Crystalline part of the Galois cohomology of crystalline representations

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Abstract. For an unramified extension $F/\mathbb{Q}_p$ with perfect residue field, we define a syntomic complex with coefficients in a Wach module. We show that our complex computes the crystalline part of the Galois cohomology (in the sense of Bloch and Kato) of the crystalline representation, of the absolute Galois group of F, associated to the Wach module.

1. Introduction

Let p be a fixed prime number and let κ denote a perfect field of characteristic p; set $O_F := W(\kappa)$ to be the ring of p-typical Witt vectors with coefficients in κ and $F := \mathrm{Frac}(O_F)$. Let $\overline{F}$ denote a fixed algebraic closure of F and $G_F := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. Let V be a p-adic crystalline representation of G_F (see [Fon82]). In [BK90], the authors defined Bloch-Kato Selmer groups of V as a subspace inside the continuous G_F-cohomology of V, i.e. $H^k_f(G_F, V) \subset H^k(G_F, V)$, for $k \in \mathbb{N}$. Bloch-Kato Selmer group picks out the "crystalline part" of the Galois cohomology of V. More precisely, let $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F)$ denote the category of crystalline representations, then we have $H^1_f(G_F, V) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F)}(\mathbb{Q}_p, V)$ (see Remark 2.2). Furthermore, let $\mathbf{D}_{\mathrm{cris}}(V)$ denote the filtered φ-module attached to V (see [Fon82]). Then from [BK90], we have the following:

Proposition 1.1 (Corollary 2.4)**.** *Let V be a p-adic crystalline representation of G_F. Then the complex*

$$\mathscr{D}^\bullet(\mathbf{D}_{\mathrm{cris}}(V)) : \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{1-\varphi} \mathbf{D}_{\mathrm{cris}}(V), \tag{1.1}$$

computes the crystalline part of the Galois cohomology of V, i.e. $H^k(\mathscr{D}^\bullet(\mathbf{D}_{\mathrm{cris}}(V))) \xrightarrow{\sim} H^k_f(G_F, V)$ for each $k \in \mathbb{N}$.

Now let $F_\infty := \cup_n F(\mu_{p^n})$ with $\Gamma_F := \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times$ and choose a generator γ of Γ_F such that γ^{p-1} is a topological generator of $\mathrm{Gal}(F_\infty/F(\zeta_p))$. Let $\mathbf{A}_F^+ := O_F[\![\mu]\!]$ equipped with continuous (φ, Γ_F)-action (see §2.1). Let T be a G_F-stable $\mathbb{Z}_p$-lattice inside V. By the works of Fontaine [Fon90], Wach [Wac96], Colmez [Col99] and Berger [Ber04], one can functorially attach to T, a finite free $\mathbf{A}_F^+$-module $\mathbf{N}_F(T)$ equipped with continuous (φ, Γ_F)-action satisfying nice properties (see Definition 3.1 and Theorem 3.2). The module $\mathbf{N}_F(T)$ is called the Wach module associated to T. In fact, Wach modules can be defined abstractly (see Definition 3.1), and we denote the category of such objects, with (φ, Γ_F)-compatible $\mathbf{A}_F^+$-linear morphisms, by $(\varphi, \Gamma)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{A}_F^+}$. Let $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_F. Then the Wach module functor induces an equivalence of categories $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Z}_p}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{A}_F^+}$ (see Theorem 3.2). Moreover, after inverting p, i.e. on passing to associated isogeny categories, the Wach module functor induces an exact equivalence of categories $\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F) \xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{B}_F^+}$, where $\mathbf{B}_F^+ := \mathbf{A}_F^+[1/p]$ (see Remark 3.3). Hence, it is reasonable

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to expect that one could define a complex in terms of Wach modules and whose cohomology computes the crystalline part of the Galois cohomology of V. The goal of this paper is to realise this expectation.

Let $N := \mathbf{N}_F(T)$ be the Wach module over $\mathbf{A}_F^+$ associated to T. Define a decreasing filtration on N called the *Nygaard filtration*, for $k \in \mathbb{Z}$, as $\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}$. Define an operator on N as $\nabla_q := \frac{\gamma-1}{\mu} : N \to N$. Then for each $k \in \mathbb{Z}$ we have $\nabla_q(\mathrm{Fil}^k N) \subset \mathrm{Fil}^{k-1} N$ (see Remark 3.7).

Definition 1.2. Define the *syntomic complex* with coeffcients in N as

$$\mathcal{S}^\bullet(N) : \mathrm{Fil}^0 N \xrightarrow{(\nabla_q, 1-\varphi)} \mathrm{Fil}^{-1} N \oplus N \xrightarrow{(1-[p]_q\varphi, \nabla_q)^\intercal} N,$$

where the first map is $x \mapsto (\nabla_q(x), (1-\varphi)x)$ and the second map is $(x, y) \mapsto (1-[p]_q\varphi)x - \nabla_q(y)$.

Remark 1.3. Definiton 1.2 can be modified to obtain a subcomplex of the Fontaine-Herr complex from [Her98] (see §4 and Remark 4.3). The reader should note that the Fontaine-Herr complex computes the Galois cohomology of a representation, while the complex in Definition 1.2 or Remark 4.3 is concerned with capturing the crystalline part inside the space of Galois cohomology. Complexes similar to the modified complex in Remark 4.3 were studied in [Abh23c] and named syntomic complexes. Hence, we also call the complex in Definition 1.2 as the syntomic complex with coefficients in N.

With notations as above, the main result of this article is as follows:

Theorem 1.4 (Theorem 4.2)**.** *For each $k \in \mathbb{N}$, we have a natural isomorphism*

$$H^k(\mathcal{S}^\bullet(N))[1/p] \xrightarrow{\sim} H^k_f(G_F, V).$$

Proof of Theorem 1.4 for H^0 and H^2 follow from direct computations (see Lemma 4.4 and Proposition 4.6 repsectively). Proof for H^1 follows from studying extension classes in the category of Wach modules over $\mathbf{A}_F^+$ (see Proposition 4.5).

In [Bha23, §6], Bhatt and Lurie have defined syntomic cohomology of prismatic F-gauges on the stack $\mathbb{Z}_p^{\mathrm{syn}}$ and, in case of reflexive F-gauges, compared it to the Bloch-Kato Selmer groups of the associated crystalline representation of $\mathrm{Gal}(\overline{F}/\mathbb{Q}_p)$ (see [Bha23, Proposition 6.7.3]). In light of Theorem 1.4 and prismatic interpretation of Wach modules (cf. [Abh23b]), a natural and interesting question is to ask for a direct (integral) relationship between Definition 1.2 and the definition of [Bha23]. The aforementioned question and generalisation of the theory above to the relative case, i.e. Definition 1.2 and its relationship with Galois cohomology, will be investigated in a future work.

