\mathbb{Z}_p} T$, as claimed. Moreover, from §2.5, recall that $A^+ = A_{\inf}(\overline{R}) \cap A \subset \tilde{A}$, therefore, from Proposition 3.17 it also follows that the isomorphism above is compatible with Frobenius after base change along $A^+[1/\mu] \to A$. ■

Proposition 3.19. *Let N be an effective Wach module over A_R^+ and $T := \mathbf{T}_R(N)$ the associated finite free $\mathbb{Z}_p$-representation of G_R. Then we have (φ, Γ_R)-equivariant inclusions $\mu^s \mathbf{D}_R^+(T) \subset N \subset \mathbf{D}_R^+(T)$ (see §2.6 for notations).*

Proof. The proof follows in a manner similar to the proof of Proposition 3.17, so we will freely use the notation of that proof. Inverting p in (3.9) we get (φ, G_R)-equivariant inclusions

$$\begin{aligned} \mu^s\Big(\prod_{\mathfrak{p}\in\mathcal{S}} (A_{\inf}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T)\Big)\big[\tfrac{1}{p}\big] &\subset \Big(\prod_{\mathfrak{p}\in\mathcal{S}} (A_{\inf}(\mathbb{C}^+(\mathfrak{p})) \otimes_{A_R^+} N_R)\Big)\big[\tfrac{1}{p}\big] \\ &\subset \Big(\prod_{\mathfrak{p}\in\mathcal{S}} (A_{\inf}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Z}_p} T)\Big)\big[\tfrac{1}{p}\big]. \end{aligned} \tag{3.12}$$

The last term of (3.12) can be written as $\big(\prod_{\mathfrak{p}\in\mathcal{S}} A_{\inf}(\mathbb{C}^+(\mathfrak{p}))\big)[\frac{1}{p}] \otimes_{\mathbb{Q}_p} V$ and similarly for the first term. Moreover, we have $\prod_{\mathfrak{p}\in\mathcal{S}}(A_{\inf}(\mathbb{C}^+(\mathfrak{p})) \otimes_{A_R^+} N_R) = (\prod_{\mathfrak{p}\in\mathcal{S}} A_{\inf}(\mathbb{C}^+(\mathfrak{p}))) \otimes_{A_R^+} N_R$, so the middle term of (3.12) can be written as $\big(\prod_{\mathfrak{p}\in\mathcal{S}} A_{\inf}(\mathbb{C}^+(\mathfrak{p}))\big)[\frac{1}{p}] \otimes_{B_R^+} N_R[\frac{1}{p}]$. Furthermore, by inverting p in (3.5), we have the following (φ, G_R)-equivariant comparison isomorphism:

$$W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}] \otimes_{B_R^+} N_R[\tfrac{1}{p}] \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V. \tag{3.13}$$

Using Lemma 2.7, we embed all terms in (3.12) and (3.13) inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big)[1/p] \otimes_{B_R^+} N_R[1/p] \xrightarrow{\sim} \big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big)[1/p] \otimes_{\mathbb{Q}_p} V$, compatible with respective actions of φ and G_R. Since $N_R[1/p]$ is finite projective over B_R^+, the intersection of the middle term in (3.12) and the left-hand term in (3.13), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big)[1/p] \otimes_{B_R^+} N_R[1/p]$, gives

$$\Big(W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}] \otimes_{B_R^+} N_R[\tfrac{1}{p}]\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} A_{\inf}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{p}] \otimes_{B_R^+} N_R[\tfrac{1}{p}]\Big) = A_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{B_R^+} N_R[\tfrac{1}{p}],$$

where the equality follows from Lemma 2.7. Similarly, the intersection of the right-hand terms in (3.10) and (3.13), inside $\big(\prod_{\mathfrak{p}\in\mathcal{S}} W(\mathbb{C}(\mathfrak{p})^\flat)\big)[1/p] \otimes_{\mathbb{Q}_p} V$, gives

$$\Big(W(\mathbb{C}(\overline{R})^\flat)[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V\Big) \cap \Big(\Big(\prod_{\mathfrak{p}\in\mathcal{S}} A_{\inf}(\mathbb{C}^+(\mathfrak{p}))\Big)[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V\Big) = A_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V,$$

where the equality again follows from Lemma 2.7. Therefore, from (3.12) and (φ, G_R)-equivariance of (3.13), we obtain the following (φ, G_R)-equivariant inclusions

$$\mu^s(A_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V) \subset A_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{B_R^+} N_R[\tfrac{1}{p}] \subset A_{\inf}(\overline{R})[\tfrac{1}{p}] \otimes_{\mathbb{Q}_p} V. \tag{3.14}$$

Inverting p in the isomorphism obtained in Corollary 3.18 and by taking its intersection with (3.14), inside $W(\mathbb{C}(\overline{R})^\flat)[1/p] \otimes_{B_R^+} N_R[1/p] \xrightarrow{\sim} W(\mathbb{C}(\overline{R})^\flat)[1/p] \otimes_{\mathbb{Q}_p} V$, we obtain the following (φ, G_R)-equivariant inclusions

$$\mu^s(B^+ \otimes_{\mathbb{Q}_p} V) \subset B^+ \otimes_{B_R^+} N_R[\tfrac{1}{p}] \subset B^+ \otimes_{\mathbb{Q}_p} V.$$

In the preceding equation, by taking H_R-invariants and its intersection with $\mathbf{D}_R(T) = N_R[1/\mu]^\wedge$, inside $\mathbf{D}_R(V)$, we obtain $\mu^s \mathbf{D}_R^+(T) \subset N_R \subset \mathbf{D}_R^+(T)$, since $N_R = N_R[1/p] \cap N_R[1/\mu]^\wedge$ from Lemma 3.5 and $\mathbf{D}_R^+(T) = \mathbf{D}_R(T) \cap \mathbf{D}_R^+(V) \subset \mathbf{D}_R(V)$ by definition. Hence, the proposition is proved. ■

3.4. Finite $[p]_q$-height representations. In this section we will generalise the definition of finite $[p]_q$-height representations from [Abh21, Definition 4.9] in the relative case.

Definition 3.20. A finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_R is a finite free $\mathbb{Z}_p$-module T admitting a linear and continuous action of G_R such that there exists a finitely generated A_R^+-submodule $\mathbf{N}_R(T) \subset \mathbf{D}_R(T)$, stable under the action of Γ_R on $\mathbf{D}_R(T)$, and such that $\mathbf{N}_R(T)$, equipped with the induced actions of φ and Γ_R, satisfies the following:

(1) $\mathbf{N}_R(T)$ is a Wach module in the sense of Definition 3.8.

(2) A_R-linearly extending the inclusion $\mathbf{N}_R(T) \to \mathbf{D}_R(T)$ induces a (φ, Γ_R)-equivariant isomorphism $A_R \otimes_{A_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$.

The height of T is defined to be the height of $\mathbf{N}_R(T)$. Say that T is *positive* if $\mathbf{N}_R(T)$ is effective.

A finite $[p]_q$-height p-adic representation of G_R is a finite dimensional $\mathbb{Q}_p$-vector space admitting a linear and continuous action of G_R such that there exists a G_R-stable $\mathbb{Z}_p$-lattice $T \subset V$, with T of finite $[p]_q$-height. We set $\mathbf{N}_R(V) := \mathbf{N}_R(T)[1/p]$ satisfying properties analogous to (1) and (2) above. The height of V is defined to be the height of T. Say that V is positive if $\mathbf{N}_R(V)$ is effective.

Lemma 3.21. *Let T be a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_R then the A_R^+-module $\mathbf{N}_R(T)$, associated to T in Definition 3.20, is unique.*

Proof. By definition, $A_R \otimes_{A_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$ and this scalar extension induces a fullly faithful functor in Proposition 3.15. So from (2.8) we obtain the uniqueness of $\mathbf{N}(T)$. Alternatively, the uniquess can also be deduced using Proposition 3.19 and [Abh21, Proposition 4.13]. ∎

Remark 3.22. Let V be a finite $[p]_q$-height p-adic representation of G_R and $T \subset V$ a finite $[p]_q$-height G_R-stable $\mathbb{Z}_p$-lattice. Then we have $\mathbf{N}_R(V) = \mathbf{N}_R(T)[1/p]$ and from Proposition 3.19 we get that if V is positive then $\mu^s \mathbf{D}_R^+(V) \subset \mathbf{N}_R(V) \subset \mathbf{D}_R^+(V)$. Moreover, similar to [Abh21, Remark 3.10], we can show that $\mathbf{N}_R(V)$ is unique, in particular, it is independent of choice of the lattice T by Corollary 3.16.

Remark 3.23. By the definition of finite $[p]_q$-height representations, Lemma 3.21 and the fully faithful functor in (3.3) it follows that the data of a finite height representation is equivalent to the data of a Wach module.

3.5. Nygaard filtration on Wach modules. In this section we consider the Nygaard filtration on Wach modules as follows:

Definition 3.24. Let N be a Wach module over A_R^+. Define a decreasing filtration on N called the *Nygaard filtration*, for $k \in \mathbb{Z}$, as

$$\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}.$$

From the definition it is clear that N is effective if and only if $\mathrm{Fil}^0 N = N$. Similarly, we define Nygaard filtration on $M := N[1/p]$ and it easily follows that $\mathrm{Fil}^k M = (\mathrm{Fil}^k N)[1/p]$.

Lemma 3.25. *Let N be a Wach module A_R^+.*

(1) *For any $k, r \in \mathbb{Z}$, and the Wach module $\mu^{-r}N(r)$ over A_R^+, we have that $\mathrm{Fil}^k(\mu^{-r}N(r)) = \mu^{-r}(\mathrm{Fil}^{r+k}N)(r)$.*

(2) *For all $k \in \mathbb{Z}$, we have that $\mathrm{Fil}^k N \cap \mu N = \mu \mathrm{Fil}^{k-1} N \subset N$.*

Similar statements are also true for the B_R^+-module $N[1/p]$.

Proof. The proof follows from arguments similar to [Abh23b, Lemma 3.3 & Lemma 3.4]. In (1), the inclusion $\mu^{-r}(\mathrm{Fil}^{r+k}N)(r) \subset \mathrm{Fil}^k(\mu^{-r}N(r))$ is obvious. For the converse, let $\mu^{-r}x \otimes \epsilon^{\otimes r}$ be an element of $\mathrm{Fil}^k(\mu^{-r}N(r))$, where x is an element of N and $\epsilon^{\otimes r}$ is a $\mathbb{Z}_p$-basis of $\mathbb{Z}_p(r)$. By assumption, note that $\varphi(\mu^{-r}x \otimes \epsilon^{\otimes r}) = ([p]_q\mu)^{-r}\varphi(x) \otimes \epsilon^{\otimes r}$ is in $[p]_q^k \mu^{-r} N(r)$. Therefore, $\varphi(x)$ belongs to $[p]_q^{r+k}N$, i.e. x is in $\mathrm{Fil}^{r+k}N$. For (2), note that using (1) we may assume that N is effective. The claim is obvious if $\mathrm{Fil}^{k-1}N = N$, so we assume that $k \geq 2$. Let x be an element of $\mathrm{Fil}^k N \cap \mu N$ and write $x = \mu y$, for some y in N. Now, as $\varphi(x)$ is in $[p]_q^k N$, therefore, $\mu\varphi(y)$ is in $[p]_q^{k-1}N$, i.e. $\mu\varphi(y) = [p]_q^{k-1}z$, for some z in N. In particular, we have that $[p]_q^{r-1}z = p^{r-1}z = 0 \bmod \mu N$. But $N/\mu N$ is p-torsion free, so it follows that $z = 0 \bmod \mu N$, i.e. y is in $\mathrm{Fil}^{k-1}N$. The other inclusion is obvious because $\mu\mathrm{Fil}^{k-1}N \subset \mathrm{Fil}^k N$. This allows us to conclude. ■

Remark 3.26. The Nygaard filtration from Definition 3.24, on a Wach module N over A_R^+, is stable under the action of Γ_R. Therefore, using Lemma 3.25 (2) we see that for any g in Γ_R and $k \in \mathbb{Z}$, we have that $(g-1)\mathrm{Fil}^k N \subset (\mathrm{Fil}^k N) \cap \mu N = \mu\mathrm{Fil}^{k-1}N$.

The reason for considering the Nygaard filtration as above is the following: note that $A_{\mathrm{cris}}(\overline{R})$ is equipped with a filtration by divided power ideals and the embedding $A_{\mathrm{inf}}(\overline{R}) \subset A_{\mathrm{cris}}(\overline{R})$ induces a filtration on $A_{\mathrm{inf}}(\overline{R})$ given as $\mathrm{Fil}^k A_{\mathrm{inf}}(\overline{R}) := \xi^k A_{\mathrm{inf}}(\overline{R})$ for $k \in \mathbb{N}$. We equip A^+ with the induced filtration $\mathrm{Fil}^k A^+ := A^+ \cap \mathrm{Fil}^k A_{\mathrm{inf}}(\overline{R}) \subset A_{\mathrm{inf}}(\overline{R})$.

Lemma 3.27. *Let N be an effective Wach module over A_R^+ and let $T := \mathbf{T}_R(N)$ denote the associated $\mathbb{Z}_p$-representation of G_R. Then, for $k \in \mathbb{N}$, we have $(\mathrm{Fil}^k A^+ \otimes_{\mathbb{Z}_p} T) \cap N = \mathrm{Fil}^k N$.*

Proof. Let $V := T[1/p]$, $M := N[1/p]$ and $\mathrm{Fil}^k B^+ := (\mathrm{Fil}^k A^+)[1/p]$. Then it is enough to show that $(\mathrm{Fil}^k B^+ \otimes_{\mathbb{Q}_p} V) \cap M = \mathrm{Fil}^k M$. Indeed, from Definition 3.24 we have $\mathrm{Fil}^k N := \mathrm{Fil}^k M \cap N = (\mathrm{Fil}^k B^+ \otimes_{\mathbb{Q}_p} V) \cap M \cap N = (\mathrm{Fil}^k A^+ \otimes_{\mathbb{Z}_p} T) \cap N$ since $\mathrm{Fil}^k B^+ \cap A^+ = \mathrm{Fil}^k A^+$. Now the inclusion $\mathrm{Fil}^k M \subset (\mathrm{Fil}^k B^+ \otimes_{\mathbb{Q}_p} V)$ is obvious and for the converse it is enough to show $([p]_q^k B^+ \otimes_{\mathbb{Q}_p} V) \cap M = [p]_q^k M$. Indeed, if we have x in $(\mathrm{Fil}^k B^+ \otimes_{\mathbb{Q}_p} V) \cap M$ then $\varphi(x)$ is in $([p]_q^k B^+ \otimes_{\mathbb{Q}_p} V) \cap M = [p]_q^k M$, i.e. x is in $\mathrm{Fil}^k M$. For the reduced claim, note that the inclusion $[p]_q^k M \subset ([p]_q^k B^+ \otimes_{\mathbb{Q}_p} V) \cap M$ is obvious. To show the converse, let x in $B^+ \otimes_{\mathbb{Q}_p} V$ such that $[p]_q^k x$ is in M, in particular, x is in $M[1/[p]_q]$. Then it follows that $h(x) = x$ for all $h \in H_R$, i.e. x is in $(B^+ \otimes_{\mathbb{Q}_p} V)^{H_R} =: \mathbf{D}_R^+(V) = \mathbf{D}_R^+(T)[1/p]$. From Proposition 3.19 recall that $\mu^s\mathbf{D}_R^+(V) \subset M$, where s is the $[p]_q$-height of N. So we get that $\mu^s x$ is in M, in particular, x is in $M[1/\mu]$. Combining this with the previous observation we get that x is in $M[1/\mu] \cap M[1/[p]_q] \subset B_R \otimes_{B_R^+} M$. But from Proposition 3.11 we know that M is finite projective over B_R^+ and note that $B_R^+ = B_R^+[1/\mu] \cap B_R^+[1/[p]_q] \subset B_R$, since $[p]_q = p \bmod \mu B_R^+$. Hence, it follows that we have x is in $M[1/\mu] \cap M[1/[p]_q] = M$, as desired. ■