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2. Period rings and p-adic representations

Let p be a fixed prime number and let κ denote a perfect field of characteristic p; set $O_F := W(\kappa)$ to be the ring of p-typical Witt vectors with coefficients in κ and $F := \mathrm{Frac}(O_F)$. Let $\overline{F}$ denote a fixed algebraic closure of F, $\mathbb{C}_p := \widehat{\overline{F}}$ the p-adic completion and $G_F := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. Moreover, let $F_\infty := \cup_n F(\mu_{p^n})$ with $\Gamma_F := \mathrm{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times$ and $H_F := \mathrm{Gal}(\overline{F}/F_\infty)$. Choose a generator γ of Γ_F such that γ^{p-1} is a topological generator of $\mathrm{Gal}(F_\infty/F(\zeta_p))$.

2.1. Period rings. In this section we briefly recall the period rings to be used in this paper (see [Fon90; Fon94] for details). Let $\mathbf{A}_{\inf}(O_{F_\infty}) := W(O_{F_\infty}^\flat)$ and $\mathbf{A}_{\inf}(O_{\overline{F}}) := W(O_{\overline{F}}^\flat)$ admitting the Frobenius on Witt vectors and continuous G_F-action (for the weak topology). Moreover, we have $\mathbf{A}_{\inf}(O_{F_\infty}) = \mathbf{A}_{\inf}(O_{\overline{F}})^{H_F}$ (see [And06, Proposition 7.2]). We fix $\overline{\mu} := \varepsilon - 1$, where $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in O_{F_\infty}^\flat$ and let $\mu := [\varepsilon] - 1, \xi := \mu/\varphi^{-1}(\mu) \in \mathbf{A}_{\inf}(O_{F_\infty})$. For $g \in G_F$, we have $g(1+\mu) = (1+\mu)^{\chi(g)}$, where χ is the p-adic cyclotomic character. Furthermore, we have a G_F-equivariant surjection $\theta : \mathbf{A}_{\inf}(O_{\overline{F}}) \to \mathbb{C}_p$ with $\mathrm{Ker}\theta = \xi\mathbf{A}_{\inf}(O_{\overline{F}})$. The map θ further induces a Γ_F-equivariant surjection $\theta : \mathbf{A}_{\inf}(O_{F_\infty}) \to O_{\widehat{F}_\infty}$.

We set $\mathbf{A}_{\mathrm{cris}}(O_{F_\infty}) := \mathbf{A}_{\inf}(O_{F_\infty})\langle \xi^k/k!, k \in \mathbb{N}\rangle$. Let $t := \log(1+\mu) \in \mathbf{A}_{\mathrm{cris}}(O_{F_\infty})$ and set $\mathbf{B}^+_{\mathrm{cris}}(O_{F_\infty}) := \mathbf{A}_{\mathrm{cris}}(O_{F_\infty})[1/p]$ and $\mathbf{B}_{\mathrm{cris}}(O_{F_\infty}) := \mathbf{B}^+_{\mathrm{cris}}(O_{F_\infty})[1/t]$. These rings are equipped with a Frobenius endomorphism and continuous Γ_F-action, a decreasing filtration and an appropriate extension of the map θ. Moreover, we set $\mathbf{B}^+_{\mathrm{dR}}(O_{F_\infty}) = \lim_n(\mathbf{A}_{\inf}(O_{F_\infty})[1/p])/(\mathrm{Ker}\theta)^n$ and $\mathbf{B}_{\mathrm{dR}}(O_{F_\infty}) = \mathbf{B}^+_{\mathrm{dR}}(O_{\overline{F}})[1/t]$. These rings are equipped with a Γ_F-action, a decreasing filtration and an appropriate extension of the map θ. We have (φ, Γ_F)-equivariant inclusions $\mathbf{B}^+_{\mathrm{cris}}(O_{F_\infty}) \subset \mathbf{B}^+_{\mathrm{dR}}(O_{F_\infty})$ and $\mathbf{B}_{\mathrm{cris}}(O_{F_\infty}) \subset \mathbf{B}_{\mathrm{dR}}(O_{F_\infty})$. One can define variations of these rings over $O_{\overline{F}}$ as well. From [MT20, Corollary 4.34] we have a (φ, Γ_F)-equivariant isomorphism $\mathbf{A}_{\mathrm{cris}}(O_{F_\infty}) \xrightarrow{\sim} \mathbf{A}_{\mathrm{cris}}(O_{\overline{F}})^{H_F}$. Moreover, from [Fon94, Théorème 5.3.7] we have the following (φ, G_F)-equivariant *fundamental exact sequence*

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \xrightarrow{1-\varphi} \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \longrightarrow 0. \tag{2.1}$$

Let $\mathbf{A}_F^+ := O_F[\![\mu]\!]$ equipped with a Frobenius endomorphism φ, given by Witt vector Frobenius on O_F and setting $\varphi(\mu) = (1+\mu)^p - 1$ and a continuous action of Γ_F given as $g(\mu) = (1+\mu)^{\chi(g)} - 1$ for $g \in \Gamma_F$ and χ the p-adic cyclotomic character. We have a natural embedding $\mathbf{A}_F^+ \subset \mathbf{A}_{\inf}(O_{F_\infty})$ compatible with Frobenius and Γ_F-action. Set $\mathbf{A}_F := \mathbf{A}_F^+[1/\mu]^\wedge$, where $^\wedge$ denotes the p-adic completion. The Frobenius endomorphism φ and continuous action of Γ_F on $\mathbf{A}_F^+$ naturally extend to $\mathbf{A}_F$. Let $\tilde{\mathbf{A}} := W(\mathbb{C}_p^\flat)$ and $\tilde{\mathbf{B}} := \tilde{\mathbf{A}}[1/p]$ admitting the Frobenius on Witt vectors and continuous G_F-action (for the weak topology). We have natural Frobenius and Γ_F-equivariant embeddings $\mathbf{A}_F^+ \subset \mathbf{A}_{\inf}(O_{F_\infty})$ which extends to an embedding $\mathbf{A}_F \subset \tilde{\mathbf{A}}^{H_F}$ and we set $\mathbf{B}_F = \mathbf{A}_F[1/p]$. Let $\mathbf{A}$ denote the p-adic completion of the maximal unramified extension of $\mathbf{A}_F$ inside $\tilde{\mathbf{A}}$ and set $\mathbf{B} := \mathbf{A}[1/p] \subset \tilde{\mathbf{B}}$. The rings $\mathbf{A}$ and $\mathbf{B}$ are stable under Frobenius and G_F-action on $\tilde{\mathbf{B}}$ and we equip them with induced structures. We have $\mathbf{A}_F = \mathbf{A}^{H_F}$ and $\mathbf{B}_F = \mathbf{B}^{H_F}$. Moreover, set $\mathbf{A}^+ := \mathbf{A}_{\inf}(O_{\overline{F}}) \cap \mathbf{A} \subset \tilde{\mathbf{A}}$ and $\mathbf{B}^+ := \mathbf{A}^+[1/p]$ stable under the Frobenius and G_F-action on $\mathbf{B}$, then we have $\mathbf{A}_F^+ = (\mathbf{A}^+)^{H_F}$ and $\mathbf{B}_F^+ = (\mathbf{B}^+)^{H_F}$.

2.2. p-adic representations. Let T be a finite free $\mathbb{Z}_p$-representation of G_F. By the theory of étale (φ, Γ_F)-modules (see [Fon90]) one can functorially associate to T a finite free étale (φ, Γ_F)-module $\mathbf{D}_F(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_F}$ over $\mathbf{A}_F$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$. Moreover, we have a natural (φ, Γ_F)-equivariant isomorphism $\mathbf{A} \otimes_{\mathbf{A}_F} \mathbf{D}_F(T) \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T$. These constructions are functorial in $\mathbb{Z}_p$-representations and induce an exact equivalence of $\otimes$-categories

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_F)\text{-}\mathrm{Mod}^{\text{ét}}_{\mathbf{A}_F}, \tag{2.2}$$

with an exact $\otimes$-compatible quasi-inverse given as $\mathbf{T}_F(D) := (\mathbf{A} \otimes_{\mathbf{A}_F} D)^{\varphi=1} = (\tilde{\mathbf{A}} \otimes_{\mathbf{A}_F} D)^{\varphi=1}$. Similar statements are also true for p-adic representations of G_F. Furthermore, let $\mathbf{D}_F^+(T) := (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_F}$ be the (φ, Γ_F)-module over $\mathbf{A}_F^+$ associated to T and for $V := T[1/p]$ let $\mathbf{D}_F^+(V) := \mathbf{D}_F^+(T)[1/p]$ be the (φ, Γ_F)-module over $\mathbf{B}_F^+$ associated to V.

From p-adic Hodge theory of G_F (see [Fon82]), one can attach to a p-adic representation V a filtered φ-module over F of rank $\leq \dim_{\mathbb{Q}_p} V$ given as $\mathbf{D}_{\mathrm{cris}}(V) := (\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)^{G_F}$. The representation V is said to be crystalline if the natural map $\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_F \mathbf{D}_{\mathrm{cris}}(V) \to \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V$ is an isomorphism, or equivalently, V is crystalline if and only if $\dim_F \mathbf{D}_{\mathrm{cris}}(V) =$

$\dim_{\mathbb{Q}_p} V$. Restricting $\mathbf{D}_{\mathrm{cris}}$ to the category of crystalline representations of G_F and writing $\mathrm{MF}_F^{\mathrm{wa}}(\varphi)$ for the category of weakly admissible filtered φ-modules over F (see [CF00]), we obtain an exact equivalence of $\otimes$-categories (see [CF00, Théorème A])

$$\mathbf{D}_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_F) \xrightarrow{\sim} \mathrm{MF}_F^{\mathrm{wa}}(\varphi), \tag{2.3}$$

with an exact $\otimes$-compatible quasi-inverse given as $\mathbf{V}_{\mathrm{cris}}(D) := (\mathrm{Fil}^0(\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_F D))^{\varphi=1}$.

2.3. Fontaine-Herr complex. Let T be a $\mathbb{Z}_p$-representation of G_F, and let $\mathbf{D}_F(T)$ denote the associated étale (φ, Γ_F)-module over $\mathbf{A}_F$. In [Her98], Herr obtained a three term complex computing continuous G_F-cohomology of T in terms of $\mathbf{D}_F(T)$. More precisely, recall that γ is a topological generator of Γ_F and consider the following complex:

$$\mathscr{C}^\bullet(\mathbf{D}_F(T)) : \mathbf{D}_F(T) \xrightarrow{(\gamma-1, 1-\varphi)} \mathbf{D}_F(T) \oplus \mathbf{D}_F(T) \xrightarrow{(1-\varphi, \gamma-1)^{\intercal}} \mathbf{D}_F(T), \tag{2.4}$$

where the first map is $x \mapsto ((\gamma-1)x, (1-\varphi)y)$ and the second map is $(x, y) \mapsto (1-\varphi)x - (\gamma-1)y$. Then the complex $\mathscr{C}^\bullet(T)$ computes the continuous G_F-cohomology of T in each cohomological degree, i.e. for each $k \in \mathbb{N}$, we have natural (in T) isomorphims $H^k(\mathscr{C}^\bullet(T)) \xrightarrow{\sim} H^k_{\mathrm{cont}}(G_F, T)$. From the complex it is clear that $H^k_{\mathrm{cont}}(G_F, T) = 0$ for $i \geq 3$. To ease notations, from now onwards we will write $H^k(G_F, V)$ instead of $H^k_{\mathrm{cont}}(G_F, T)$.

Note that for a $\mathbb{Z}_p$-representation T of G_F, the space $H^1(G_F, T)$ classifies all extension classes of $\mathbb{Z}_p$ by T in the category of $\mathbb{Z}_p$-representations of G_F. Similarly, for an étale (φ, Γ_F)-module D, the space $H^1(\mathscr{C}^\bullet(D))$ classifies all extension classs of $\mathbf{A}_F$ by D in the category of étale (φ, Γ_F)-modules over $\mathbf{A}_F$. In particular, we have natural isomorphisms

$$H^1(G_F, T) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}_{\mathbb{Z}_p}(G_F)}(\mathbb{Z}_p, T) \xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma_F)\text{-}\mathrm{Mod}^{\mathrm{\acute{e}t}}_{\mathbf{A}_F}}(\mathbf{A}_F, \mathbf{D}_F(T)) \xleftarrow{\sim} H^1(\mathscr{C}^\bullet(\mathbf{D}_F(T))).$$

2.4. Bloch-Kato Selmer groups. In this section we will recall the definition of Bloch-Kato Selmer groups from [BK90]. Let V be a p-adic crystalline representation of G_F. Then we have a natural G_F-equivariant map $V \to \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V$ sending $x \mapsto 1 \otimes x$. Considering the continuous G_F-cohomology groups, we obtain natural maps $H^k(G_F, V) \to H^k(G_F, \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)$, for each $k \in \mathbb{N}$.

Definition 2.1. Define the Bloch-Kato Selmer groups of V denoted $H^k_f(G_F, V) \subset H^k(G_F, V)$, for $k \in \mathbb{N}$, by setting $H^0_f(G_F, V) = H^0(G_F, V)$, $H^k_f(G_F, V) = 0$ for $k \geq 2$ and

$$H^1_f(G_F, V) := \mathrm{Ker}\left(H^1(G_F, V) \to H^1(G_F, \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)\right).$$

Remark 2.2. For $k \in \mathbb{N}$, the subspace $H^k_f(G_F, V) \subset H^k(G_F, V)$ are also referred to as the crystalline part of the Galois cohomology of V. Notably, the subspace $H^1_f(G_F, V) \subset H^1(G_F, V)$ classifies all crystalline extension classes of $\mathbb{Q}_p$ by V, i.e. we have natural isomorphisms

$$H^1_f(G_F, V) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F)}(\mathbb{Q}_p, V) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{MF}^{\mathrm{wa}}_F(\varphi)}(F, \mathbf{D}_{\mathrm{cris}}(V)),$$

where the last isomorphism follows from exactness of functors $\mathbf{D}_{\mathrm{cris}}$ and $\mathbf{V}_{\mathrm{cris}}$ (see §2.2).