3.5.1. Nygaard filtration on scalar extension of Wach modules. Let N_R be a Wach module over A_R^+. Then by Remark 1.4 we know that $N_L := A_L^+ \otimes_{A_R^+} N_R$ is a Wach module over A_L^+ equipped with the natural action of φ and $\Gamma_L \xrightarrow{\sim} \Gamma_R$ (in the sense of [Abh23a, Definition 3.1]). Note that similar to Definition 3.24, we can equip N_L with the Nyggard filtration (see [Abh23a, Definition 3.2]), and we claim the following:

Lemma 3.28. *For each $k \in \mathbb{Z}$, we have that*

$$A_L^+ \otimes_{A_R^+} \mathrm{Fil}^k N_R \xrightarrow{\sim} \mathrm{Fil}^k N_L. \tag{3.15}$$

Proof. Using Lemma 3.25 (1), note that it is enough to prove the claim for effective Wach modules, in particular, we will assume that $\mathrm{Fil}^0 N_R = N_R$. Note that the case $k = 0$ is trivial. We will first prove the claim rationally and use it to deduce the integral claim. Set $M_R := N_R[1/p]$ and $M_L := N_L[1/p]$, equipped with induced structures. Now, consider the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Fil}^{k+1}M_R & \longrightarrow & \mathrm{Fil}^k M_R & \longrightarrow & \mathrm{gr}^k M_R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^{k+1}M_L & \longrightarrow & \mathrm{Fil}^k M_L & \longrightarrow & \mathrm{gr}^k M_L & \longrightarrow & 0, \end{array} \tag{3.16}$$

where the left and the middle vertical arrows are natural inclusions. For the induced right vertical arrow note that we have $\mathrm{Fil}^k M_R \cap \mathrm{Fil}^{k+1} M_L = \mathrm{Fil}^{k+1} M_R \subset M_L$, since $[p]_q^k M_R \cap [p]_q^{k+1} M_L = (B_R^+ \cap [p]_q B_L^+) \otimes_{B_R^+} [p]_q^k M_R = [p]_q^{k+1} M_R$. So we get that the right vertical arrow of (3.16) is injective. From the preceding conclusion, it follows that $B_L^+ \otimes_{B_R^+} \mathrm{gr}^k M_R \xrightarrow{\sim} L \otimes_{R[1/p]} \mathrm{gr}^k M_R$ is a finitely generated module over $\mathrm{gr}^0 B_L^+ = B_L^+/\mu B_L^+ \xrightarrow{\sim} L$, i.e. a finite dimensional L-vector space, and further it is easy to see that the natural map $L \otimes_{R[1/p]} \mathrm{gr}^k M_R \to \mathrm{gr}^k M_L$ is injective. Now, consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B_L^+ \otimes_{B_R^+} \mathrm{Fil}^{k+1} M_R & \longrightarrow & B_L^+ \otimes_{B_R^+} \mathrm{Fil}^k M_R & \longrightarrow & L \otimes_{R[1/p]} \mathrm{gr}^k M_R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Fil}^{k+1} M_L & \longrightarrow & \mathrm{Fil}^k M_L & \longrightarrow & \mathrm{gr}^k M_L & \longrightarrow & 0, \end{array} \tag{3.17}$$

4 where the top row is the scalar extension of the top exact row in (3.16) along the natural flat map $B_R^+ \to B_L^+$ and the vertical maps are induced from the isomorphism $A_L^+ \otimes_{A_R^+} N_R \xrightarrow{\sim} N_L$. Now, we will proceed by induction on k, i.e. assume that the middle vertical arrow of (3.17) is an isomorphism for some $k \geq 0$. Then, it follows that the right vertical arrow is surjective and it is injective by the discussion after (3.16), hence bijective. So, we conclude that the left vertical arrow is bijective as well, i.e. $B_L^+ \otimes_{B_R^+} \mathrm{Fil}^{k+1} M_R \xrightarrow{\sim} \mathrm{Fil}^{k+1} M_L$. Finally, note the natural map $A_R^+ \to A_L^+$ is flat, therefore, for any $k \in \mathbb{Z}$ we have that

$$\begin{aligned} A_L^+ \otimes_{A_R^+} \mathrm{Fil}^k N_R &= A_L^+ \otimes_{A_R^+} (\mathrm{Fil}^k M_R \cap N_R) \\ &= (A_L^+ \otimes_{A_R^+} \mathrm{Fil}^k M_R) \cap (A_L^+ \otimes_{A_R^+} N_R) \xrightarrow{\sim} \mathrm{Fil}^k M_L \cap N_L = \mathrm{Fil}^k N_L. \end{aligned}$$

This completes our proof. ∎

Remark 3.29. The ideas emplyed in the proof of Lemma 3.28 enables us to relate the Nygaard filtration on N_L to classical Wach modules. Indeed, let $O_{\breve{L}} := (\cup_{i=1}^d O_L[X_i^{1/p^\infty}])^\wedge$, where $^\wedge$ denotes the p-adic completion. The O_L-algebra $O_{\breve{L}}$ is a complete discrete valuation ring with perfect residue field, uniformiser p and fraction field $\breve{L} := O_{\breve{L}}[1/p]$. The Witt vector Frobenius on $O_{\breve{L}}$ is given by the Frobenius on O_L described in §1.4 and setting $\varphi(X_i^{1/p^n}) = X_i^{1/p^{n-1}}$ for all $1 \leq i \leq d$ and $n \in \mathbb{N}$. Let $\breve{L}_\infty := \breve{L}(\mu_{p^\infty})$ and let $\overline{\breve{L}} \supset \overline{L}$ denote a fixed algebraic closure of $\breve{L}$. We have the Galois groups $G_{\breve{L}} := \mathrm{Gal}(\overline{\breve{L}}/\breve{L}) \xrightarrow{\sim} \mathrm{Gal}(\overline{L}/\cup_{i=1}^d L(X_i^{1/p^\infty}))$ and $\Gamma_{\breve{L}} := \mathrm{Gal}(\breve{L}_\infty/\breve{L}) \xrightarrow{\sim} \mathrm{Gal}(L(\mu_{p^\infty})/L) \xrightarrow{\sim} \mathbb{Z}_p^\times$. Note that $G_{\breve{L}}$ can be identified with a subgroup of G_L and $\Gamma_{\breve{L}}$ can be identified with a quotient of Γ_L. Next, recall that from [Ber04], we have the theory of Wach modules over $A_{\breve{L}}^+ = O_{\breve{L}}[\![\mu]\!]$ (see [Abh23a, §4.1] for a quick recollection). Now, if N_L is a Wach module over A_L^+, then $N_{\breve{L}} := A_{\breve{L}}^+ \otimes_{A_L^+} N_L$ is naturally a Wach module over $A_{\breve{L}}^+$ (see [Abh23a, Corollary 4.27]). Euipping $N_{\breve{L}}$ with the Nygaard filtration as in Definition 3.24 and employing an argument similar to Lemma 3.28 shows that, for each $k \in \mathbb{Z}$, we have that $A_{\breve{L}}^+ \otimes_{A_L^+} \mathrm{Fil}^k N_L \xrightarrow{\sim} \mathrm{Fil}^k N_{\breve{L}}$.

3.5.2. Reduction modulo μ of the Nygaard filtration.

Let N_R be a Wach module over A_R^+ and note that $(N_R/\mu N_R)[1/p]$ is a φ-module over $R[1/p]$ since $[p]_q = p \bmod \mu A_R^+$, and $N_R/\mu N_R$ is equipped with a filtration $\mathrm{Fil}^k(N_R/\mu N_R)$ given as the image of $\mathrm{Fil}^k N_R$ under the surjection $N_R \twoheadrightarrow N_R/\mu N_R$. We equip $(N_R/\mu N_R)[1/p]$ with the induced filtration $\mathrm{Fil}^k((N_R/\mu N_R)[1/p]) := \mathrm{Fil}^k(N_R/\mu N_R)[1/p]$, and note that it is a filtered φ-module over $R[1/p]$.

Lemma 3.30. *For each $k \in \mathbb{Z}$, the following sequence is exact:*

$$0 \longrightarrow \mathrm{Fil}^{k-1} N_R \xrightarrow{\mu} \mathrm{Fil}^k N_R \longrightarrow \mathrm{Fil}^k(N_R/\mu N_R) \longrightarrow 0. \tag{3.18}$$

Moreover, by taking the associated graded pieces, we get that $\mathrm{gr}^k N_R \xrightarrow{\sim} \mathrm{gr}^k(N_R/\mu N_R)$. Similar statements are also true for the B_R^+-module $N[1/p]$. Furthermore, similar claims hold for Wach modules over A_L^+ and $A_{\breve{L}}^+$ (see Remark 3.29 for the latter ring) as well as after inverting p.

Proof. Exactness of (3.18) easily follows from Lemma 3.25 (2). Then, by taking the associated graded pieces, we obtain the following exact sequence:

$$0 \longrightarrow \mathrm{gr}^{k-1} N_R \xrightarrow{\mu} \mathrm{gr}^k N_R \longrightarrow \mathrm{gr}^k(N_R/\mu N_R) \longrightarrow 0.$$

It is clear that the map $\mathrm{gr}^{k-1} N_R \xrightarrow{\mu} \mathrm{gr}^k N_R$ is trivial, i.e. $\mathrm{gr}^k N_R \xrightarrow{\sim} \mathrm{gr}^k(N_R/\mu N_R)$. Rest is obvious. ∎

Let $N_L := A_L^+ \otimes_{A_R^+} N_R$ equipped with the natural action of φ and $\Gamma_L \xrightarrow{\sim} \Gamma_R$ and set $M_R := N_R[1/p]$ and $M_L := N_L[1/p]$.

Lemma 3.31. *The natural isomorphism of φ-modules $O_L \otimes_R (N_R/\mu N_R) \xrightarrow{\sim} N_L/\mu N_L$ is compatible with filtrations, i.e. for each $k \in \mathbb{Z}$ we have that*

$$O_L \otimes_R \mathrm{Fil}^k(N_R/\mu N_R) \xrightarrow{\sim} \mathrm{Fil}^k(N_L/\mu N_L). \tag{3.19}$$

In particular, $L \otimes_R (M_R/\mu M_R) \xrightarrow{\sim} M_L/\mu M_L$ is a natural isomorphism of filtered φ-modules over L.

Proof. Consider the following diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A_L^+ \otimes_{A_R^+} \mu\mathrm{Fil}^{k-1} N_R & \longrightarrow & A_L^+ \otimes_{A_R^+} \mathrm{Fil}^k N_R & \longrightarrow & O_L \otimes_R \mathrm{Fil}^k(N_R/\mu N_R) & \longrightarrow & 0 \\ & & \wr\downarrow (3.15) & & \wr\downarrow (3.15) & & \downarrow (3.19) & & \\ 0 & \longrightarrow & \mu\mathrm{Fil}^{k-1} N_L & \longrightarrow & \mathrm{Fil}^k N_L & \longrightarrow & \mathrm{Fil}^k(N_L/\mu N_L) & \longrightarrow & 0, \end{array} \tag{3.20}$$

where the top row is the extension of (3.18) along the flat map $A_R^+ \to A_L^+$ and the bottom row is the exact sequence (analogous to (3.18)) for Wach modules over A_L^+. Note that the map in (3.19) is compatible with the natural map in (3.15), i.e. the the right square in (3.20) commutes. Hence, the right vertical arrow, i.e. (3.19) is bijective. ∎

Remark 3.32. In the notation of Remark 3.29, let N_L be a Wach module over A_L^+ and set $N_{\breve{L}} := A_{\breve{L}}^+ \otimes_{A_L^+} N_L$. Then then natural isomorphism of φ-modules $O_{\breve{L}} \otimes_{O_L} (N_L/\mu N_L) \xrightarrow{\sim} N_{\breve{L}}/\mu N_{\breve{L}}$ is compatible with filtrations, i.e. for each $k \in \mathbb{Z}$ we have that $O_{\breve{L}} \otimes_{O_L} \mathrm{Fil}^k(N_L/\mu N_L) \xrightarrow{\sim} \mathrm{Fil}^k(N_{\breve{L}}/\mu N_{\breve{L}})$. In particular, $\breve{L} \otimes_{O_L} (N_L/\mu N_L) \xrightarrow{\sim} (N_{\breve{L}}/\mu N_{\breve{L}})[1/p]$ is a natural isomorphism of filtered φ-modules over $\breve{L}$.

3.6. Wach modules are crystalline. The goal of this subsection is to prove Theorem 3.34. In order to prove our results, we will need auxiliary period rings $A_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ from [Abh21, §4.3.1]. We briefly recall their definitions. Let $\varpi := \zeta_p - 1$ and set $A_{R,\varpi}^+ := A_R^+[\varphi^{-1}(\mu)] \subset A_{\mathrm{inf}}(R_\infty)$, stable under the (φ, Γ_R)-action on the latter. By restricting the map θ on $A_{\mathrm{inf}}(R_\infty)$, to $A_{R,\varpi}^+$ (see §2.2), we obtain a surjective ring homomorphism $\theta : A_{R,\varpi}^+ \twoheadrightarrow R[\varpi]$. We define $A_{R,\varpi}^{\mathrm{PD}}$ to be the p-adic completion of the divided power envelope of the map θ with respect to $\mathrm{Ker}\,\theta$. Furthermore, the map θ extends R-linearly to a surjective ring homomorphism $\theta_R : R \otimes_{\mathbb{Z}} A_{R,\varpi}^+ \twoheadrightarrow R[\varpi]$, given as $x \otimes y \mapsto x\theta(y)$. Similar to above, we define $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ to be the p-adic completion of the divided power envelope of the map θ_R with respect to $\mathrm{Ker}\,\theta_R$. The morphisms θ and θ_R naturally extend to respective surjections $\theta : A_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow R[\varpi]$ and $\theta_R : \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \twoheadrightarrow R[\varpi]$. Now, from loc. cit., we have natural inclusions $A_{R,\varpi}^{\mathrm{PD}} \subset A_{\mathrm{cris}}(R_\infty)$ and $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}A_{\mathrm{cris}}(R_\infty)$, and it is easy to verify that the former rings are stable under respective actions of φ and Γ_R on the latter rings. Therefore, we equip $A_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ with induced structures, in particular, a filtration and an $A_{R,\varpi}^{\mathrm{PD}}$-linear connection ∂_A on $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ satisfying Griffiths transversality with respect to the filtration, and it is easy to show that $(\mathcal{O}A_{R,\varpi}^{\mathrm{PD}})^{\partial_A=0} = A_{R,\varpi}^{\mathrm{PD}}$. Note that the aforementioned filtration on $A_{R,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ coincide with the divided power filtration by $\mathrm{Ker}\,\theta$ and $\mathrm{Ker}\,\theta_R$ respectively (see [Abh21, Remark 4.23]).

Remark 3.33. Let us first remark that the ring $A_{R,\varpi}^{\mathrm{PD}}$ is flat over A_R^+. Indeed, note that $A_{R,\varpi}^{\mathrm{PD}}$ is the p-adic completion of a divided power algebra over $A_{R,\varpi}^+$, given as $A_{R,\varpi}^+[\xi^k/k!, k \in \mathbb{N}]$, where $\xi = \mu/\varphi^{-1}\mu$. Now, since (p, ξ) is a regular sequence on $A_{R,\varpi}^+$, therefore, using [BS22, Lemma 2.38 and Lemma 2.43], it follows that $A_{R,\varpi}^{\mathrm{PD}}$ is p-completely flat over $A_{R,\varpi}^+$, therefore, flat since $A_{R,\varpi}^+$ is noetherian (see [Sta23, Tag 0912]). As $A_{R,\varpi}^+$ is flat over A_R^+, it follows that $A_{R,\varpi}^{\mathrm{PD}}$ is flat over A_R^+. Furthermore, from [Abh21, Lemma 4.20], note that $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ is a PD-polynomial algebra over $A_{R,\varpi}^{\mathrm{PD}}$ in d variables. So again from [BS22, Lemma 2.38 and Lemma 2.43], it follows that $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ is p-completely flat over $A_{R,\varpi}^{\mathrm{PD}}$. As $A_{R,\varpi}^{\mathrm{PD}}$ is flat over A_R^+, we get that $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ is p-completely flat over A_R^+, hence flat. Next, note that for any $k \in \mathbb{N}$, the graded quotient $\mathrm{gr}^k(A_{R,\varpi}^{\mathrm{PD}}) = \mathrm{Fil}^k(A_{R,\varpi}^{\mathrm{PD}})/\mathrm{Fil}^{k+1}(A_{R,\varpi}^{\mathrm{PD}})$ is isomorphic to $\xi^{[k]}R[\varpi]$, in particular, we have that $\mathrm{gr}^k(A_{R,\varpi}^{\mathrm{PD}})$ is a free R-module. Now, since $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ is a PD-polynomial algebra over $A_{R,\varpi}^{\mathrm{PD}}$, we also get that for any $k \in \mathbb{N}$, the graded quotient $\mathrm{gr}^k(\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) = \mathrm{Fil}^k(\mathcal{O}A_{R,\varpi}^{\mathrm{PD}})/\mathrm{Fil}^{k+1}(\mathcal{O}A_{R,\varpi}^{\mathrm{PD}})$ is a free R-module. Moreover, we have $A_R^+/\mu \xrightarrow{\sim} R$, so the flat dimension of R as an A_R^+-module is 1, and it follows that the flat dimension of $\mathrm{gr}^k\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ as an A_R^+-module is also 1. Since $\mathrm{Fil}^0\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} = \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$, therefore, using induction on $k \in \mathbb{N}$, we conclude that $\mathrm{Fil}^k\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ is flat as an A_R^+-module.

Theorem 3.34. *Let N be a Wach module over A_R^+ and let $T := \mathbf{T}_R(N)$ be the associated finite free $\mathbb{Z}_p$-representation of G_R. Then $V := T[1/p]$ is a p-adic crystalline representation of G_R and we have a natural isomorphism of $R[1/p]$-modules $(\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} N[1/p])^{\Gamma_R} \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ compatible with the respective Frobenii and connections.*

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and $\mathbf{T}_R(\mu^r N(-r)) = \mathbf{T}_R(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which Γ_R acts via χ^{-r}, where χ is the p-adic cyclotomic character). Therefore, it follows that it is enough to show the claim for effective Wach modules. So, in the rest of the proof, we will assume that N is effective. Now, let us set $\mathcal{O}D_R := (\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} N[1/p])^{\Gamma_R} \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, and using Proposition 3.35, we note that $\mathcal{O}D_R$ is a finite projective $R[1/p]$-module of rank $= \mathrm{rk}_{B_R^+} N[1/p]$. Moreover, $\mathcal{O}D_R$ is equipped with the tensor product Frobenius. Next, we note that $\mathcal{O}D_R$ is equipped with a connection induced from the connection on $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$. Using Proposition 3.35, note that we have a natural isomorphism $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}D_R \xrightarrow{\sim} \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} N[1/p]$. Now, consider the following diagram:

$$\begin{array}{ccc} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}D_R & \xrightarrow[\sim]{(3.23)} & \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{B_R^+} N[1/p] \\ {\scriptstyle (3.24)}\downarrow & & \downarrow\wr\,{\scriptstyle (3.4)} \\ \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) & \longrightarrow & \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V, \end{array} \tag{3.21}$$

where the left vertical arrow is the extension of the $R[1/p]$-linear injective map $\mathcal{O}D_R \to \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, from (3.24), along the faithfully flat ring homomorphism $R[1/p] \to \mathcal{O}B_{\mathrm{cris}}(\overline{R})$ (see [Bri08, Thèoréme 6.3.8]), the top horizontal arrow is the extension along $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}[1/p] \to \mathcal{O}B_{\mathrm{cris}}(\overline{R})$ of the isomorphism (3.23) in Proposition 3.35, the right vertical arrow is the extension along $A^+[1/\mu] \to \mathcal{O}B_{\mathrm{cris}}(\overline{R})$ of the isomorphism in Proposition 3.17 and the bottom horizontal arrow is the natural injective map (see [Bri08, Proposition 8.2.6]). Commutativity of the diagram (3.21) and compatibility of its arrows with the respective actions of (φ, G_R) and connections follow from (3.24). Since the top horizontal and right vertical arrows in (3.21) are bijective, we conclude that its left vertical arrow and the bottom horizontal arrow are also bijective. Therefore, V is a p-adic crystalline representation of G_R, and by taking G_R fixed part of the left vertical arrow in (3.21), we obtain an isomorphism of $R[1/p]$-modules

$$\mathcal{O}D_R \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \tag{3.22}$$

compatible with the respective Frobenii and connections. This concludes our proof. ∎

The following observation was used above:

Proposition 3.35. *Let N be an effective Wach module over A_R^+, then $\mathcal{O}D_R := (\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} N[1/p])^{\Gamma_R}$ is a finite projective $R[1/p]$-module of rank $= \mathrm{rk}_{B_R^+} N[1/p]$ equipped with a Frobenius and a connection. Moreover, we have a natural comparison isomorphism*

$$\begin{aligned} f : \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}D_R &\xrightarrow{\sim} \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} N[1/p] \\ a \otimes b \otimes x &\longmapsto ab \otimes x, \end{aligned} \tag{3.23}$$

compatible with the respective Frobenii, connections and actions of Γ_R.

Remark 3.36. In (3.23), the Frobenius on each term is given as $\varphi \otimes \varphi$; the connection on the right-hand term is given as the natural $A_{R,\varpi}^{\mathrm{PD}}$-linear differential operator $\partial \otimes 1$ and on the left-hand term, it is given as $\partial \otimes 1 + 1 \otimes \partial_D$, where ∂_D is the connection on $\mathcal{O}D_R$; the action of any g in Γ_R on the left-hand term is given as $g \otimes 1$ and on the the right-hand term, it is given as $g \otimes g$.

Proof of Proposition 3.35. We will adapt the proof of [Abh21, Proposition 4.28]. Recall the following rings from [Abh21, §4.4.1]: for $n \in \mathbb{N}$, we take a p-adically complete ring $S_n^{\mathrm{PD}} := A_R^+\langle \frac{\mu}{p^n}, \frac{\mu^2}{2!p^{2n}}, \ldots, \frac{\mu^k}{k!p^{kn}}, \ldots\rangle$. We have a Frobenius homomorphism $\varphi : S_n^{\mathrm{PD}} \to S_{n-1}^{\mathrm{PD}}$, in particular, $\varphi^n(S_n^{\mathrm{PD}}) \subset A_{R,\varpi}^{\mathrm{PD}}$ and the ring S_n^{PD} is further equipped with a continuous (for the p-adic topology) action of Γ_R. The reader should note that in [Abh21, §4.4.1] we consider a further completion of S_n^{PD}, with respect to certain filtration by PD-ideals, which we denote as $\widehat{S}_n^{\mathrm{PD}}$ in loc. cit. However, such a completion is not strictly necessary and all the proofs of loc. cit. can be carried out without it. In particular, many good properties of $\widehat{S}_n^{\mathrm{PD}}$ restrict to good properties on S_n^{PD} as well (for example, (φ, Γ_R)-action above).

Let us now consider the O_F-linear homomorphism of rings $\iota : R \to S_n^{\mathrm{PD}}$, defined by sending $X_j \mapsto [X_j^\flat]$, for $1 \leq j \leq d$. Using ι we can define an O_F-linear homomorphism of rings $f : R \otimes_{O_F} S_n^{\mathrm{PD}} \to S_n^{\mathrm{PD}}$, sending $a \otimes b \mapsto \iota(a)b$. Let $\mathcal{O}S_n^{\mathrm{PD}}$ denote the p-adic completion of the divided power envelope of $R \otimes_{O_F} S_n^{\mathrm{PD}}$, with respect to Ker f. The tensor product Frobenius induces $\varphi : \mathcal{O}S_n^{\mathrm{PD}} \to \mathcal{O}S_{n-1}^{\mathrm{PD}}$, such that $\varphi^n(\mathcal{O}S_n^{\mathrm{PD}}) \subset \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$, and the action of Γ_R extends to a continuous (for the p-adic topology) action on $\mathcal{O}S_n^{\mathrm{PD}}$. Moreover, we have a (φ, Γ_R)-equivariant embedding $S_n^{\mathrm{PD}} \subset \mathcal{O}S_n^{\mathrm{PD}}$ and the latter is equipped with a Γ_R-equivariant S_n^{PD}-linear integrable connection given as the universal continuous S_n^{PD}-linear de Rham differential $d : \mathcal{O}S_n^{\mathrm{PD}} \to \Omega^1_{\mathcal{O}S_n^{\mathrm{PD}}/S_n^{\mathrm{PD}}}$. Furthermore, we have $R = (\mathcal{O}S_n^{\mathrm{PD}})^{\Gamma_R}$ and with $V_j = \frac{X_j \otimes 1}{1 \otimes [X_j^\flat]}$, for $1 \leq j \leq d$, we have the p-adically complete divided power ideals of $\mathcal{O}S_n^{\mathrm{PD}}$ as follows:

$$J^{[i]}\mathcal{O}S_n^{\mathrm{PD}} := \Big\langle \frac{\mu^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d} (1 - V_j)^{[k_j]},\ \mathbf{k} = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1} \text{ such that } \sum_{j=0}^{d} k_j \geq i \Big\rangle.$$

We equip $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{A_R^+} N$ with the tensor product Frobenius and the connection on $\mathcal{O}S_n^{\mathrm{PD}}$ induces an S_n^{PD}-linear integrable connection on $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{A_R^+} N$. Then $D_n := \big(\mathcal{O}S_n^{\mathrm{PD}} \otimes_{A_R^+} N[1/p]\big)^{\Gamma_R}$ is an $R[1/p]$-module equipped with a Frobenius $\varphi : D_n \to D_{n-1}$ and an integrable connection. In particular, it follows that $\varphi^n(D_n) \subset \mathcal{O}D_R = \big(\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} N[1/p]\big)^{\Gamma_R} \subset \big(\mathcal{O}A_{\mathrm{cris}}(\overline{R}) \otimes_{A_R^+} N[1/p]\big)^{H_R}$, where we have $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}A_{\mathrm{cris}}(R_\infty) = \mathcal{O}A_{\mathrm{cris}}(\overline{R})^{H_R}$ (see [MT20, Corollary 4.34] for the equality). Let $T := \mathbf{T}_R(N)$ denote the finite free $\mathbb{Z}_p$-representation of G_R, associated to N, and set $V := T[1/p]$, then we have

$$\begin{aligned} \mathcal{O}D_R \subset \big(\mathcal{O}B_{\mathrm{cris}}^+(\overline{R}) \otimes_{B_R^+} N[1/p]\big)^{G_R} &\subset \big(\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{B_R^+} N[1/p]\big)^{G_R} \\ &\xrightarrow{\sim} \big(\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V\big)^{G_R} = \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V), \end{aligned} \tag{3.24}$$

where the isomorphism follows by taking G_R-fixed elements of the isomorphism (3.4) in Proposition 3.17, after extending scalars along $A_{\inf}(\overline{R})[1/\mu] \to \mathcal{O}B_{\mathrm{cris}}(\overline{R})$. Since $\varphi^n(D_n) \subset \mathcal{O}D_R$, or equivalently, the $R[1/p]$-linear map $1 \otimes \varphi^n : R[1/p] \otimes_{\varphi^n, R[1/p]} D_n \to \mathcal{O}D_R$ is injective, we get that $R[1/p] \otimes_{\varphi^n, R[1/p]} D_n$ is a finitely generated $R[1/p]$-module. Moreover, recall that $\varphi^n : R[1/p] \to R[1/p]$ is finite flat (see §1.4), so it follows that D_n is finitely generated over the source of φ^n, i.e. D_n is a finitely generated $R[1/p]$-module equipped with an integrable connection, in particular, it is finite projective over $R[1/p]$ by [Bri08, Proposition 7.1.2]. Furthermore, recall that $N[1/p]$ is a finite projective B_R^+-module (see Proposition 3.11), therefore $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{A_R^+} N[1/p]$ is a finite projective $\mathcal{O}S_n^{\mathrm{PD}}[1/p]$-module, and from [AGT16, Lemma IV.3.2.2], it follows that $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{A_R^+} N$ is p-adically complete. Now, for $n \geq 1$, similar to the proof of [Abh21, Lemmas 4.32 & 4.36], it is easy to show that $\log \gamma_i := \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1}$ converges as a series of operators on $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{A_R^+} N$, where $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ_R (see §2).

Lemma 3.37. *Let $m \geq 1$ (let $m \geq 2$ if $p = 2$), then we have a Γ_R-equivariant isomorphism via the natural map $a \otimes b \otimes x \mapsto ab \otimes x$:*

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_R D_m \xrightarrow{\sim} \mathcal{O}S_m^{\mathrm{PD}} \otimes_{A_R^+} N[1/p]. \tag{3.25}$$

Proof. The map in (3.25) is obviously compatible with the respective actions of Γ_R, so we need to check that it is bijective. Let us first check the injectivity of (3.25). We have a composition of injective homomorphisms $\mathcal{O}S_m^{\mathrm{PD}}[1/p] \xrightarrow{\varphi^m} \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}[1/p] \to \mathcal{O}B_{\mathrm{cris}}(\overline{R})$. As D_m is finite projective over $R[1/p]$, the map

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_R D_m = \mathcal{O}S_m^{\mathrm{PD}}[1/p] \otimes_{R[1/p]} D_m \longrightarrow \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, R[1/p]} D_m, \tag{3.26}$$

is injective. Next, we have $V = T[1/p]$ and we consider the following composition,

$$\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, R[1/p]} D_m \xrightarrow{1\otimes\varphi^m} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}D_R \longrightarrow \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V). \tag{3.27}$$

As $R[1/p] \to \mathcal{O}B_{\mathrm{cris}}(\overline{R})$ is faithfully flat (see [Bri08, Théorème 6.3.8]) and (3.24) is injective, so in (3.27), the second map is injective and the first map is injective because $1 \otimes \varphi^m : R[1/p] \otimes_{\varphi^m, R[1/p]} D_m \to \mathcal{O}D_R$ is injective, in particular, we see that (3.27) is injective. Moreover, since $N[1/p]$ is a finite projective B_R^+-module, therefore, similar to (3.26), it can be shown that the map $\mathcal{O}S_m^{\mathrm{PD}} \otimes_{A_R^+} N[1/p] = \mathcal{O}S_m^{\mathrm{PD}}[1/p] \otimes_{B_R^+} N[1/p] \longrightarrow \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, B_R^+} N[1/p]$ is injective. Furthermore, from the definition of Wach modules (see Definition 3.8), we have an isomorphism $1 \otimes \varphi : B_R^+ \otimes_{\varphi, B_R^+} N[1/p, 1/[p]_q] \xrightarrow{\sim} N[1/p, 1/[p]_q]$. Therefore, iterating it m times and by extending scalars along $B_R^+ \to \mathcal{O}B_{\mathrm{cris}}(\overline{R})$, we obtain an isomorphism $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, B_R^+} N[1/p] \xrightarrow{\sim} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{B_R^+} N[1/p]$, since $[p]_q$ is unit in $\mathcal{O}B_{\mathrm{cris}}(\overline{R})$. So, from the preceding observations, it follows that the following composition,

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_{A_R^+} N[1/p] \longrightarrow \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, B_R^+} N[1/p] \underset{\sim}{\xrightarrow{1\otimes\varphi^m}} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{B_R^+} N[1/p], \tag{3.28}$$

is injective. Let us now consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{O}S_m^{\mathrm{PD}} \otimes_R D_m & \xrightarrow{(3.26)} & \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\varphi^m, R[1/p]} D_m & \xrightarrow{(3.27)} & \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \\
{\scriptstyle (3.25)}\downarrow & & & & \downarrow \\
\mathcal{O}S_m^{\mathrm{PD}} \otimes_{A_R^+} N[1/p] & \xrightarrow{(3.28)} & \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{B_R^+} N[1/p] & \xrightarrow{(3.4)} & \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V,
\end{array}$$

where the right vertical arrow is the natural injective map (see [Bri08, Proposition 8.2.6]). From the definitions, it easily follows that the diagram commutes, therefore, we see that the left vertical arrow, i.e. (3.25) is injective.