Now we note that we have a natural G_F-equivariant map $V \to \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V$ sending $x \mapsto 1 \otimes x$ and it induces a natural map $H^1(G_F, V) \to H^1(G_F, \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)$.

Proposition 2.3. *The following natural map is an isomorphism:*

$$\mathrm{Ker}\left(H^1(G_F, V) \to H^1(G_F, \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V)\right) \xrightarrow{\sim} H^1_f(G_F, V).$$

Proof. By naturality of G_F-action we have a commutative diagram

$$\begin{tikzcd} H^1(G_F, V) \arrow[r] \arrow[dr] & H^1(G_F, \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V) \arrow[d] \\ & H^1(G_F, \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V). \end{tikzcd} \tag{2.5}$$

To show the claim it is enough to show that the right vertical arrow is injective. Now consider the following exact sequence:

$$0 \longrightarrow \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \longrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \longrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}})/\mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \longrightarrow 0.$$

Upon tensoring this exact sequence with V and taking continuous G_F-cohomology we obtain an injective map of F-vector spaces

$$\alpha : \mathbf{D}_{\mathrm{cris}}(V)/\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \left(\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}})/\mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V\right)^{G_F}, \tag{2.6}$$

and we see that the vertical map in (2.5) is injective if and only if (2.6) is bijective. Since $\mathbf{B}_{\mathrm{dR}}(O_{\overline{F}}) = \mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}}(O_{\overline{F}}) + \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}})^{\varphi=1}$ (see [BK90, Proposition 1.17]), we have G_F-equivariant isomorphisms

$$\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}})/\mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \xrightarrow{\sim} \mathbf{B}_{\mathrm{dR}}(O_{\overline{F}})/\mathrm{Fil}^0\mathbf{B}_{\mathrm{dR}}(O_{\overline{F}}) \xrightarrow{\sim} \oplus_{k<0}\mathbb{C}_p \cdot t^k,$$

where the last isomorphism follows from [Fon94, §1.5.5]. Therefore, the codomain of the map (2.6) can be written as $\left(\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}})/\mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V\right)^{G_F} = (\oplus_{k<0} t^k \mathbb{C}_p \otimes_{\mathbb{Q}_p} V)^{G_F} = \oplus_{k<0}\mathrm{gr}^k\mathbf{D}_{\mathrm{cris}}(V)$. Counting dimensions, we note that we have

$$\dim_F(\mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V)) + \dim_F(\oplus_{k<0}\mathrm{gr}^k\mathbf{D}_{\mathrm{cris}}(V)) = \dim_F \mathbf{D}_{\mathrm{cris}}(V),$$

so the domain and codomain of the F-linear injective map in (2.6) have the same dimension. Hence, (2.6) is bijective. ∎

Corollary 2.4. *Let V be a p-adic crystalline representation of G_F. Then the following complex*

$$\mathscr{D}^\bullet(\mathbf{D}_{\mathrm{cris}}(V)) : \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{1-\varphi} \mathbf{D}_{\mathrm{cris}}(V), \tag{2.7}$$

computes the crystalline part of the Galois cohomology of V, i.e. $H^k(\mathscr{D}^\bullet(\mathbf{D}_{\mathrm{cris}}(V))) \xrightarrow{\sim} H^k_f(G_F, V)$ for each $k \in \mathbb{N}$.

Proof. Tensoring the fundamental exact sequence in (2.1) with V, we obtain a G_F-equivariant exact sequence

$$0 \longrightarrow V \longrightarrow \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V \xrightarrow{1-\varphi} \mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V \longrightarrow 0.$$

By taking the continuous Galois cohomoloy of the terms above, we obtain a long exact sequence

$$\begin{aligned} 0 \longrightarrow H^0(G_F, V) \longrightarrow \mathrm{Fil}^0\mathbf{D}_{\mathrm{cris}}(V) \xrightarrow{1-\varphi} \mathbf{D}_{\mathrm{cris}}(V) \longrightarrow H^1(G_F, V) \longrightarrow \\ \longrightarrow H^1(G_F, \mathrm{Fil}^0\mathbf{B}_{\mathrm{cris}}(O_{\overline{F}}) \otimes_{\mathbb{Q}_p} V). \end{aligned} \tag{2.8}$$

The claim now follows from Proposition 2.3. ∎

3. Wach modules

In this section we will recall the definition of Wach modules, their relation to p-adic crystalline representations and prove some results on the Nygaard filtration on Wach modules. From §2.1 recall that we have $\mathbf{A}_F^+ = O_F[\![\mu]\!]$ equipped with a Frobenius endomorphism φ and a continuous action of Γ_F. Set $q := 1 + \mu$ and $[p]_q := \tilde{\xi} = \varphi(\mu)/\mu$.

Definition 3.1. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}_F^+$ with weights in the interval $[a, b]$ is a finite free $\mathbf{A}_F^+$-module N equipped with a continuous and semilinear action of Γ_F satisfying the following:

(1) Γ_F acts trivially on $N/\mu N$.

(2) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ commuting with the action of Γ_F such that $\varphi(\mu^b N) \subset \mu^b N$ and cokernel of the induced injective map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is killed by $[p]_q^{b-a}$.

Define the $[p]_q$*-height* of N to be the largest value of $-a$ for $a \in \mathbb{Z}$ as above. Say that N is *effective* if one can take $b = 0$ and $a \leq 0$. A Wach module over $\mathbf{B}_F^+$ is a finite module M equipped with a Frobenius-semilinear operator $\varphi : M[1/\mu] \to M[1/\varphi(\mu)]$ commuting with the action of Γ_F such that there exists a φ-stable (after inverting μ) and Γ_F-stable $\mathbf{A}_F^+$-submodule $N \subset M$ with N a Wach module over $\mathbf{A}_F^+$ (equipped with induced (φ, Γ_F)-action) and $N[1/p] = M$.

Denote the category of Wach modules over $\mathbf{A}_F^+$ as $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_F^+$-linear, Γ_F-equivariant and φ-equivariant (after inverting μ) morphisms. Extending scalars along $\mathbf{A}_F^+ \to \mathbf{A}_F$ induces a fully faithful functor $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q} \to (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_F}^{\text{ét}}$ (see [Abh23a, Proposition 3.3]).

3.1. Relation to crystalline representations.

Let $\text{Rep}_{\mathbb{Z}_p}^{\text{cris}}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_F. To any T in $\text{Rep}_{\mathbb{Z}_p}^{\text{cris}}(G_F)$, Berger functorially attaches a unique Wach module $\mathbf{N}_F(T)$ over $\mathbf{A}_F^+$ in [Ber04]. Then we have the following:

Theorem 3.2 ([Wac96, Wach], [Col99, Colmez], [Ber04, Berger])**.** *The Wach module functor induces an equivalence of $\otimes$-catgeories*

$$\text{Rep}_{\mathbb{Z}_p}^{\text{cris}}(G_F) \xrightarrow{\sim} (\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}, \qquad T \longmapsto \mathbf{N}_F(T),$$

with a $\otimes$-compatible quasi-inverse given as $N \mapsto \mathbf{T}_F(N) = \big(W(\mathbb{C}_p^\flat) \otimes_{\mathbf{A}_F^+} N\big)^{\varphi=1}$.