Next, let us check the surjectivity of the map in (3.25). We define the following operators on $\mathcal{O}N_m^{\mathrm{PD}} := \mathcal{O}S_m^{\mathrm{PD}} \otimes_{A_R^+} N[1/p]$,

$$\partial_i := \begin{cases} -(\log \gamma_0)/t & \text{for } i = 0, \\ (\log \gamma_i)/(tV_i) & \text{for } 1 \leq i \leq d, \end{cases}$$

where $V_i = \frac{X_i \otimes 1}{1 \otimes [X_i^\flat]}$, for $1 \leq i \leq d$ (see [Abh21, §4.4.2]). Note that for any $g \in \Gamma_R$ and any $x \in \mathcal{O}S_m^{\mathrm{PD}} \otimes_{A_L^+} N$, we have $(g-1)(ax) = (g-1)a \cdot x + g(a)(g-1)x$. Then, from the identity $\log(\gamma_i) = \lim_{n \to +\infty} (\gamma_i^{p^n} - 1)/p^n$, it easily follows that the operators ∂_i satisfy the Leibniz rule for all $0 \leq i \leq d$. In particular, the operator $\partial : \mathcal{O}N_m^{\mathrm{PD}} \to \mathcal{O}N_m^{\mathrm{PD}} \otimes_{\mathcal{O}S_m^{\mathrm{PD}}} \Omega^1_{\mathcal{O}S_m^{\mathrm{PD}}/R}$, given by $x \mapsto \partial_0(x)dt + \sum_{i=1}^d \partial_i(x) d[X_i^\flat]$, defines a connection on $\mathcal{O}N_m^{\mathrm{PD}}$. Furthermore, from [Abh21, Lemma 4.38] the operators ∂_i commute with each other, so the connection ∂ is integrable and using the finite $[p]_q$-height property of N, similar to [Abh21, Lemma 4.39], it is easy to show that ∂ is p-adically quasi-nilpotent. Now, similar to the proof of [Abh21, Lemma 4.39 & Lemma 4.41], it follows that for $x \in N[1/p]$, the following sum converges in $D_m = (\mathcal{O}N_m^{\mathrm{PD}})^{\Gamma_R} = (\mathcal{O}N_m^{\mathrm{PD}})^{\partial=0}$:

$$y = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \partial_0^{k_0} \circ \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x) \frac{t^{[k_0]}}{p^{mk_0}} (1-V_1)^{[k_1]} \cdots (1-V_d)^{[k_d]}. \tag{3.29}$$

Using the construction above we define an $\mathcal{O}S_m^{\mathrm{PD}}[1/p]$-linear transformation α on the finite projective module $\mathcal{O}N_m^{\mathrm{PD}}$ and claim that α is an automorphism of $\mathcal{O}N_m^{\mathrm{PD}}$. Indeed, let us first choose a presentation $\mathcal{O}N_m^{\mathrm{PD}} \oplus N' = (\mathcal{O}S_m^{\mathrm{PD}})^r$, for some $r \in \mathbb{N}$. Then, on a chosen basis of $(\mathcal{O}S_m^{\mathrm{PD}})^r$, we can define a linear transformation β using (3.29) over $\mathcal{O}N_m^{\mathrm{PD}}$ and the identity on N'. Note that the transformation β preserves $\mathcal{O}N_m^{\mathrm{PD}}$ and we set $\det \alpha = \det \beta$, which is independent of the chosen presentation (see [Gol61, Proposition 1.2]). Now by an argument similar to the proof of [Abh21, Lemma 4.43], it easily follows that for some $N \in \mathbb{N}$ large enough, one can write $p^N \det \alpha = p^N \det \beta \in 1 + J^{[1]}\mathcal{O}S_m^{\mathrm{PD}}$, in particular, we get that $\det \alpha$ is a unit in $\mathcal{O}S_m^{\mathrm{PD}}[1/p]$, so α defines an automorphism of $\mathcal{O}N_m^{\mathrm{PD}}$ (see [Gol61, Proposition 1.3]). Since the formula considered in (3.29) converges in D_m, we conclude that the natural map $\mathcal{O}S_m^{\mathrm{PD}} \otimes_R D_m \to \mathcal{O}S_m^{\mathrm{PD}} \otimes_{A_R^+} N[1/p]$, is surjective. Hence, (3.25) is bijective, proving the lemma. ∎

Recall that $\mathcal{O}D_R$ is an $R[1/p]$-module equipped with an integrable connection and it is finite over $R[1/p]$ since we have an inclusion $\mathcal{O}D_R \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ of $R[1/p]$-modules from (3.24). In particular, we see that $\mathcal{O}D_R$ is a finite projective module over $R[1/p]$ by [Bri08, Proposition 7.1.2]. Moreover, $\mathcal{O}D_R$ is equipped with a Frobenius-semilinear operator φ. Now consider the following diagram:

$$\begin{array}{ccc} \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{\varphi^m,R} D_m & \xrightarrow{1\otimes\varphi^m} & \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}D_R \\ {\scriptstyle (3.25)}\downarrow\wr & & {\scriptstyle (3.23)}\downarrow \\ \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{\varphi^m,A_R^+} N[1/p] & \xrightarrow{\sim} & \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} N[1/p], \end{array} \tag{3.30}$$

where the left vertical arrow is the extension along $\varphi^m : \mathcal{O}S_m^{\mathrm{PD}} \to \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ of the isomorphism (3.25) in Lemma 3.37 and the bottom horizontal isomorphism follows from an argument similar to [Abh21, Lemma 4.46]. By the description of the arrows, it follows that the diagram is (φ, Γ_R)-equivariant and commutative. Taking Γ_R-invariants of the diagram (3.30), we obtain an isomorphism of $R[1/p]$-modules $1 \otimes \varphi^m : R \otimes_{\varphi^m,R[1/p]} D_m \xrightarrow{\sim} \mathcal{O}D_R$. In particular, it follows that the top horizontal arrow of (3.30) is an isomorphism. Hence, we conclude that the right vertical arrow of (3.30) is bijective as well, in particular, the comparison in (3.23) is an isomorphism compatible with the respective Frobenii, connections and actions of Γ_R. This finishes our proof. ∎

Remark 3.38. Let us make an observation that will be useful for the proof of Theorem 5.6. In the basis $\{d\log(X_1), \ldots, d\log(X_d)\}$ of Ω_R^1, let $\partial_{A,i}$ denote the i^{th} component of the connection on $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$, for $1 \leq i \leq d$, and let $\partial_{D,i}$ denote the induced operator on $\mathcal{O}D_R$. Moreover, employing arguments similar to [Abh23b, Lemmas 4.12, 5.17 & 5.18], we can show that, for $1 \leq i \leq d$, the operator $\nabla_i = (\log \gamma_i)/t = \frac{1}{t}\sum_{k\in\mathbb{N}}(-1)^k \frac{(\gamma_i-1)^{k+1}}{k+1}$ converges as a series of operators on $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_L^+} N$. Now, using (3.29) and the top horizontal arrow in diagram (3.30), we note that for any $x \in N[1/p]$, there exists $w \in \mathcal{O}D_R$ and $z \in (\mathrm{Fil}^1\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_{A_R^+} N[1/p]$, such that $x = f(w) + z$, where f is the isomorphism in (3.23). Then an easy computation shows that $\nabla_i(x) - f(\partial_{D,i}(w)) = \nabla_i(z) + \partial_{A,i}(z) \in (\mathrm{Fil}^1\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_{A_R^+} N[1/p]$.

4. Crystalline implies finite height

The goal of this section is to prove the following claim:

Theorem 4.1. *Let T be a finite free $\mathbb{Z}_p$-representation of G_R such that $V := T[1/p]$ is a p-adic crystalline representation of G_R. Then there exists a unique Wach module $\mathbf{N}_R(T)$ over A_R^+ attached to T. In other words, T is of finite $[p]_q$-height.*

Proof. For a p-adic representation, the property of being crystalline and of finite $[p]_q$-height is invariant under twisting the representation by χ^r, where χ is the p-adic cyclotomic character and $r \in \mathbb{N}$. Therefore, we can assume that V is positive crystalline. Note that V is also a positive crystalline representation of G_L, and therefore, it is also positive and of finite $[p]_q$-height as a p-adic representation of G_L (see [Abh23a, Definition 3.7]). In particular, we also get that T is positive and of finite $[p]_q$-height as a $\mathbb{Z}_p$-representation of G_L. Moreover, associated to T, from loc. cit. we have the Wach module $\mathbf{N}_L(T)$ over A_L^+ and we set $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T) \subset \mathbf{D}_L(T)$ as an A_R^+-module. From Proposition 4.7, the module $\mathbf{N}_R(T)$ satisfies all the axioms of Definition 3.8 and Definition 3.20. Hence, it follows that $\mathbf{N}_R(T)$ is the unique Wach module attached to T, or equivalently, T is of finite $[p]_q$-height. ∎

Remark 4.2. From Theorem 4.1, note that T is a $\mathbb{Z}_p$-representation of G_L such that $V := T[1/p]$ is crystalline for G_L. Then, from [Abh23a, Theorem 4.1] it follows T is of finite $[p]_q$-height as a representation of G_L, i.e. there exists a unique Wach module $\mathbf{N}_L(T)$ over A_L^+ attached to T. Moreover, note that $A_L^+ \otimes_{A_R^+} \mathbf{N}_R(T)$ is also a Wach module over A_L^+ attached to T, where we use $\Gamma_L \xrightarrow{\sim} \Gamma_R$. Now, using Proposition 4.12, we have that $A_L \otimes_{A_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} A_L \otimes_{A_R} \mathbf{D}_R(T) \xrightarrow{\sim} \mathbf{D}_L(T)$ as étale (φ, Γ_L)-modules over A_L. Hence, by the uniqueness of the Wach module attached to T over A_L^+ in [Abh23a, Lemma 3.9] it follows that $A_L^+ \otimes_{A_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{N}_L(T)$ as (φ, Γ_L)-modules over A_L^+.

4.1. Consequences of Theorem 4.1. Let $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_R)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_R. Then, by combining Theorem 3.34 and Theorem 4.1, we obtain the following:

Corollary 4.3. *The Wach module functor induces an equivalence of categories*

$$\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{A_R^+}$$
$$T \longmapsto \mathbf{N}_R(T),$$

with a quasi-inverse given as $N \mapsto \mathbf{T}_R(N) := (W(\overline{R}^\flat[1/p^\flat]) \otimes_{A_R^+} N)^{\varphi=1}$.

Passing to associated isogeny categories, we obtain the following:

Corollary 4.4. *The Wach module functor induces an exact equivalence of* $\otimes$*-categories* $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-}\mathrm{Mod}^{[p]_q}_{B_R^+}$, *via* $V \mapsto \mathbf{N}_R(V)$, *and with an exact* $\otimes$*-compatible quasi-inverse given as* $M \mapsto \mathbf{V}_R(M) := (W(\overline{R}^\flat[1/p^\flat]) \otimes_{A_R^+} M)^{\varphi=1}$.

Proof. The equivalence of categories follows from Theorem 4.1. For the rest of the proof, let us remark that for a p-adic crystalline representation V of G_R, from Proposition 4.7, we have $\mathbf{N}_R(V) = \mathbf{N}_L(V) \cap \mathbf{D}_R(V) \subset \mathbf{D}_L(V)$ as finite projective (φ, Γ_R)-modules over B_R^+. Moreover, from Proposition 4.12 and Remark 4.2, note that $B_L^+ \otimes_{B_R^+} \mathbf{N}_R(V) \xrightarrow{\sim} \mathbf{N}_L(V)$ and $B_R \otimes_{B_R^+} \mathbf{N}_R(V) \xrightarrow{\sim} \mathbf{D}_R(V)$ compatible with resepctive natural actions of (φ, Γ_R).

Now, let V_1 and V_2 be two crystalline representations of G_R, then $V_1 \otimes_{\mathbb{Q}_p} V_2$ is again crystalline (see [Bri08, Théorèm 8.4.2]). We have

$$\begin{aligned}
\mathbf{N}_R(V_1) \otimes_{B_R^+} \mathbf{N}_R(V_2) &= \mathbf{N}_R(V_1) \otimes_{B_R^+} (\mathbf{N}_L(V_2) \cap \mathbf{D}_R(V_2)) \\
&= (\mathbf{N}_R(V_1) \otimes_{B_R^+} \mathbf{N}_L(V_2)) \cap (\mathbf{N}_R(V_1) \otimes_{B_R^+} \mathbf{D}_R(V_2)) \\
&= (\mathbf{N}_L(V_1) \otimes_{B_L^+} \mathbf{N}_L(V_2)) \cap (\mathbf{D}_R(V_1) \otimes_{B_L^+} \mathbf{D}_R(V_2)) \\
&= \mathbf{N}_L(V_1 \otimes_{\mathbb{Q}_p} V_2) \cap \mathbf{D}_R(V_1 \otimes_{\mathbb{Q}_p} V_2) = \mathbf{N}_R(V_1 \otimes V_2),
\end{aligned}$$

where the first equality follows from the discussion above, the second equality follows since $\mathbf{N}_R(V_1)$ is projective, the third equality again follows from the discussion above and the last equality follows from [Abh23a, Corollary 4.3] and (2.8). This shows the compatibility of $\mathbf{N}_R$ with tensor products. Conversely, let N_1 and N_2 be two Wach modules over A_R^+ and set $N_3 := (N_1 \otimes_{A_R^+} N_2)/(p\text{-torsion})$ as a finitely generated A_R^+-module. Then, note that we have $N_3 \subset N_3[1/p] = N_1[1/p] \otimes_{B_R^+} N_2[1/p]$, where the right-hand term is a projective B_R^+-module. Therefore, N_3 is torsion free and by definition N_3/μ is also p-torsion free, in particular, the sequence $\{p, \mu\}$ is strictly N_3-regular by Remark 3.2. Furthermore, assumptions for the (φ, Γ_R)-action on N_3, as in Definition 3.8, can be verified similar to [Abh21, Proposition 4.14]. So it follows that N_3 is a Wach module over A_R^+. Since, $N_3[1/p] = N_1[1/p] \otimes_{B_R^+} N_2[1/p]$, compatibility of the functor $\mathbf{V}_R$ with tensor products now follows from (2.8).

It remains to show the exactness of $\mathbf{N}_R$ since exactness of the quasi-inverse functor $\mathbf{V}_R$ follows from Proposition 3.15 and the exact equivalence in (2.8). So, let us consider an exact sequence of p-adic crystalline representations of G_R as $0 \to V_1 \to V_2 \to V_3 \to 0$, and we wish to show that the sequence

$$0 \longrightarrow \mathbf{N}_R(V_1) \longrightarrow \mathbf{N}_R(V_2) \longrightarrow \mathbf{N}_R(V_3) \longrightarrow 0, \tag{4.1}$$

is exact. Let $T_2 \subset V_2$ be a G_R-stable $\mathbb{Z}_p$-lattice, then $T_1 := V_1 \cap T_2 \subset V_2$ is a G_R-stable $\mathbb{Z}_p$-lattice inside V_1 and set $T_3 := T_2/T_1 \subset V_3$ as a G_R-stable $\mathbb{Z}_p$-lattice. By definition, we have Wach modules $\mathbf{N}_R(T_1)$, $\mathbf{N}_R(T_2)$ and $\mathbf{N}_R(T_3)$ and we set $N := \mathbf{N}_R(T_2)/\mathbf{N}_R(T_1)$ as a finitely generated A_R^+-module equipped with a Frobenius $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ and a continuous action of Γ_R induced from the corresponding structures on $\mathbf{N}_R(T_2)$. We claim that $N[1/p] \xrightarrow{\sim} \mathbf{N}_R(V_3)$ as (φ, Γ_R)-modules over B_R^+.