Remark 3.3. In Theorem 3.2 note that we do not expect the functor $\mathbf{N}_F$ to be exact. However, after inverting p, the Wach module functor induces an exact equivalence of $\otimes$-categories $\text{Rep}_{\mathbb{Q}_p}^{\text{cris}}(G_F) \xrightarrow{\sim} (\varphi, \Gamma)\text{-Mod}_{\mathbf{B}_F^+}^{[p]_q}$, via $V \mapsto \mathbf{N}_F(V)$ and with an exact $\otimes$-compatible quasi-inverse given as $M \mapsto \mathbf{V}_F(M) = \big(W(\mathbb{C}_p^\flat) \otimes_{\mathbf{A}_F^+} M\big)^{\varphi=1}$ (see [Abh23a, Corollary 4.3]).

Remark 3.4. Let N be a Wach module over $\mathbf{A}_F^+$ and $T = \mathbf{T}_F(N)$ the associated $\mathbb{Z}_p$-representation of G_F from Theorem 3.2. Then for each $r \in \mathbb{Z}$, we have that $\mu^{-r} N(r)$ is a Wach module over $\mathbf{A}_F^+$ and $\mathbf{T}_F(\mu^{-r} N(r)) \xrightarrow{\sim} T(r)$, where (r) denotes twist by χ^r.

3.2. Nygaard filtration on Wach modules. Let N be a Wach module over $\mathbf{A}_F^+$. Define a decreasing filtration on N called the *Nygaard filtration*, for $k \in \mathbb{Z}$, as $\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}$. From the definition it is clear that N is effective if and only if $\mathrm{Fil}^0 N = N$. Similarly, we define Nygaard filtration on $M := N[1/p]$ and it satisfies $\mathrm{Fil}^k M = (\mathrm{Fil}^k N)[1/p]$.

Now note that $(N/\mu N)[1/p]$ is a φ-module over F since $[p]_q = p \bmod \mu N$ and $N/\mu N$ is equipped with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $(N/\mu N)[1/p]$ with induced filtration, in particular, it is a filtered φ-module over F. From [Ber04, Théorème III.4.4] and [Abh23a, Theorem 1.7 & Remark 1.8] we have:

Theorem 3.5. *Let N be a Wach module over $\mathbf{A}_F^+$ and $V := \mathbf{T}_F(N)[1/p]$ the associated crystalline representation from Theorem 3.2. Then we have $(N/\mu N)[1/p] \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris}}(V)$ as filtered φ-modules over F.*

From Theorem 3.5 we have a surjection $\mathrm{Fil}^k N[1/p] \twoheadrightarrow \mathrm{Fil}^k \mathbf{D}_{\mathrm{cris}}(V)$ and we would like to determine its kernel.

Lemma 3.6. *Let N be a Wach module over $\mathbf{A}_F^+$ and $j, k \in \mathbb{N}_{\geq 1}$. Then we have*

$$\mu^{-j}\mathrm{Fil}^k N \cap \mu^{-j+1} N = \mu^{-j+1}\mathrm{Fil}^{k-1} N.$$

Similar statement is true for the Wach module $N[1/p]$ over $\mathbf{B}_F^+$. Moreover, for each $k \in \mathbb{Z}$, we have an exact sequence

$$0 \longrightarrow \mu\mathrm{Fil}^{k-1}N \longrightarrow \mathrm{Fil}^k N \longrightarrow \mathrm{Fil}^k(N/\mu N) \longrightarrow 0. \tag{3.1}$$

In particular, $\mathrm{Ker}(\mathrm{Fil}^k N[1/p] \twoheadrightarrow \mathrm{Fil}^k \mathbf{D}_{\mathrm{cris}}(V)) = \mu \mathrm{Fil}^{k-1} N[1/p]$.

Proof. First part of the claim follows from [Abh23c, Lemma 3.4] and the exactness of (3.1) easily follows from the first part. Rest is obvious. ∎

Remark 3.7. The Nygaard filtration on a Wach module N over $\mathbf{A}_F^+$ is stable under the action of Γ_F. Therefore, for $g \in \Gamma_F$ and $k \in \mathbb{Z}$, we have $(g-1)\mathrm{Fil}^k N \subset (\mathrm{Fil}^k N) \cap \mu N = \mu \mathrm{Fil}^{k-1} N$.

Finally, we will check the compatibility of Nygaard filtration with exact sequences of Wach modules over $\mathbf{A}_F^+$. So consider an exact sequence of Wach modules over $\mathbf{A}_F^+$ as

$$0 \longrightarrow N_1 \longrightarrow N_2 \longrightarrow N_3 \longrightarrow 0. \tag{3.2}$$

Lemma 3.8. *For $k \in \mathbb{Z}$, we have $N_1 \cap \mathrm{Fil}^k N_2 = \mathrm{Fil}^k N_1$.*

Proof. Let $D_i := \mathbf{A}_F \otimes_{\mathbf{A}_F^+} N_i$, for $i = 1, 2$. Note that we have $N_1 := D_1 \cap N_2 \subset D_2$. So if $x \in N_1 \cap \mathrm{Fil}^k N_2$ then $\varphi(x) \in D_1 \cap [p]_q^k N_2$, i.e. $[p]_q^{-k}\varphi(x) \in D_1 \cap N_2 = N_1$. Hence, $x \in \mathrm{Fil}^k N_1$. ∎

Remark 3.9. For $j, k \in \mathbb{Z}$, we have $N_1 \cap \mu^j \mathrm{Fil}^k N_2 = \mu^j \mathrm{Fil}^k N_1$. Indeed, using the same notation as in the proof of Lemma 3.8, we note that if $x \in N_1 \cap \mu^j N_2$ then $x = \mu^j y$ for $y \in N_2$ and $y = \mu^{-j} x \in D_1 \cap N_2 = N_1$, i.e. $x \in \mu^j N_1$. Combining this with Lemma 3.8 we get the claim.

We can obtain a stronger statement after inverting p. More precisely, let $M_i := N_i[1/p]$ for $i = 1, 2, 3$, be Wach modules over $\mathbf{B}_F^+$, where N_i are as in (3.2). Then we have,

Lemma 3.10. *The following sequence is exact for each $k \in \mathbb{Z}$*

$$0 \longrightarrow \mathrm{Fil}^k M_1 \longrightarrow \mathrm{Fil}^k M_2 \longrightarrow \mathrm{Fil}^k M_3 \longrightarrow 0. \tag{3.3}$$

Proof. For $i = 1, 2, 3$ and $r \in \mathbb{Z}$, from Remark 3.4 note that $\mu^{-r}N_i(r)$, where (r) denotes a twist by χ^r, are again Wach modules over $\mathbf{A}_F^+$ and (3.2) is exact if and only if the following is exact

$$0 \longrightarrow \mu^{-r}N_1(r) \longrightarrow \mu^{-r}N_2(r) \longrightarrow \mu^{-r}N_3(r) \longrightarrow 0.$$

Moreover, from [Abh21, Lemma 4.17] we have $\mathrm{Fil}^{k-r}(\mu^{-r}M_i(r)) = \mu^{-r}\mathrm{Fil}^k M_i(r)$ and therefore (3.3) is exact if and only if the following is exact

$$0 \longrightarrow \mathrm{Fil}^{k-r}(\mu^{-r}M_1(r)) \longrightarrow \mathrm{Fil}^{k-r}(\mu^{-r}M_2(r)) \longrightarrow \mathrm{Fil}^{k-r}(\mu^{-r}M_2(r)) \longrightarrow 0.$$

In particular, without loss of generality we may assume that M_i are effective Wach modules over $\mathbf{B}_F^+$ for each $i = 1, 2, 3$, in particular, $\mathrm{Fil}^0 M_i = M_i$. We will prove the claim by induction on $k \in \mathbb{N}$. So let us assume the claim for $k - 1$ and consider the following diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \mu\mathrm{Fil}^{k-1}M_1 \arrow[r] \arrow[d] & \mathrm{Fil}^k M_1 \arrow[r] \arrow[d] & \mathrm{Fil}^k(M_1/\mu M_1) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mu\mathrm{Fil}^{k-1}M_2 \arrow[r] \arrow[d] & \mathrm{Fil}^k M_2 \arrow[r] \arrow[d] & \mathrm{Fil}^k(M_2/\mu M_2) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mu\mathrm{Fil}^{k-1}M_3 \arrow[r] \arrow[d] & (\mathrm{Fil}^k M_2)/(\mathrm{Fil}^k M_1) \arrow[r] \arrow[d] & \mathrm{Fil}^k(M_3/\mu M_3) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd} \tag{3.4}$$

Note that first and second rows are exact by Lemma 3.6 and the first column is exact by induction assumption. In the second column, using $M_1 = (\mathbf{B}_F \otimes_{\mathbf{B}_F^+} M_1) \cap M_2 \subset \mathbf{B}_F \otimes_{\mathbf{B}_F^+} M_2$, it easily follows that $\mathrm{Fil}^k M_1 \subset \mathrm{Fil}^k M_2$. Now let $V_i := \mathbf{V}_F(M_i)$, for $i = 1, 2, 3$. Then from Theorem 3.5 we have filtered isomorphisms $\mathrm{Fil}^k(M_i/\mu M_i) \xrightarrow{\sim} \mathrm{Fil}^k\mathbf{D}_{\mathrm{cris}}(V_i)$. Recall that $\mathbf{D}_{\mathrm{cris}}$ is an exact functor and in the category $\mathrm{MF}_F^{\mathrm{wa}}(\varphi)$ exact sequences are compatible with filtration. So we get that the third column is also exact. Hence, it follows that the third row is exact and from Lemma 3.6 we conclude that $(\mathrm{Fil}^k M_2)/(\mathrm{Fil}^k M_1) \xrightarrow{\sim} \mathrm{Fil}^k M_3$, proving the claim. ∎

4. Syntomic complex and Galois cohomology

In this section we will define a syntomic complex with coefficients in a Wach module and show that, after inverting p, it computes the crystalline part of the Galois cohomology of the associated crystalline representation.

Let N be a Wach module over $\mathbf{A}_F^+$ and define an operator $\nabla_q := \frac{\gamma-1}{\mu} : N \to N$. From Remark 3.7 note that we have $\nabla_q(\mathrm{Fil}^k N) \subset \mathrm{Fil}^{k-1} N$ for each $k \in \mathbb{Z}$.

Definition 4.1. Define the *syntomic complex* with coeffcients in N as

$$\mathcal{S}^\bullet(N) : \mathrm{Fil}^0 N \xrightarrow{(\nabla_q, 1-\varphi)} \mathrm{Fil}^{-1} N \oplus N \xrightarrow{(1-[p]_q\varphi, \nabla_q)^\intercal} N, \tag{4.1}$$

where the first map is $x \mapsto (\nabla_q(x), (1-\varphi)x)$ and the second map is $(x, y) \mapsto (1-[p]_q\varphi)x - \nabla_q(y)$.

The goal of this section is to show the following claim:

Theorem 4.2. *Let N be a Wach module over $\mathbf{A}_F^+$ and $V = \mathbf{T}_F(N)[1/p]$ the associated p-adic crystalline representation of G_F from Theorem 3.2. Then we have a natural isomorphism for each $k \in \mathbb{N}$*

$$H^k(\mathcal{S}^\bullet(N))[1/p] \xrightarrow{\sim} H^k_f(G_F, V).$$

Proof. The claim for H^0_f follows from Lemma 4.4. For H^1_f recall that from Remark 2.2 we have a natural (in V) isomorphism

$$H^1_f(G_F, V) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F)}(\mathbb{Q}_p, V).$$

Moreover, from Remark 3.3 the functors $\mathbf{N}_F$ and its quasi-inverse $\mathbf{V}_F$ are exact. Therefore, we have a natural (in V) isomorphism

$$\mathrm{Ext}^1_{(\varphi,\Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{B}^+_F}}(\mathbf{B}^+_F, N[1/p]) \xrightarrow{\sim} \mathrm{Ext}^1_{\mathrm{Rep}^{\mathrm{cris}}_{\mathbb{Q}_p}(G_F)}(\mathbb{Q}_p, V).$$

Combining these observations with Proposition 4.5 (after inverting p) we get a natural (in V) isomorphism

$$H^1(\mathcal{S}^\bullet(N, r))[1/p] \xrightarrow{\sim} H^1_f(G_F, V).$$

Finally, note that the Wach module $N[1/p]$ over $\mathbf{B}^+_F$ can always be written as a twist of an effective Wach module over $\mathbf{B}^+_F$ and similarly, the representation V is the twist of the corresponding positive crystalline representation by a power of the cyclotomic character (see Remark 3.4). Therefore, the claim for H^2_f follows from Proposition 4.6. ∎

Remark 4.3. The complex in (4.1) is isomorphic to the complex

$$\mathrm{Fil}^0 N \xrightarrow{(\gamma-1, 1-\varphi)} \mu\mathrm{Fil}^{-1} N \oplus N \xrightarrow{(1-\varphi, \gamma-1)^\intercal} \mu N,$$

which can be seen as a subcomplex of the Fontaine-Herr complex of $\mathbf{A}_F \otimes_{\mathbf{A}^+_F} N$ (see §2.3).

4.1. Comparing H^0 and H^1.

In this section we will compute H^0 and H^1 of the complex $\mathcal{S}^\bullet(N)$.