Indeed, first recall that $\mathbf{D}_R$ is an exact functor from the category of $\mathbb{Z}_p$-representations of G_R to the category of étale (φ, Γ_R)-modules over A_R (see §2.6). So we get that the natural map $N = \mathbf{N}_R(T_2)/\mathbf{N}_R(T_1) \to$

$\mathbf{N}_R(T_3)$ is injective, and since $A_R^+ \to A_R$ is flat, therefore, we have $A_R \otimes_{A_R^+} N \xrightarrow{\sim} \mathbf{D}_R(T_2)/\mathbf{D}_R(T_1) \xrightarrow{\sim} \mathbf{D}_R(T_3) \xleftarrow{\sim} A_R \otimes_{A_R^+} \mathbf{N}_R(T_3)$ as étale (φ, Γ_R)-modules over A_R. Moreover, since $\varphi : A_R^+ \to A_R^+$ is flat, therefore, using the finite $[p]_q$-height property of $\mathbf{N}_R(T_1)$ and $\mathbf{N}_R(T_2)$, we get that N is of finite $[p]_q$-height, i.e. $1 \otimes \varphi : (\varphi^* N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. In particular, $N[1/p]$ is finite projective over B_R^+ by Proposition A.1. Next, for $i \in \{1, 2, 3\}$, considering V_i as a p-adic crystalline representation of G_L, from [Abh23a, Corollary 4.3], we have an exact sequence $0 \to \mathbf{N}_L(V_1) \to \mathbf{N}_L(V_2) \to \mathbf{N}_L(V_3) \to 0$ of Wach modules over $B_L^+ = A_L^+[1/p]$. Note that the natural map $B_R^+ \to B_L^+$ is flat, so we get that $B_L^+ \otimes_{B_R^+} N[1/p] \xrightarrow{\sim} \mathbf{N}_L(V_3)$ as (φ, Γ_L)-modules over B_L^+. Moreover, from Remark 4.2, we have that $B_L^+ \otimes_{B_R^+} \mathbf{N}_R(V_i) \xrightarrow{\sim} \mathbf{N}_L(V_i)$, for $i \in \{1, 2, 3\}$. Now, since $N[1/p]$ is finite projective over B_R^+, therefore, as submodules of $\mathbf{D}_L(V_3)$, we obtain an isomorphism of (φ, Γ_R)-modules over B_R^+ as follows:

$$N[1/p] = (B_L^+ \otimes_{B_R^+} N[1/p]) \cap (B_R \otimes_{B_R^+} N[1/p]) \xrightarrow{\sim} \mathbf{N}_L(V_3) \cap \mathbf{D}_R(V_3) \xleftarrow{\sim} \mathbf{N}_R(V_3).$$

Hence, (4.1) is exact, concluding our proof. ■

We obtain applications of Theorem 4.1 as follows:

Theorem 4.5. *Let V be a p-adic representation of G_R. Then the following are equivalent:*

(1) *V is crystalline as a representation of G_R;*

(2) *V is crystalline as a representation of G_L;*

(3) $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$.

Proof. Let $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$, then obviously we have that $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_L)$. Conversely, let $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_L)$ and choose a G_R-stable $\mathbb{Z}_p$-lattice $T \subset V$ such that T is of finite $[p]_q$-height as a representation of G_L. Then, using Proposition 4.7, note that T is of finite $[p]_q$-height as a representation of G_R. Therefore, $V = T[1/p]$ is a crystalline representation of G_R by Theorem 3.34. This shows the equivalence of (1) and (2).

Next, if $V \in \mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_R)$, then $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$ (see §2.6), proving that (1) implies (3). So it remains to show that (3) implies (2). Let V be a p-adic representation of G_R such that $\mathrm{rk}_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$. From [Bri06, §3.3] recall that V is crystalline for G_L if and only if $\dim_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = \dim_{\mathbb{Q}_p} V$. So we will show that $\dim_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = \dim_{\mathbb{Q}_p} V$ by constructing a natural isomorphism of L-vector spaces $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Since $\dim_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \leq \dim_{\mathbb{Q}_p} V$, it is enough to construct a natural L-linear injective map $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \to \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and the claim would follow by considering L-dimensions.

From Remark 2.16, note that we have a natural (φ, G_R)-equivariant L-linear injective map $L \otimes_{R[1/p]} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p} \in \mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))$. Tensoring this map with V and considering the diagonal action of G_R, we obtain a (φ, G_R)-equivariant injective map

$$L \otimes_{R[1/p]} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V \longrightarrow \Big(\prod_{\mathfrak{p} \in \mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))\Big) \otimes_{\mathbb{Q}_p} V = \prod_{\mathfrak{p} \in \mathcal{S}} (\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V). \tag{4.2}$$

The map in (4.2) further induces a natural map $L \otimes_{R[1/p]} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V \to \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V$, compatible with the respective Frobenii, filtrations and connections (see Remark 2.16). Now, we take the G_R-invariant part of (4.2) and note that product commutes with the left exact functors, in particular, with taking G_R-invariants. So we obtain φ-equivariant L-linear injective maps

$$\begin{aligned} L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) &\longrightarrow \Big(\prod_{\mathfrak{p} \in \mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V\Big)^{G_R} \\ &= \prod_{\mathfrak{p} \in \mathcal{S}} (\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R} \\ &\longrightarrow \prod_{\mathfrak{p} \in \mathcal{S}} (\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_{R(\mathfrak{p})}}, \end{aligned} \tag{4.3}$$

where note that the last arrow is injective since $G_R(\mathfrak{p}) \subset G_R$ is a subgroup. Moreover, since G_R acts transitively on $\mathcal{S}$, it transitively permutes the components of $\prod_{\mathfrak{p}\in\mathcal{S}}(\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})}$, i.e. if $0 \neq x \in L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, then its image $(x_\mathfrak{p})_{\mathfrak{p}\in\mathcal{S}}$ under the composition (4.3) satisfies that $x_\mathfrak{p} \neq 0$, for all $\mathfrak{p} \in \mathcal{S}$. Therefore, for each $\mathfrak{p} \in \mathcal{S}$, composing (4.3) with the natural φ-equivariant L-linear projection $\prod_{\mathfrak{p}\in\mathcal{S}}(\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})} \to (\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})}$ gives a natural φ-equivariant L-linear injective map

$$L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \longrightarrow (\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})}, \tag{4.4}$$

compatible with the respective Frobenii, filtrations and connections (see above and Remark 2.16), where the left-hand term is equipped with the tensor product Frobenius, the filtration on $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ and natural connection.

Finally, from Lemma 2.13, recall that we have a natural L-linear $(\varphi, \widehat{G}_R(\mathfrak{p}))$-equivariant injective map $\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+)$ compatible with respective filtrations and connections and where the $\widehat{G}_R(\mathfrak{p})$-action on the left term factors through $\widehat{G}_R(\mathfrak{p}) \twoheadrightarrow G_R(\mathfrak{p})$. Tensoring the preceding injective map with V, equipping each term with the diagonal action of $\widehat{G}_R(\mathfrak{p})$ and taking $\widehat{G}_R(\mathfrak{p})$-invariants produces a natural L-linear injective map $(\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p}))\otimes_{\mathbb{Q}_p} V)^{G_R(\mathfrak{p})} \to \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, compatible with the respective Frobenii, filtrations and connections. Composing (4.4) with the preceding L-linear map gives a natural L-linear injective map

$$L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \longrightarrow \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \tag{4.5}$$

compatible with the respective Frobenii, filtrations and connections. By considering L-dimensions, it follows that (4.5) is bijective (see Corollary 4.6 for a stronger statement). Hence, $\dim_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = \mathrm{rk}_{R[1/p]}\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) = \dim_{\mathbb{Q}_p} V$, showing that (3) implies (2). This concludes our proof. ∎

Corollary 4.6. *Let V be a p-adic representation of G_R. Under the equivalent conditions of Theorem 4.5, the map in (4.5) induces a natural isomorphism of filtered (φ, ∂)-modules over L.*

Proof. Assume that V is a crystalline representation of G_R, so that we have a natural $\mathcal{O}B_{\mathrm{cris}}(\overline{R})$-linear isomorphism $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{O}B_{\mathrm{cris}}(\overline{R}) \otimes_{\mathbb{Q}_p} V$, compatible with the respective Frobenii, filtrations, connections and G_R-actions (see [Bri08, Proposition 8.4.3]). For any $\mathfrak{p} \in \mathcal{S}$, by base changing the preceding isomorphism along the composition $\mathcal{O}B_{\mathrm{cris}}(\overline{R}) \to \prod_{\mathfrak{p}\in\mathcal{S}} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}^+(\mathfrak{p})) \to \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+)$, we get a $\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+)$-linear isomorphism

$$\mathcal{O}B_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{\sim} \mathcal{O}B_{\mathrm{cris}}(\mathbb{C}_\mathfrak{p}^+) \otimes_{\mathbb{Q}_p} V, \tag{4.6}$$

compatible with the respective Frobenii, filtrations, connections and $\widehat{G}_R(\mathfrak{p})$-actions. In (4.6), by taking $\widehat{G}_R(\mathfrak{p})$-invariants we get (4.5), i.e. $L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, and by construction, the preceding isomorphism is compatible with the respective Frobenii, filtrations and connections. Hence, the claim follows. ∎

4.2. Main ingredients for the proof of Theorem 4.1.

In this subsection, let T be a finite free $\mathbb{Z}_p$-representation of G_R such that T is a finite $[p]_q$-height representation of G_L (see Definition 3.20 and [Abh23a, Definition 3.7]). In particular, we can attach to T a (φ, Γ_R)-module $\mathbf{D}_R(T)$ over A_R, as well as, a Wach module $\mathbf{N}_L(T)$ over A_L^+. Our goal is to prove the following claim:

Proposition 4.7. *The A_R^+-module $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T) \subset \mathbf{D}_L(T)$ satisfies all the axioms of Definition 3.20. In particular, T is a finite $[p]_q$-height representation of G_R.*

Proof. It is immediate that $\mathbf{N}_R(T)$ is p-torsion free and μ-torsion free. From Lemma 4.8 and its proof, note that $\mathbf{N}_R(T)$ is finitely generated over A_R^+ and we have that $\mathbf{N}_R(T)/p \subset (\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p) \subset \mathbf{D}_L(T)/p$, in particular, $\mathbf{N}_R(T)/p$ is μ-torsion free. Next, from Lemma 4.9, we know that $\mathbf{N}_R(T)$ is of finite $[p]_q$-height, i.e. the cokernel of the injective map $1\otimes\varphi : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$, where s is the height of $\mathbf{N}_L(T)$. Furthermore, from Proposition 4.12 we have that $A_R \otimes_{A_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$. Finally, recall that the action of Γ_L is trivial on $\mathbf{N}_L(T)/\mu\mathbf{N}_L(T)$ and $\Gamma_L \xrightarrow{\sim} \Gamma_R$, so for any $g \in \Gamma_R$, we have $(g-1)\mathbf{N}_L(T) \subset \mu\mathbf{N}_L(T)$. Therefore, we get that $(g-1)\mathbf{N}_R(T) \subset (\mu\mathbf{N}_L(T)) \cap \mathbf{D}_R(T) = \mu\mathbf{N}_R(T)$, so it follows that Γ_R acts trivially on $\mathbf{N}_R(T)/\mu\mathbf{N}_R(T)$. This concludes our proof. ∎

Lemma 4.8. *The A_R^+-module $\mathbf{N}_R(T) := \mathbf{N}_L(T) \cap \mathbf{D}_R(T)$ is finitely generated.*

Proof. We first claim that for each $n \in \mathbb{N}_{\geq 1}$, the natural A_R^+/p^n-linear map $\mathbf{N}_R(T)/p^n \to (\mathbf{N}_L(T)/p^n) \cap (\mathbf{D}_R(T)/p^n) \subset \mathbf{D}_L(T)/p^n$ is injective and the intersection $(\mathbf{N}_L(T)/p^n) \cap (\mathbf{D}_R(T)/p^n)$ is a finitely generated A_R^+/p^n-module. Since $\mathbf{N}_R(T)$, $\mathbf{N}_L(T)$ and $\mathbf{D}_R(T)$ are p-torsion free, it is enough to show the claim for $n = 1$ and the claim for $n \geq 1$ can be deduced by an easy induction. So we are reduced to showing that the natural map $\mathbf{N}_R(T)/p \to (\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p) \subset \mathbf{D}_L(T)/p$ is injective and the module $(\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p)$ is finitely generated over $A_R^+/p =: E_R^+$. Note that we have $p\mathbf{N}_L(T) \cap \mathbf{N}_R(T) \hookrightarrow p\mathbf{D}_L(T) \cap \mathbf{D}_R(T) = p\mathbf{D}_R(T)$, so we get that $p\mathbf{N}_L(T) \cap \mathbf{N}_R(T) \hookrightarrow p\mathbf{N}_L(T) \cap p\mathbf{D}_R(T) = p\mathbf{N}_R(T)$, in particular, $\mathbf{N}_R(T)/p \hookrightarrow \mathbf{N}_L(T)/p$. Similarly, we have $p\mathbf{D}_R(T) \cap \mathbf{N}_R(T) \hookrightarrow p\mathbf{D}_L(T) \cap \mathbf{N}_L(T) = p\mathbf{N}_L(T)$, so we get that $p\mathbf{D}_R(T) \cap \mathbf{N}_R(T) \hookrightarrow p\mathbf{D}_R(T) \cap p\mathbf{N}_L(T) = p\mathbf{N}_R(T)$, in particular, $\mathbf{N}_R(T)/p \hookrightarrow \mathbf{D}_R(T)/p$.

Next, we will show that $(\mathbf{N}_L(T)/p) \cap (\mathbf{D}_R(T)/p)$ is a finitely generated E_R^+-module. Assume that $\overline{D}_R := \mathbf{D}_R(T)/p$ is finite free (a priori it is finite projective) of rank h over $E_R := A_R/p$. Let $\mathbf{e} = \{e_1, \ldots, e_h\}$ be a basis of $\overline{N}_L := \mathbf{N}_L(T)/p$ over $E_L^+ := A_L^+/p$ and $\mathbf{f} = \{f_1, \ldots, f_h\}$ a basis of $\overline{D}_R$ over E_R. Then, for $E_L := A_L/p$, we have $\mathbf{f} = A\mathbf{e}$, for some $A := (a_{ij}) \in \mathrm{GL}(h, E_L)$, and write $A^{-1} = (b_{ij}) \in \mathrm{GL}(h, E_L)$. Set $M := \oplus_{i=1}^h E_R^+ f_i$, so that $M[1/\mu] = \overline{D}_R$. Let $x \in M[1/\mu] \cap \overline{N}_L$ and write $x = \sum_{i=1}^h c_i e_i = \sum_{i=1}^h d_i f_i$ with $c_i \in E_L^+$ and $d_i \in E_R$, for all $1 \leq i \leq h$. So we obtain that $d_i = \sum_{j=1}^h b_{ji} c_j$, for all $1 \leq i \leq h$. In particular, for some k large enough, we have $d_i \in \mu^{-k} E_L^+$, for all $1 \leq i \leq h$. Note that $\mu^{-k} E_L^+ \cap E_R = \mu^{-k} E_R^+$, so we obtain that $d_i \in \mu^{-k} E_R^+$. Hence, $M[1/\mu] \cap \overline{N}_L \subset \mu^{-k} M$, in particular, $M[1/\mu] \cap \overline{N}_L = \overline{D}_R \cap \overline{N}_L$ is finitely generated over E_R^+.

In general, when $\overline{D}_R$ is finite projective, we choose an E_R^+-module D' such that $\overline{D}_R \oplus D' = E_R^{\oplus k}$, for some $k \in \mathbb{N}$. Let $D'_L := E_L \otimes_{E_R} \overline{D}_R$, so we have that $\overline{D}_L \oplus D'_L = E_L^{\oplus k}$. Note that since E_L is a field with ring of integers E_L^+, therefore, we can choose a lattice of D'_L over E_L^+, i.e. there exists a free E_L^+-submodule $N'_L \subset D'_L$ such that $N'_L[1/\mu] = D'_L$. So, we get that $\overline{N}_L \oplus N'_L$ is a free E_L^+-module such that $E_L \otimes_{E_L^+} (\overline{N}_L \oplus N'_L) = \overline{D}_L \oplus D'_L = E_L^{\oplus k}$. Inside $E_L^{\oplus k}$, consider the inclusion of E_R^+-modules

$$(\overline{D}_R \cap \overline{N}_L) \oplus (D' \cap N'_L) = (\overline{D}_R \oplus D') \cap (\overline{N}_L \oplus N'_L) \subset E_R^{\oplus k} \cap (\overline{N}_L \oplus N'_L). \tag{4.7}$$

Using the conclusion in the free case from the previous paragraph, we get that the last term in (4.7) is a finite E_R^+-module. Hence, $\overline{D}_R \cap \overline{N}_L$ is also a finite E_R^+-module, proving the claim.

To prove the lemma, it remains to show that $\mathbf{N}_R(T)$ is p-adically complete. Indeed, from the claim above note that, for all $n \in \mathbb{N}_{\geq 1}$, $\mathbf{N}_R(T)/p^n$ is a finitely generated A_R^+/p^n-module. Since A_R^+ is noetherian, therefore, for each $n \in \mathbb{N}$ and $k \in \mathbb{N}$ as in the previous paragraph, we have a presentation $0 \to M_n \to (A_R^+/p^n)^{\oplus k} \to \mathbf{N}_R(T)/p^n \to 0$, where M_n is a finitely generated A_R^+/p^n-module. By taking a finite presentation of M_n as an A_R^+/p^n-module, it is easy to see that the system $\{M_n\}_{n \in \mathbb{N}_{\geq 1}}$ is Mittag-Leffler. In particular, it follows that $\lim_n \mathbf{N}_R(T)/p^n$ is a finitely generated module over $\lim_n A_R^+/p^n = A_R^+$. Now consider the following natural A_R^+-linear maps:

$$\begin{aligned} f : \mathbf{N}_R(T) \longrightarrow \lim_n \mathbf{N}_R(T)/p^n &\longrightarrow \lim_n((\mathbf{N}_L(T)/p^n) \cap (\mathbf{D}_R(T)/p^n)) \\ &\longrightarrow (\lim_n \mathbf{N}_L(T)/p^n) \cap (\lim_n \mathbf{D}_R(T)/p^n) \\ &\xrightarrow{\sim} \mathbf{N}_L(T) \cap \mathbf{D}_R(T) = \mathbf{N}_R(T), \end{aligned}$$

where the first arrow is the natural projection map, the second arrow is injective by the claim proved above, the third arrow is injective by definition and the fourth arrow is bijective since $\mathbf{N}_L(T)$ and $\mathbf{D}_R(T)$ are p-adically complete. Chasing an element of $x \in \mathbf{N}_R(T)$ through the composition, we see that $f(x) = x$. Hence, we get that $\mathbf{N}_R(T) \xrightarrow{\sim} \lim_n \mathbf{N}_R(T)/p^n$, in particular, it is a finitely generated A_R^+-module. ∎

Lemma 4.9. *The A_R^+-module $\mathbf{N}_R(T)$ is of finite $[p]_q$-height, i.e. the cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$, for some $s \in \mathbb{N}$.*

Proof. Note that $\varphi : A_R^+ \to A_R^+$ is finite and faithfully flat of degree p^{d+1} (see §2.2). Moreover, from §2.2 we have that $\varphi^*(A_R) \xrightarrow{\sim} A_R^+ \otimes_{\varphi, A_R^+} A_R$ and $\varphi^*(A_L^+) := A_L^+ \otimes_{\varphi, A_L^+} A_L^+ \xrightarrow{\sim} \oplus_\alpha \varphi(A_L^+) u_\alpha = (\oplus_\alpha \varphi(A_R^+) u_\alpha) \otimes_{\varphi(A_R^+)} \varphi(A_L^+) \xleftarrow{\sim} A_R^+ \otimes_{\varphi, A_R^+} A_L^+$. Therefore, we also obtain that $\varphi^*(\mathbf{N}_L(T)) := A_L^+ \otimes_{\varphi, A_L^+}$

$\mathbf{N}_L(T) \xrightarrow{\sim} A_R^+ \otimes_{\varphi, A_R^+} \mathbf{N}_L(T)$ and $\varphi^*(\mathbf{D}_R(T)) := A_R \otimes_{\varphi, A_R} \mathbf{D}_R(T) \xrightarrow{\sim} A_R^+ \otimes_{\varphi, A_R^+} \mathbf{D}_R(T)$. Hence, as A_R^+-submodules of $\varphi^*(\mathbf{D}_L(T))$, we have that

$$\begin{aligned}\varphi^*(\mathbf{N}_R(T)) &:= A_R^+ \otimes_{\varphi, A_R^+} \mathbf{N}_R(T) = A_R^+ \otimes_{\varphi, A_R^+} (\mathbf{N}_L(T) \cap \mathbf{D}_R(T)) \\ &= (A_R^+ \otimes_{\varphi, A_R^+} \mathbf{N}_L(T)) \cap (A_R^+ \otimes_{\varphi, A_R^+} \mathbf{D}_R(T)) \xrightarrow{\sim} \varphi^*(\mathbf{N}_L(T)) \cap \varphi^*(\mathbf{D}_R(T)).\end{aligned}$$

Since the cokernel of the injective map $(1 \otimes \varphi) : \varphi^*(\mathbf{N}_L(T)) \to \mathbf{N}_L(T)$ is killed by $[p]_q^s$, for some $s \in \mathbb{N}$, and $(1 \otimes \varphi) : \varphi^*(\mathbf{D}_R(T)) \xrightarrow{\sim} \mathbf{D}_R(T)$, therefore, it easily follows that the cokernel of $(1 \otimes \varphi) : \varphi^*(\mathbf{N}_R(T)) \to \mathbf{N}_R(T)$ is killed by $[p]_q^s$ as well. ∎

Finally, we will show that $A_L^+ \otimes_{A_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{N}_L(T)$ and $A_R \otimes_{A_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$, using an approach parallel to [DLMS22, Proposition 4.24 & Lemma 4.25]. For $n \in \mathbb{N}_{\geq 1}$, let $N_{R,n} := \mathbf{N}_R(T)/p^n$, $D_{R,n} := \mathbf{D}_R(T)/p^n$, $N_{L,n} := \mathbf{N}_L(T)/p^n$, $D_{L,n} := \mathbf{D}_L(T)/p^n$ and $M_n := N_{L,n} \cap D_{R,n} \subset D_{L,n}$. Then have the following commutative diagram,

$$\begin{array}{ccc} M_n & \xrightarrow{f_n} & M_1 \\ \downarrow & & \downarrow \\ D_{R,n} & \overset{f_n}{\twoheadrightarrow} & D_{R,1}, \end{array}$$

where the vertical arrows are natural inclusions, the bottom horizontal arrow f_n is the natural projection map and the top arrow is the induced map. We have a similar diagram with the bottom row replaced by $N_{L,n} \twoheadrightarrow N_{L,1}$.

Lemma 4.10. *We have the following,*

(1) *M_n is a finitely generated A_R^+/p^n-module and $\mathbf{N}_R(T) \xrightarrow{\sim} \lim_n M_n$.*

(2) *M_n is of finite $[p]_q$-height s, for $s \in \mathbb{N}$ as in Lemma 4.9.*

(3) *$M_n[1/\mu] = A_R \otimes_{A_R^+} M_n \xrightarrow{\sim} D_{R,n}$ and $A_L^+ \otimes_{A_R^+} M_n \xrightarrow{\sim} N_{L,n}$.*

Proof. The claim in (1) follows from the proof of Lemma 4.8 and the claim in (2) follows similar to Lemma 4.9. As the maps $A_R^+ \to A_R$ and $A_R^+ \to A_L^+$ are flat, the last claim follows from the following equalities:

$$\begin{aligned} A_R \otimes_{A_R^+} M_n &= (A_R \otimes_{A_R^+} D_{R,n}) \cap (A_R \otimes_{A_R^+} N_{L,n}) = (A_R \otimes_{A_R^+} D_{R,n}) \cap (A_L^+ \otimes_{A_R^+} D_{R,n}) = D_{R,n}, \\ A_L^+ \otimes_{A_R^+} M_n &= (A_L^+ \otimes_{A_R^+} D_{R,n}) \cap (A_L^+ \otimes_{A_R^+} N_{L,n}) = (A_R \otimes_{A_R^+} N_{L,n}) \cap (A_L^+ \otimes_{A_R^+} N_{L,n}) = N_{L,n}. \end{aligned}$$

Hence, the lemma is proved. ∎

Let $\mathcal{S}$ denote the set of A_R^+-submodules $M' \subset M_1$ such that M' is stable under the action of φ, it is of finite $[p]_q$-height s and $M'[1/\mu] = M_1[1/\mu] = D_{R,1} = \mathbf{D}_R(T)/p$. Set $M^\circ := \cap_{M' \in \mathcal{S}} M' \subset M_1$.

Lemma 4.11. *The A_R^+-module M° belongs to $\mathcal{S}$ and $f_n(M_n)$ is also in $\mathcal{S}$, for all $n \in \mathbb{N}_{\geq 1}$.*

Proof. The idea of the proof is motivated from [DLMS22, Lemma 4.25]. Let M' be an element of $\mathcal{S}$. For the first claim, we need to show that there exists $r \in \mathbb{N}$ such that $\mu^r M_1 \subset M' \subset M_1$. Let $M'' := M_1/M'$ such that $M'' \neq 0$ and let $k = p(p-1)s \in \mathbb{N}$. Also, let $\varphi^*(M'') := \varphi^*(M_1)/\varphi^*(M')$ and let $1 \otimes \varphi_{M''} : \varphi^*(M'') \to M''$ denote the map induced from $1 \otimes \varphi_M$. Since M_1 (resp. M') is of finite $[p]_q$-height k (since $s < k$), we define $\psi_M : M_1 \xrightarrow{\mu^k} \mu^k M_1 \to \varphi^*(M_1)$ (resp. $\psi_{M'} : M' \xrightarrow{\mu^k} \mu^k M' \to \varphi^*(M')$) to be the unique A_R^+/p-linear map such that $\psi_M \circ (1 \otimes \varphi_M) = \mu^k \mathrm{Id}_{\varphi_M^*}$ (resp. $\psi_{M'} \circ (1 \otimes \varphi_{M'}) = \mu^k \mathrm{Id}_{\varphi_{M'}^*}$). Let $\psi_{M''} : M'' \to \varphi^*(M'')$ denote the map induced from ψ_M. Now, consider the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi^*(M') & \longrightarrow & \varphi^*(M_1) & \longrightarrow & \varphi^*(M'') & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\varphi_{M'}} & & \downarrow{\scriptstyle 1\otimes\varphi_{M}} & & \downarrow{\scriptstyle 1\otimes\varphi_{M''}} & & \\
0 & \longrightarrow & M' & \longrightarrow & M_1 & \longrightarrow & M'' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi_{M'}} & & \downarrow{\scriptstyle \psi_{M}} & & \downarrow{\scriptstyle \psi_{M''}} & & \\
0 & \longrightarrow & \varphi^*(M') & \longrightarrow & \varphi^*(M_1) & \longrightarrow & \varphi^*(M'') & \longrightarrow & 0.
\end{array}$$

Note that $[p]_q = \mu^{p-1} \mod p$, $\varphi(\mu) = \mu^p \mod p$ and $\varphi([p]_q) = \mu^{p(p-1)} \mod p$. Since $M_1[1/\mu] = M'[1/\mu]$, let $i \in \mathbb{N}_{\geq 1}$ such that $\mu^{pi}M'' = 0$ and $\mu^{p(i-1)}M'' \neq 0$. Let $x \in M''$ such that $\mu^{pi}x \neq 0$ and set $y = 1 \otimes x \in \varphi^*(M'')$. Then $\varphi(\mu^{pi})y = 1 \otimes \mu^{pi}x = 0$, but $\mu^{p^2(i-1)}y = \varphi(\mu^{p(i-1)})y = 1 \otimes \mu^{p(i-1)}x \neq 0$. Let $z = (1 \otimes \varphi_{M''})y \in M''$, then $\mu^{pi}z = 0$. So we have $0 = \psi_{M''}(\mu^{pi}z) = \mu^{pi}(\psi_{M''} \circ (1 \otimes \varphi_{M''})y) = \mu^{pi+k}y$. Therefore, we get that $pi + k = pi + p(p-1)s > p^2(i-1)$, i.e. $i < s + \frac{p}{p-1}$. Hence, $\mu^{s+1}M'' = 0$. Since the constant obtained is independent of M', we also get that $\mu^{s+1}M_1 \subset M^\circ \subset M_1$ and $M^\circ[1/\mu] = M_1[1/\mu]$.

Next, we will show that M° is of finite height s. Let $x \in M^\circ$, so that $x \in M'$ for each $M' \in \mathcal{S}$ and there exists some $y \in \varphi^*(M') \subset \varphi^*(M_1)$ such that $(1 \otimes \varphi)y = [p]_q^s x$. Note that y is unique in $\varphi^*(M_1)$ and since $\varphi : A_R^+ \to A_R^+$ is flat, we get that $y \in \cap_{M' \in \mathcal{S}}(A_R^+ \otimes_{\varphi, A_R^+} M') = A_R^+ \otimes_{\varphi, A_R^+} (\cap_{M' \in \mathcal{S}} M') = \varphi^*(M^\circ)$. Therefore, we concldue that $M^\circ \in \mathcal{S}$.

For the second part of the claim note that $M_n[1/\mu] = D_{R,n}$ and $f_n(D_{R,n}) = D_{R,n}/p = \mathbf{D}_R(T)/p$ (see Lemma 4.10). So we get that $f_n(M_n[1/\mu]) = \mathbf{D}_R(T)/p$ and we are left to show that $f_n(M_n)$ is of finite height s. Note that we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \varphi^*(\text{kernel}) & \longrightarrow & \varphi^*(M_n) & \longrightarrow & \varphi^*(f_n(M_n)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 1\otimes\varphi} & & \downarrow{\scriptstyle 1\otimes\varphi} & & \downarrow{\scriptstyle 1\otimes\varphi} & & \\
0 & \longrightarrow & \text{kernel} & \longrightarrow & M_n & \xrightarrow{f_n} & f_n(M_n) & \longrightarrow & 0.
\end{array}$$

The rightmost vertical arrow is injective since $f_n(M_n) \subset D_{R,n}$ and the cokernel of the middle vertical arrow is killed by $[p]_q^s$ (see Lemma 4.10). Hence, the cokernel of the rightmost vertical arrow is also killed by $[p]_q^s$. This concludes our proof. ■

Proposition 4.12. *The natural inclusion $\mathbf{N}_R(T) \subset \mathbf{D}_R(T)$ extends to a (φ, Γ_R)-equivariant isomorphism $A_R \otimes_{A_R^+} \mathbf{N}_R(T) \xrightarrow{\sim} \mathbf{D}_R(T)$.*

Proof. Since everything is p-adically complete and $\mathbf{D}_R(T)$ and $\mathbf{N}_R(T)$ are p-torsion free, it is enough to show the claim modulo p. Recall that we have $\mathbf{N}_R(T)/p \subset M_1 = \mathbf{D}_R(T)/p \cap \mathbf{N}_L(T)/p \subset \mathbf{D}_L(T)/p$ and from Lemma 4.11 we have $M^\circ \subset \mathbf{N}_R(T)/p$. Therefore, we get that $\mathbf{D}_R(T)/p = M^\circ[1/\mu] \subset A_R/p \otimes_{A_R^+/p} \mathbf{N}_R(T)/p \subset M_1[1/\mu] = \mathbf{D}_R(T)/p$. ■

5. Wach modules and q-connections

In this section we will interpret Wach modules over A_R^+ (resp. B_R^+) as modules with q-connection and show that Wach modules over B_R^+ can be seen as q-deformation of filtered (φ, ∂)-modules over $R[1/p]$, coming from p-adic crystalline representations of G_R (see Theorem 5.6). For our definitions, we will follow [MT20, §2], with slight modifications.

5.1. Formalism on q-connection. Let D be a commutative ring and consider a D-algebra A equipped with d commuting D-algebra automorphisms $\gamma_1 \ldots, \gamma_d$, i.e. an action of $\mathbb{Z}^d$. Moreover, fix an element $q \in D$ such that $q-1$ is a nonzerodivisor of D and $\gamma_i = 1 \mod (q-1)A$, for all $1 \leq i \leq d$. Assume that we have units $U_1, \ldots, U_d \in A^\times$ such that $\gamma_i(U_j) = qU_j$, if $i = j$ or U_j if $i \neq j$. We fix these choices for the rest of the section.

Definition 5.1 ([MT20, Definition 2.1])**.** Let $q\Omega^\bullet_{A/D} := \oplus_{k=0}^d q\Omega^k_{A/D}$ be a differential graded D-algebra defined as:

- $q\Omega^0_{A/D} := A$ and $q\Omega^1_{A/D}$ is a free left A-module on formal basis elements $d\log(U_i)$.

- The right A-module structure on $q\Omega^1_{A/D}$ is twisted by the rule $d\log(U_i) \cdot f = \gamma_i(f) d\log(U_i)$.
- $d\log(U_i) d\log(U_j) = -d\log(U_j) d\log(U_i)$ if $i \neq j$ and 0 if $i = j$.
- The following map is an isomorphism of A-modules:

$$\begin{aligned} \oplus_{\mathbf{i} \in I_k} A &\xrightarrow{\sim} q\Omega^k_{A/D} \\ (f_{\mathbf{i}}) &\longmapsto \textstyle\sum_{\mathbf{i} \in I_k} f_{\mathbf{i}} d\log(U_{i_1}) \cdots d\log(U_{i_k}), \end{aligned}$$

  where $I_k = \{\mathbf{i} = (i_1, \ldots, i_k) \in \mathbb{N}^k \text{ such that } 1 \leq i_1 < \cdots < i_k \leq d\}$.
- The 0^{th} differential $d_q : A \to \Omega^1_{A/D}$ is given as $f \mapsto \sum_{i=1}^{d} \frac{\gamma_i(f) - f}{q-1} d\log(U_i)$.
- The elements $d\log(U_i) \in q\Omega^1_{D/A}$ are cocycles, for all $1 \leq i \leq d$.