Lemma 4.4. *Let N be a Wach module over $\mathbf{A}^+_F$ and $T = \mathbf{T}_F(N)$ the associated $\mathbb{Z}_p$-representation of G_F from Theorem 3.2 such that $T[1/p]$ is crystalline. Then we have a natural isomorphism*

$$H^0(\mathcal{S}^\bullet(N)) = (\mathrm{Fil}^0 N)^{\varphi=1, \nabla_q=0} \xrightarrow{\sim} T^{G_F}.$$

Proof. A simple computation shows that $(\mathrm{Fil}^0 N)^{\varphi=1,\nabla_q=0} = (\mathrm{Fil}^0 N)^{\varphi=1,\gamma=1} = N^{\varphi=1,\gamma=1} = N^{\varphi=1,\Gamma_F}$, where the last equality follows from the continuity of Γ_F-action on N. Now from [Ber04, Proposition II.1.1] we have $N[1/\mu] \xrightarrow{\sim} \mathbf{D}^+_F(T)[1/\mu]$. Note that $(\mathbf{A}^+[1/\mu])^{\varphi=1} = \mathbb{Z}_p$, therefore it follows that

$$(N[1/\mu])^{\varphi=1,\gamma=1} = (\mathbf{D}^+_F(T)[1/\mu])^{\varphi=1,\gamma=1} \xrightarrow{\sim} T^{G_F}.$$

We claim that $N^{\varphi=1,\gamma=1} = (N[1/\mu])^{\varphi=1,\gamma=1}$, which is enough to prove the lemma. Indeed, let $(x/\mu^k) \in N[1/\mu]^{\varphi=1,\gamma=1}$ for some $x \in N$ and $k \in \mathbb{Z}$. Then it is enough to show that $x \in \mu^k N$. Note that $g = \gamma^{p-1}$ is a topological generator of $\mathrm{Gal}(F_\infty/F(\zeta_p))$ and we have $g(x) = (g(\mu)^k/\mu^k)x$. Reducing modulo μ, we obtain $g(x) = \chi(g)^k x \mod \mu N$. Since Γ_F acts trivially on $N/\mu N$ and $\chi(g)^k - 1$ is a unit in $\mathbf{B}^+_F$, we obtain that $x \in \mu N[1/p] \cap N = \mu N$. Iterating this k times we obtain $x \in \mu^k N$ as desired. ∎

Proposition 4.5. *Let N be a Wach module over $\mathbf{A}^+_F$. Then we have a natural (in N) isomorphism*

$$H^1(\mathcal{S}^\bullet(N)) \xrightarrow{\sim} \mathrm{Ext}^1_{(\varphi,\Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{A}^+_F}}(\mathbf{A}^+_F, N).$$

Proof. We will construct a map

$$\alpha : H^1(\mathcal{S}^\bullet(N)) \longrightarrow \mathrm{Ext}^1_{(\varphi,\Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{A}_F^+}}(\mathbf{A}_F^+, N), \tag{4.2}$$

and show that it is bijective by constructing an inverse map. Let (x, y) represent a class in $H^1(\mathcal{S}^\bullet(N))$, i.e. we have $x \in \mathrm{Fil}^{-1}N$ and $y \in N$ such that $(1 - [p]_q\varphi)x = \nabla_q(y)$. Set $E_1 = N \oplus \mathbf{A}_F^+ \cdot e$ with $\gamma(e) = \mu x + e$ and $\varphi(e) = y + e$. Clearly, E_1 is a Wach module over $\mathbf{A}_F^+$. Moreover, by sending e to $1 \in \mathbf{A}_F^+$ we have an exact sequence of Wach modules over $\mathbf{A}_F^+$

$$0 \longrightarrow M \longrightarrow E_1 \longrightarrow \mathbf{A}_F^+ \longrightarrow 0,$$

This represents an extension class of $\mathbf{A}_F^+$ by N in the category $(\varphi, \Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{A}_F^+}$ and we set $\alpha[(x, y)] = [E_1]$, where we represent cohomological classes with "[]". To show that α is well-defined we must show that the extension class $[E_1]$ is independent of the choice of presentation (x, y). Indeed, let (x', y') denote another presentation such that $x'-x = \nabla_q(w)$, $y'-y = (1-\varphi)w$ for some $w \in \mathrm{Fil}^0N$. Then similar to above $E_2 = N \oplus \mathbf{A}_F^+ \cdot e'$, with $\gamma(e') = \mu x' + e'$ and $\varphi(e') = y' + e'$, is a Wach module over $\mathbf{A}_F^+$ and an extension of $\mathbf{A}_F^+$ by N. Let us define a map $f : E_1 \to E_2$ given as identity on N and letting $f(e) = e' - w$. Then f is bijective since we have $f^{-1} : E_2 \to E_1$ given as identity on M and letting $f^{-1}(e') = e + w$ and $f \circ f^{-1} = id$ and $f^{-1} \circ f = id$. From the formulas $x' - x = \nabla_q(w)$ and $y' - y = (1 - \varphi)y$ it is easy to verify that f and f^{-1} are (φ, Γ_F)-equivariant. Now consider the following diagram with $\mathbf{A}_F^+$-linear maps and exact rows

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r] \arrow[d, equal] & E_1 \arrow[r] \arrow[d, "f", "\wr"'] & \mathbf{A}_F^+ \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & N \arrow[r] & E_2 \arrow[r] & \mathbf{A}_F^+ \arrow[r] & 0.
\end{tikzcd}$$

The left square commutes by definition of f. Moreover, the $\mathbf{A}_F^+$-linear map $E_1 \to \mathbf{A}_F^+$ sends $e \mapsto 1$ and the $\mathbf{A}_F^+$-linear map $E_2 \to \mathbf{A}_F^+$ sends $e' \mapsto 1$, therefore its follows that right square commutes as well. Hence, E_1 and E_2 represent the same extension class of $\mathbf{A}_F^+$ by N in the category $(\varphi, \Gamma_F)\text{-}\mathrm{Mod}^{[p]_q}_{\mathbf{A}_F^+}$. In particular, α is well-defined.

Now we will construct an inverse of α which we will denote by β. Consider an extension of Wach modules over $\mathbf{A}_F^+$ as

$$0 \longrightarrow N \longrightarrow E_1 \longrightarrow \mathbf{A}_F^+ \longrightarrow 0.$$