The data $d_q : A \to q\Omega^1_{A/D}$ forms a differential ring over D, i.e. $q\Omega^1_{A/D}$ is a D-bimodule and d_q is D-linear satisfying the Leibniz rule $d_q(fg) = d_q(f)g + f d_q(g)$ (see [And01, §II.1.2.1]).

Definition 5.2 ([MT20, Definition 2.2])**.** A module with *q-connection* over A is a right A-module N equipped with a D-linear map $\nabla_q : N \to N \otimes_A q\Omega^1_{A/D}$ satisfying the Leibniz rule $\nabla_q(xf) = \nabla_q(x)f + x \otimes d_q(f)$, for all $f \in A$ and $x \in N$. The q-connection ∇_q extends uniquely to a map of graded D-modules $\nabla_q : N \otimes_A q\Omega^\bullet_{A/D} \to N \otimes_A q\Omega^{\bullet+1}_{A/D}$ satisfying $\nabla_q((n \otimes \omega) \cdot \omega') = \nabla_q(n \otimes \omega) \cdot \omega' + (-1)^{\deg \omega}(n \otimes \omega) \cdot d_q(\omega')$. The q-connection ∇_q is said to be *flat* or *integrable* if $\nabla_q \circ \nabla_q = 0$.

Now, assume that D is equipped with an endomorphism $\varphi : D \to D$ such that it is a lift of the absolute Frobenius on D/pD and $\varphi(q) = q^p$. Further, assume that A is equipped with a compatible (with φ on D) endomorphism $\varphi : A \to A$ such that it is a lift of the absolute Frobenius on A/p and commutes with the action of $\gamma_1, \ldots, \gamma_d$ on A. The endomorphism φ induces an endomorphism φ_Ω on $q\Omega^1_{A/D}$ given as $\varphi_\Omega(\sum_{i=1}^{d} f_i d\log(U_i)) = [p]_q \sum_{i=1}^{d} \varphi(f_i) d\log(U_i)$. In particular, from [MT20, Lemma 2.12] the following diagram commutes

$$\begin{array}{ccc} A & \xrightarrow{d_q} & q\Omega^1_{A/D} \\ {\scriptstyle\varphi}\downarrow & & \downarrow{\scriptstyle\varphi_\Omega} \\ A & \xrightarrow{d_q} & q\Omega^1_{A/D}. \end{array}$$

It follows that given a q-connection (N, ∇_q) we can define the base change via Frobenius, of the q-connection, denoted $\varphi^*\nabla_q$ on $\varphi^*N := N \otimes_{A,\varphi} A$, as

$$\begin{aligned} \varphi^*\nabla_q : \varphi^*N &\longrightarrow N \otimes_{A,\varphi} q\Omega^1_{A/D} = \varphi^*N \otimes q\Omega^1_{A/D} \\ x \otimes f &\longmapsto (1 \otimes \varphi_\Omega)(\nabla_q(x)) \cdot f + n \otimes d_q(f). \end{aligned}$$

A φ-module with q-connection is a pair (N, ∇_q) as above equipped with an A-linear isomorphism $\varphi_N : (\varphi^*N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ such that the following diagram commutes:

$$\begin{array}{ccc} (\varphi^*N)[1/[p]_q] & \xrightarrow{\varphi^*\nabla_q} & (\varphi^*N)[1/[p]_q] \otimes q\Omega^1_{A/D} \\ {\scriptstyle\varphi_N}\downarrow & & \downarrow{\scriptstyle\varphi_N \otimes 1} \\ N[1/[p]_q] & \xrightarrow{\nabla_q} & N[1/[p]_q] \otimes q\Omega^1_{A/D}. \end{array} \tag{5.1}$$

5.2. Wach modules as q-deformations. In this subsection, we take $D := O_F[\![\mu]\!]$, $A := A_R^+$ equipped with the action of Γ_R and $\{\gamma_1, \ldots, \gamma_d\}$ as topological generators of Γ'_R, the geometric part of Γ_R (see §2). Then, by setting $q := 1 + \mu$ and $U_i := [X_i^\flat]$, for $1 \leq i \leq d$, we have $\gamma_i = 1 \bmod \mu A_R^+$, for all $1 \leq i \leq d$. In particular, A_R^+ satisfies the hypotheses of Definition 5.1. Moreover, the Frobenius endomorphism on A_R^+ extends the Frobenius on D given by identity on $\mathbb{Z}_p$ and $\varphi(\mu) = (1 + \mu)^p - 1$.

Furthermore, in this case, $q\Omega^1_{A_R^+/D}$ identifies with $\Omega^1_{A_R^+/D}$ given as (p,μ)-adic completion of the module of Kähler differentials of A_R^+ with respect to D.

Note that we have a Frobenius-equivariant isomorphism of rings $A_R^+/\mu \xrightarrow{\sim} R$, so from [MT20, Remarks 2.4 & 2.10], reduction modulo $q-1$ of the differential ring $d_q : A_R^+ \to \Omega^1_{A_R^+/D}$ is the usual de Rham differential $d : R \to \Omega^1_R$. Similarly, the reduction modulo $q-1$ of a module with q-connection over A_R^+ (Definition 5.2) is an R-module with connection. We say that a q-connection is $(p,[p]_q)$-*adically quasi-nilpotent* (equivalently, $(p,q-1)$-*adically* quasi-nilpotent) if ∇_q mod $q-1$ is p-adically quasi-nilpotent.

Proposition 5.3. *Let N be a Wach module over A_R^+. Then the geometric q-connection*

$$\nabla_q : N \longrightarrow N \otimes_{A_R^+} \Omega^1_{A_R^+/D}$$
$$x \longmapsto \sum_{i=1}^{d} \tfrac{\gamma_i(x)-x}{\mu} d\log([X_i^\flat]),$$

describes (N,∇_q) as a φ-module equipped with a $(p,[p]_q)$-adically quasi-nilpotent flat q-connection over A_R^+.

Proof. Flatness of the q-connection ∇_q follows from the first part of the proof of [MT20, Proposition 2.6]. Moreover, from Definition 3.8 and Lemma 3.10, note that we have $\varphi \otimes 1 : (N \otimes_{A_R^+,\varphi} A_R^+)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. So we get that the pair (N,∇_q) is a φ-module equipped with a q-connection over A_R^+. Moreover, since the action of φ and Γ'_R commute on N, therefore, it follows that the corresponding diagram (5.1) is commutative. Now, from the commutativity of the action of φ and Γ_R and the diagram (5.1), note that we have $\nabla_q \circ \varphi = [p]_q \nabla_q \circ \varphi$. Frurthermore, from the Frobenius finite height condition on N, we have that for any x in N, there exists $r \in \mathbb{N}$ large enough, such that $[p]_q^r x$ belongs to $\varphi^*(N)$. So, using the relation $\nabla_q \circ \varphi = [p]_q \nabla_q \circ \varphi$ and the fact that $[p]_q = p$ mod $q-1$, we see that $\nabla_q^k([p]_q^r x)$ mod $q-1$ converges p-adically to 0 as $k \to +\infty$. Hence, it follows that $\nabla_q^k(x) = [p]_q^{-r} \nabla_q([p]_q^r x)$ modulo $q-1$ converges p-adically to 0, i.e. ∇_q is $(p,[p]_q)$-adically quasi-nilpotent. This concludes our proof. ∎

Remark 5.4. In Proposition 5.3 we call the q-connection "geometric" because in the definition we only use the geometric part of Γ_R, i.e. Γ'_R.

Remark 5.5. From §3.6 recall that we have the ring $A_{R,\varpi}^{\mathrm{PD}} \subset A_{\mathrm{cris}}(R_\infty)$ stable under the Frobenius and the action of Γ_R. For $R = O_F$, we denote the aforementioned ring, i.e. $A_{F,\varpi}^{\mathrm{PD}}$ by D^{PD} and for general R, we denote it by $A^{\mathrm{PD}} := A_{R,\varpi}^{\mathrm{PD}}$ (we do not use D and A for these rings to avoid conflict with assumptions at the beginning of this subsection). Then, it is easy to see that the hypotheses of Definition 5.1 are satisfied for D^{PD}, A^{PD} with Γ_R-action and $U_i := [X_i^\flat]$. Now, given a Wach module N over A_R^+, similar to Propostion 5.3, one can show that for $N^{\mathrm{PD}} := A^{\mathrm{PD}} \otimes_{A_R^+} N$, the q-connection

$$\nabla_q : N^{\mathrm{PD}} \longrightarrow N^{\mathrm{PD}} \otimes_{A^{\mathrm{PD}}} \Omega^1_{A^{\mathrm{PD}}/D^{\mathrm{PD}}}, \quad x \longmapsto \textstyle\sum_{i=1}^{d} \frac{\gamma_i(x)-x}{\mu} d\log([X_i^\flat]),$$

describes $(N^{\mathrm{PD}},\nabla_q)$ as a φ-module equipped with a p-adically quasi-nilpotent flat q-connection over A^{PD}. Set $\nabla_{q,i} := (\gamma_i - 1)/\mu$, for $1 \leq i \leq d$. Furthermore, employing arguments similar to [Abh23b, Lemmas 4.12, 5.17 & 5.18] we can show that for $1 \leq i \leq d$, the operator $\nabla_i := (\log \gamma_i)/t = \frac{1}{t}\sum_{k \in \mathbb{N}}(-1)^k \frac{(\gamma_i-1)^{k+1}}{k+1}$ converges as a series of operators on N^{PD}. So using the explicit formulas described above, it is easy to see that for any $x \in N$, we have $\nabla_{q,i}(x) - \nabla_i(x) = (\frac{\gamma_i-1}{\mu} - \frac{\log \gamma_i}{t})(x) \in (\mathrm{Fil}^1 A^{\mathrm{PD}}) \otimes_{A_R^+} N$, since t/μ is a unit in A^{PD} by [Abh21, Lemma 3.14].

We are now ready to state the main result of this section. Let N be a Wach module over A_R^+ equipped with a q-connection as in Proposition 5.3 and a Nygaard filtration as in Definition 3.24. Then, from the discussion preceding Proposition 5.3, we note that $N/\mu N$ is a φ-module over R equipped with a p-adically quasi-nilpotent flat connection and a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. Using Reamrk 3.26 note that the connection on $N/\mu N$ satisfies Griffiths transversality with respect to the filtration, i.e. $\nabla(\mathrm{Fil}^k(N/\mu N)) \subset \mathrm{Fil}^{k-1}(N/\mu N) \otimes \Omega^1_R$. We equip $N[1/p]/\mu N[1/p] = (N/\mu N)[1/p]$ with the induced structures, in particular, we note that it is a filtered (φ,∂)-module over $R[1/p]$.

Theorem 5.6. *Let N be a Wach module over A_R^+ and $V := \mathbf{T}_R(N)[1/p]$ the associated crystalline representation from Theorem 3.34. Then we have $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ as filtered (φ, ∂)-modules over $R[1/p]$.*

Proof. For $r \in \mathbb{N}$ large enough, note that the Wach module $\mu^r N(-r)$ is always effective and we have that $\mathbf{T}_R(\mu^r N(-r)) = \mathbf{T}_R(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which Γ_R acts via χ^{-r}, where χ is the p-adic cyclotomic character). Therefore, it is enough to show both the claims for effective Wach modules. So we assume that N is effective and set $M := N[1/p]$ equipped with an induced action of Γ_R, a Frobenius-semilinear operator φ and the Nygaard filtration. It follows that the finite projective $R[1/p]$-module $M/\mu M$ is equipped with a Frobenius-semilinear operator φ, induced from M. Note that $[p]_q = p \mod \mu A_R^+$, therefore, we have $1 \otimes \varphi : \varphi^*(M/\mu M) \xrightarrow{\sim} M/\mu M$. Furthermore, the filtration $\mathrm{Fil}^k(M/\mu M)$ is defined to be the image of $\mathrm{Fil}^k M$ under the surjective map $M \twoheadrightarrow M/\mu M$. Next, from Theorem 3.34, we have the $R[1/p]$-module $\mathcal{O}D_R := (\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} M)^{\Gamma_R}$ equipped with a Frobenius-semilinear operator φ and a connection, and an $R[1/p]$-linear isomorphism $\mathcal{O}D_R \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ compatible with the respective Frobenii and connections (see (3.22) in Theorem 3.34). So, let us consider the following diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mu M & \longrightarrow & M & \longrightarrow & M/\mu M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & (\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_{A_R^+} M & \longrightarrow & \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} M & \longrightarrow & R[\varpi] \otimes_R (M/\mu M) & \longrightarrow & 0 \\
 & & \wr\uparrow & & \wr\uparrow (3.23) & & \wr\uparrow & & \\
0 & \longrightarrow & (\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_R \mathcal{O}D_R & \longrightarrow & \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}D_R & \longrightarrow & R[\varpi] \otimes_R \mathcal{O}D_R & \longrightarrow & 0.
\end{array}$$

Note that $(\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} M) \cap M = (\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \cap A_R^+) \otimes_{A_R^+} M = \mu M$. Then, from the exactness of the second row, it follows that the vertical maps from the first to the second row are natural inclusions. The middle vertical arrow from the third to the second row is the isomorphism (3.23) in Proposition 3.35 from which it follows that the left vertical arrow is an isomorphism as well. In particular, we get that the right vertical arrow is also an isomorphism. Taking the $\mathrm{Gal}(R[1/p][\varpi]/R[1/p]) = \mathrm{Gal}(F(\zeta_p)/F)$-invariants of the right vertical arrows gives a natural isomorphism

$$\mathcal{O}D_R \xrightarrow{\sim} M/\mu M, \tag{5.2}$$

compatible with the respective Frobenii and we claim that it is compatible with the respective connections as well. Indeed, note that the connection on $M/\mu M$ is obtained by first reducing, the q-connection ∇_q on N, modulo $\mu = q - 1$ and then inverting p. On the other hand, the connection ∂_D on $\mathcal{O}D_R = (\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} M)^{\Gamma_R}$ is induced from the natural $A_{R,\varpi}^{\mathrm{PD}}$-linear connection on $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$. Let $\nabla_{q,i}$ and $\partial_{D,i}$ respectively denote the i^{th} component of the q-connection on N and the connection on $\mathcal{O}D_R$. Now take x in M, and note that from Remark 3.38 there exists some w in $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_R \mathcal{O}D_R$ such that $x = f(w) \mod (\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_{A_R^+} M$, where f is the isomorphism in (3.23). Then it follows that to check the compatibility of the isomorphism $\mathcal{O}D_R \xrightarrow{\sim} M/\mu M$ with connections, it is enough to show that $\nabla_{q,i}(x) - f(\partial_{D,i}(w))$ belongs to $(\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_{A_R^+} M$. From Remark 3.38 for $\nabla_i = (\log \gamma_i)/t$, we know that $\nabla_i(x) - f(\partial_{D,i}(w))$ is in $(\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_{A_R^+} M$. Furthermore, from Remark 5.5 we have that $\nabla_{q,i}(x) - \nabla_i(x)$ is in $(\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_{A_R^+} M$. Upon combining the two, we get that $\nabla_{q,i}(x) - f(\partial_{D_i}(w))$ is in $(\mathrm{Fil}^1 \mathcal{O}A_{R,\varpi}^{\mathrm{PD}}) \otimes_{A_R^+} M$, i.e. the isomorphism (5.2) is compatible with the respective connections.

Now, by composing the inverse of (5.2) with (3.22) from Theorem 3.34, we get isomorphisms

$$M/\mu M \xrightarrow{\sim} \mathcal{O}D_R \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V), \tag{5.3}$$

compatible with the respective Frobenii and connections. By transport of structure, we equip $\mathcal{O}D_R$ with a filtration induced from the Hodge filtration on $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$. Then, by Lemma 5.7 we get that the isomorphisms in (5.3) are further compatible with the respective filtrations. This allows us to conclude. ∎

The following observation was used above:

Lemma 5.7. *Let N be a Wach module over A_R^+, set $M := N[1/p]$ and let $V := \mathbf{T}_R(N)[1/p]$ denote the associated crystalline representation of G_R. Then the isomorphism $f : M/\mu M \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ from Theorem 5.6 is compatible with filtrations, i.e. for each $k \in \mathbb{Z}$, we have that*

$$\mathrm{Fil}^k(M/\mu M) \xrightarrow{\sim} \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V). \tag{5.4}$$

Proof. For $r \in \mathbb{N}$ large enough, note that the Wach module $\mu^r N(-r)$ is always effective and we have $\mathbf{T}_R(\mu^r N(-r)) = \mathbf{T}_R(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which Γ_R acts via χ^{-r}, where χ is the p-adic cyclotomic character). Therefore, it is enough to show the claim for effective Wach modules. To avoid confusion, let us write $N_R := N$, $M_R := M$, $N_L := A_L^+ \otimes_{A_R^+} N_R$ (a Wach module over A_L^+) and $M_L := N_L[1/p]$, equipped with the induced structures. From Lemma 3.28, note that the natural map $N_R \to N_L$ induces natural maps $\mathrm{gr}^k M_R \to \mathrm{gr}^k M_L$ and $\mathrm{gr}^k(M_R/\mu M_R) \to \mathrm{gr}^k(M_L/\mu M_L)$, and we claim that these are injective. Indeed, the injectivity of the first map follows from the discussion after (3.16). For the second map, note that from Lemma 3.30 we have that $\mathrm{gr}^k M_R \xrightarrow{\sim} \mathrm{gr}^k(M_R/\mu M_R)$ and $\mathrm{gr}^k M_L \xrightarrow{\sim} \mathrm{gr}^k(M_L/\mu M_L)$. So it follows that the natural map $\mathrm{gr}^k(M_R/\mu M_R) \to \mathrm{gr}^k(M_L/\mu M_L)$ is injective as well. In particular, inside $M_L/\mu M_L$, we get that

$$\mathrm{Fil}^{k+1}(M_R/\mu M_R) \xrightarrow{\sim} \mathrm{Fil}^k(M_R/\mu M_R) \cap \mathrm{Fil}^{k+1}(M_L/\mu M_L). \tag{5.5}$$

Next, recall that we have the finite projective $R[1/p]$-module $\mathcal{O}D_R := (\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \otimes_{A_R^+} M_R)^{\Gamma_R}$ and similarly we have the finite dimensional L-vector space $\mathcal{O}D_L := (\mathcal{O}A_{L,\varpi}^{\mathrm{PD}} \otimes_{A_L^+} M_L)^{\Gamma_L}$, where the ring $\mathcal{O}A_{L,\varpi}^{\mathrm{PD}}$ (depending on L) is analogous to $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}}$ (see [Abh23a, §3.3] for precise definitions), and admits a natural map $\mathcal{O}A_{R,\varpi}^{\mathrm{PD}} \to \mathcal{O}A_{L,\varpi}^{\mathrm{PD}}$ compatible with supplementary structures. From [Abh23a, Theorem 1.8 & Corollary 3.16], recall that we have isomorphisms of φ-modules $M_L/\mu M_L \xrightarrow{\sim} \mathcal{O}D_L \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ over L (similar to (5.3)), and note that the constructions of loc. cit. are compatible with the constructions of this paper. Now, consider the following diagram:

$$\begin{array}{ccccc} L \otimes_{R[1/p]} (M_R/\mu M_R) & \xrightarrow{\sim} & L\otimes_{R[1/p]} & \xrightarrow[(3.22)]{\sim} & L \otimes_{R[1/p]} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \\ \downarrow \wr & & \downarrow & & \wr\downarrow (4.5) \\ M_L/\mu M_L & \xrightarrow{\sim} & \mathcal{O}D_L & \xrightarrow{\sim} & \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \end{array} \tag{5.6}$$

where the top row is (5.3) and the bottom row is as discussed above (see the proof of [Abh23a, Corollary 3.16] for details). In (5.6), the left and the middle vertical arrows are the natural maps. Then, by the discussion above we see that the left square commutes. Moreover, as the top right and the bottom right horizontal isomorphisms are induced by natural inclusions and the crystalline period rings over R and L are compatible, therefore, it follows that the right square commutes as well. Furthermore, in (5.6), the left vertical arrow is a filtered isomorphism by Lemma 3.31, the composition of the bottom arrows is a filtered isomorphism by [Abh23a, Theorem 1.8] (see Remark 5.8 for another proof) and the right vertical arrow is a filtered isomorphism by Corollary 4.6.

Note that the composition of the arrows in the top row of (5.6) is the isomorphism $f : M_R/\mu M_R \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$ and we need to show that f induces the map in (5.4) and that the induced map is bijective. We will proceed by induction on k, where the case $k = 0$ is trivial. So assume that f induces an isomorphism $\mathrm{Fil}^k(M_R/\mu M_R) \xrightarrow{\sim} \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, for some $k \geq 0$. Then, by using (5.5), the filtered isomorphism $M_L/\mu M_L \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and its compatibility with f (see (5.6)) and the induction assumption, it follows that

$$\begin{aligned} \mathrm{Fil}^{k+1}(M_R/\mu M_R) &\xrightarrow{\sim} \mathrm{Fil}^k(M_R/\mu M_R) \cap \mathrm{Fil}^k(M_L/\mu M_L) \\ &\xrightarrow{\sim} \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V) \cap \mathrm{Fil}^{k+1}\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = \mathrm{Fil}^{k+1}\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V), \end{aligned}$$

where the terms of the last row are conatined in $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ via the filtered isomorphism (4.5) (see Corollary 4.6). Hence, it follows that (5.4) is bijective for each $k \in \mathbb{Z}$. This allows us to conclude. ∎

Remark 5.8. Let N_L be a Wach module over A_L^+ and let T be the associated crystalline $\mathbb{Z}_p$-representation of G_L from [Abh23a, Theorem 1.6]. In [Abh23a, Theorem 1.8 & Corollary 3.16], we have shown that the natural isomorphism $(N_L/\mu N_L)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is compatible with the respective filtrations. We claim that the compatibility between filtrations can also be obtained by using the analogous result in the perfect residue field case from [Ber04, Théorème III.4.4]. Indeed, consider the extension $\breve{L}/L$ with perfect residue field from Remark 3.29. Then $N_{\breve{L}} := A_{\breve{L}}^+ \otimes_{A_L^+} N_L$ is a Wach module over $A_{\breve{L}}^+$ and T is a $\mathbb{Z}_p$-representation of $G_{\breve{L}}$. Set $M_L := N_L[1/p]$, $M_{\breve{L}} := N_{\breve{L}}[1/p]$ and $V := T[1/p]$. Then from Remark 3.29, Lemma 3.30 and Remark 3.32, it follows that inside $M_{\breve{L}}/\mu M_{\breve{L}}$, we have that

$$\mathrm{Fil}^{k+1}(M_L/\mu M_L) \xrightarrow{\sim} \mathrm{Fil}^k(M_L/\mu M_L) \cap \mathrm{Fil}^{k+1}(M_{\breve{L}}/\mu M_{\breve{L}}). \tag{5.7}$$

Now, consider the following diagram:

$$\begin{array}{ccc} \breve{L} \otimes_L (M_L/\mu M_L) & \xrightarrow{\sim} & \breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \\ \downarrow \wr & & \wr \downarrow \\ M_{\breve{L}}/\mu M_{\breve{L}} & \xrightarrow{\sim} & \mathbf{D}_{\mathrm{cris},\breve{L}}(V), \end{array} \tag{5.8}$$

where the left vertical arrow is the natural map, the top horizontal arrow is induced from [Abh23a, Equations (4.6) & (4.10)] and coincides the bottom row of (5.6) by [Abh23a, Equations (4.5), (4.11) & (4.15)], the right vertical arrow is the natural isomorphism of filtered φ-modules over $\breve{L}$ (see [Abh23a, Equation (2.5)]) and the bottom horizontal arrow is the inverse of the natural isomorphism of filtered φ-modules over $\breve{L}$ from [Ber04, Théorème III.4.4]. The diagram commutes by the compatibilty between the constructions of [Abh23a] and [Ber04] (more precisely, by using [Abh23a, Equations (4.6) & (4.14)]). Now, note that to obtain the claim, it is enough to show that the top horizontal arrow of (5.8) induces a map $\mathrm{Fil}^k(M_L/\mu M_L) \to \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and that the induced map is bijective. We will proceed by induction on k, where the case $k = 0$ is trivial. So assume that the top horizontal arrow of (5.8) induces an isomorphism $\mathrm{Fil}^k(M_R/\mu M_R) \xrightarrow{\sim} \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris},R}(V)$, for some $k \geq 0$. Then, by using (5.7), the filtered isomorphism $M_{\breve{L}}/\mu M_{\breve{L}} \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)$ from (5.8) and the induction assumption, it follows that

$$\begin{aligned}\mathrm{Fil}^{k+1}(M_L/\mu M_L) &\xrightarrow{\sim} \mathrm{Fil}^k(M_L/\mu M_L) \cap \mathrm{Fil}^k(M_{\breve{L}}/\mu M_{\breve{L}}) \\ &\xrightarrow{\sim} \mathrm{Fil}^k\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \cap \mathrm{Fil}^{k+1}\mathbf{D}_{\mathrm{cris},\breve{L}}(V) = \mathrm{Fil}^{k+1}\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),\end{aligned}$$

where the terms of the last row are conatined in $\mathbf{D}_{\mathrm{cris},\breve{L}}(V)$ via the right vertical filtered isomorphism in (5.8). Hence, the claim follows.

Remark 5.9. The obvious variation of Theorem 5.6 also holds true in the imperfect residue field case. Indeed, for O_L, recall that all compatibilities except for the connection part was already proven in [Abh23a, Corollary 3.15] (Remark 5.8 another proof of compatibility between filtrations). To verify the compatibility of connections, similar to Proposition 5.3, we can define a q-connection over a Wach module over A_L^+. Then, using the results of [Abh23a, §3.3], one obtains an obvious variation of Remark 5.5 over $A_{L,\varpi}^{\mathrm{PD}}$. Finally, proceeding exactly as in the proof of Theorem 5.6 (after replacing each object by analogous object for L), we obtain the desired isomorphism of filtered (φ, ∂)-modules over L.

Let us summarise the relationship between various categories considered in (2.9), Corollary 4.4 and Theorem 5.6. Recall that $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R)$ is the category of p-adic crystalline representations of G_R and $\mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial)$ denotes the essential image of the functor $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}$ restricted to $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R)$.

Corollary 5.10. *Functors in the following diagram induce exact equivalence of $\otimes$-categories*

$$\begin{array}{ccc} \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_R) & \underset{\mathbf{V}_R}{\overset{\mathbf{N}_R}{\rightleftarrows}} & (\varphi, \Gamma_R)\text{-}\mathrm{Mod}_{B_R^+}^{[p]_q} \\ {\scriptstyle \mathcal{O}\mathbf{V}_{\mathrm{cris},R}} \nwarrow \searrow {\scriptstyle \mathcal{O}\mathbf{D}_{\mathrm{cris},R}} & & \swarrow {\scriptstyle q \mapsto 1} \\ & \mathrm{MF}_R^{\mathrm{ad}}(\varphi, \partial). & \end{array}$$

Proof. The exact equivalence induced by functors $\mathbf{N}_R$ and $\mathbf{V}_R$ is from Corollary 4.4 and the exact equivalence induced by $\mathcal{O}\mathbf{D}_{\mathrm{cris},R}$ and $\mathcal{O}\mathbf{V}_{\mathrm{cris},R}$ is from [Bri08, Théorème 8.5.1]. Moreover, from Theorem 5.6, note that for a Wach module M over B_R^+ we have $M/(q-1)M = M/\mu M \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},R}(\mathbf{V}_R(M))$. Hence, from the preceding exact equivalence of $\otimes$-categories, it follows that the slanted arrow labelled "$q \mapsto 1$" is also an exact equivalence of $\otimes$-categories. ∎

A. Structure of φ-modules

We will use setup and notations from §1.4 and the rings defined in §2.2. Let q be an indeterminate and recall that we have a Frobenius-equivariant isomorphism $R[\![q-1]\!] \xrightarrow{\sim} A_R^+$, via the map $X_i \mapsto [X_i^\flat]$ and $q \mapsto 1+\mu$. We will show the following structural result:

Proposition A.1. *Let N be a finitely generated A_R^+-module and suppose that N is equipped with a Frobenius-semilinear endomorphism $\varphi : N \to N$ such that $1 \otimes \varphi : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p_q]]$. Then $N[1/p]$ is finite projective over B_R^+.*

Proof. The proof is essentially the same as [DLMS22, Proposition 4.13]. Compared to loc. cit., the Frobenius endomorphism on A_R^+ and finite height assumption on N are different and we do not assume N to be torsion free. However, one observes that torsion freeness of N is not used in the proof and one can use [Abh23a, Lemma 2.14] and Lemma A.2 instead of [BMS18, Proposition 4.3] and [DLMS22, Lemma 4.12]. ∎

Lemma A.2. *Let k be a perfect field of characteristic p and $S := W(k)[\![u_1, \dots, u_m]\!]$ equipped with a Frobenius endomorphism φ extending the Witt vector Frobenius on $W(k)$ such that $\varphi(u_i) \in S$ has zero constant term for each $1 \leq i \leq m$. Let $A := S[\![q-1]\!]$ equipped with a Frobenius endomorphism extending the one on S by $\varphi(q) = q^p$ and let N be a finitely generated A-module equipped with a Frobenius-semilinear endomorphism $\varphi : N \to N$ such that $1 \otimes \varphi : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Then $N[1/p]$ is finite projective over $A[1/p]$.*

Proof. The proof is essentially the same as [DLMS22, Lemma 4.12], except for a few changes. One proceeds by induction on m. The case $m = 0$ follows from [Abh23a, Lemma 2.14], so let $m \geq 1$. Take J to be the smallest non-zero Fitting ideal of N over A. It suffices to show that $JA[1/p] = A[1/p]$. Compatibility of Fitting ideals under base change implies that $JA[1/[p]_q] = \varphi(J)A[1/[p]_q]$ as ideals of $A[1/[p]_q]$, therefore, $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$. Let us assume $JA[1/p] \neq A[1/p]$ and we will show a contradiction.

In our setting, the Frobenius endomorphism on A and the finite height condition are different from [DLMS22, Lemma 4.12]. Therefore, we need some modifications in the arguments of loc. cit.; let us point out the differences in terms of their notations. Let $K = W(k)[1/p]$, fix $\overline{K}$ as an algebraic closure of K. Consider the $\overline{K}$-valued points of $\mathrm{Spec}\,(A[1/p]/J)$ and let $Z = \{(|u_1|, \dots, |u_m|, |q-1|) \in \mathbb{R}^{m+1}\}$ be the corresponding set of $(m+1)$-tuple norms. Define the set $Z' = \{(|u_1|, \dots, |u_m|, |q-1|) \in \mathbb{R}^{m+1}$ such that $(|\varphi(u_1), \dots |\varphi(u_m)|, |q^p - 1|) \in Z\}$ and take $\zeta_p - 1$ as the chosen uniformiser. Then, one proceeds as in loc. cit. to show that $JA[1/p] \subset (u_1, \dots, u_m, q-1)A[1/p]$ and $JA[1/p] \not\subset IA[1/p]$, where $I = (u_1, \dots, u_m) \subset A[1/p]$.

Finally, consider the Frobenius-equivariant projection $A \to \overline{A} = A/I = W(k)[\![q-1]\!]$ and let $\overline{J} \subset \overline{A}$ denote the image of J. Since $JA[1/p] \not\subset IA[1/p]$, we get that $\overline{J} \neq 0$. Moreover, $\overline{JA}[1/p] \neq \overline{A}[1/p]$ since $JA[1/p] \subset (u_1, \dots, u_m, q-1)A[1/p]$. However, the equality $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$ implies that $(\overline{A}/\overline{J})[1/[p]_q] = (\overline{A}/\varphi(\overline{J}))[1/[p]_q]$, i.e. $\overline{JA}[1/p] = \overline{A}[1/p]$ by inductive hypothesis (see [Abh23a, Lemma 2.14]). This gives a contradiction. Hence, we must have $JA[1/p] = A[1/p]$, thus proving the lemma. ∎

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