We write $E_1 = N \oplus \mathbf{A}_F^+ \cdot e$, where $e \in E_1$ is a lift of $1 \in \mathbf{A}_F^+$ and we have $(\gamma - 1)e = z$ and $(1 - \varphi)e = y$ for some $y, z \in N$. But then $\varphi(e) = e - y \in E_1$, i.e. $e \in \mathrm{Fil}^0E_1$. Therefore, $z = (\gamma - 1)e \in N \cap \mu\mathrm{Fil}^{-1}E_1 = \mu\mathrm{Fil}^{-1}N$, where the last equality follows from Remark 3.9. In particular, we have $\nabla_q(e) = \frac{\gamma-1}{\mu}e = x$, for some $x \in \mathrm{Fil}^{-1}N$. By the commutativity of φ and γ, we get that $(1 - [p]_q\varphi) \circ \nabla_q(e) = \nabla_q \circ (1 - \varphi)e$, or equivalently,

$$(1 - [p]_q\varphi)x = \nabla_q(y).$$

Therefore, (x, y) represents a cohomological class in $H^1(\mathcal{S}^\bullet(N))$ and we set $\beta([E_1]) = [(x, y)]$. Let us first show that the class $[(x, y)]$ is independent of the lift $e \in E_1$ of $1 \in \mathbf{A}_F^+$. So let $e' \in E$ denote another lift of $1 \in \mathbf{A}_F^+$, then arguing as above we have $e' \in \mathrm{Fil}^0E$ and there exist $x' \in \mathrm{Fil}^{-1}N$ and $y' \in M$ such that $\nabla_q(e') = x'$, $(1 - \varphi)e' = y'$ and $(1 - [p]_q\varphi)x = \nabla_q(y)$. Moreover, from Lemma 3.8 we note that $w = e' - e \in \mathrm{Fil}^0E \cap N = \mathrm{Fil}^0N$, in particular, we get that $x' = x + \nabla_q(w)$ and $y' = y + (1 - \varphi)w$. Since $(1 - [p]_q\varphi) \circ \nabla_q = \nabla_q \circ (1 - \varphi)$, we conclude that (x, y) and (x', y') represent the same class in $H^1(\mathcal{S}^\bullet(N))$. Now to show that α^{-1} is well-defined we must show that the class $[(x, y)]$ is independent of the presentation E_1 of the extension class.

So let E_2 denote another presentation of the extension class $[E_1]$, i.e. E_2 is a Wach module over $\mathbf{B}_F^+$ and there exists a (φ, Γ_F)-equivariant isomorphism $f : E_1 \xrightarrow{\sim} E_2$ fitting into the following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & E_1 & \longrightarrow & \mathbf{A}_F^+ & \longrightarrow & 0 \\ & & \Vert & & {\scriptstyle f}\downarrow{\scriptstyle \wr} & & \Vert & & \\ 0 & \longrightarrow & N & \longrightarrow & E_2 & \longrightarrow & \mathbf{A}_F^+ & \longrightarrow & 0. \end{array}$$

Let $e'' \in E_2$ denote a lift of $1 \in \mathbf{A}_F^+$ and arguing as above we have $e'' \in \mathrm{Fil}^0 E_2$ and there exist $x'' \in \mathrm{Fil}^{-1}N$ and $y'' \in N$ such that $\nabla_q(e'') = x''$, $(1-\varphi)e'' = y''$ and $(1-[p]_q\varphi)x'' = \nabla_q(y'')$. From the commutative diagram above we note that $f^{-1}(e'') \in E_1$ denotes a lift of $1 \in \mathbf{A}_F^+$ and it follows that $f^{-1}(e'') \in \mathrm{Fil}^0 A_1$ and $\nabla_q(f^{-1}(e'')) = x''$, $(1-\varphi)f^{-1}(e'') = y''$ and $(1-[p]_q\varphi)x'' = \nabla_q(y'')$. Using the independence from choice of a lift in E_1 of $1 \in \mathbf{A}_F^+$, it follows that (x, y) and (x'', y'') represent the same cohomological class in $H^1(\mathcal{S}^\bullet(N))$. Hence, β is well-defined.

Finally, we need to show that the two constructions described above are inverse to each other, i.e. $\alpha \circ \beta = id$ and $\beta \circ \alpha = id$. Note that starting with a class $[(x, y)]$ in $H^1(\mathcal{S}^\bullet(N))$ we can construct an E an extension of $\mathbf{A}_F^+$ by N in $(\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$, such that $[E] = \alpha[(x, y)]$, i.e. E can be described using (x, y) as above. After applying β we obtain a class $\beta([E]) = [(x', y')]$ in $H^1(\mathcal{S}^\bullet(N))$ with a presentation (x', y') depending on the choice of a lift in E of $1 \in \mathbf{A}_F^+$. By construction, E admits a description using (x, y) and (x', y') depending on the choice of a lift in E of $1 \in \mathbf{A}_F^+$. Since $[E]$ is independent of this choice, it follows that $[(x', y')] = \beta([E]) = [(x, y)]$ in $H^1(\mathcal{S}^\bullet(N))$. Next, starting with E an extension of $\mathbf{A}_F^+$ by N in $(\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$ we can construct a class $[(x, y)] = \beta([E])$ in $H^1(\mathcal{S}^\bullet(N))$. After applying α, we obtain an extension class $[E'] = \alpha[(x, y)]$ where E' is an extension of $\mathbf{A}_F^+$ by M in $(\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$. By construction, $E = N \oplus \mathbf{A}_F^+ \cdot e$ with $\nabla_q(e) = x$ and $(1-\varphi)e = y$, and $E' = N \oplus \mathbf{A}_F^+ \cdot e'$ with $\nabla_q(e') = x$ and $(1-\varphi)e' = y$. Therefore, $f : E \to E'$ defined by identity on N and letting $f(e) = e'$ is a (φ, Γ_F)-equivariant isomorphism, in particular, $[E'] = \alpha[(x, y)] = [E]$. In conclusion, we have shown that α is a natural (in N) bijective map. ■

4.2. Rational comparison.

For convenience in computations in this section, we rephrase our goal. Let V be a p-adic positive crystalline representation of G_F, i.e. all its Hodge-Tate weights ≤ 0 and let $T \subset V$ be a G_F-stable $\mathbb{Z}_p$-lattice. Set $V(r) = V \otimes_{\mathbb{Q}_p} \mathbb{Q}_p(r)$ and $T(r) = T \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$ for $r \in \mathbb{Z}$. From Theorem 3.2 we have Wach modules $\mathbf{N}_F(T)$ and $\mathbf{N}_F(V) = \mathbf{N}_F(T)[1/p]$, such that $\mathbf{N}_F(T(r)) = \mu^{-r}\mathbf{N}_F(T)(r)$ and $\mathbf{N}_F(V(r)) = \mu^{-r}\mathbf{N}_F(V)(r)$. Let us denote the complex $\mathcal{S}^\bullet(\mu^{-r}\mathbf{N}_F(T)(r))[1/p]$ by $\mathcal{S}^\bullet(\mathbf{N}_F(V), r)$. Then our goal is to show the following claim:

Proposition 4.6. *The cohomology group $H^2(\mathcal{S}^\bullet(\mathbf{N}_F(V), r))$ vanishes. In particular, we have $H^2(\mathcal{S}^\bullet(\mathbf{N}_F(V), r)) = H^2_f(G_F, V(r)) = 0$.*

Proof. Let $x \in \mathbf{N}_F(V(r))$, then to prove the claim it is enough to show that we can write $x = \nabla_q(y) - (1-[p]_q\varphi)z$ for some $y \in \mathbf{N}_F(V(r))$ and $z \in \mathrm{Fil}^{-1}\mathbf{N}_F(V(r))$. Write $x = \frac{x'}{\mu^r} \otimes \epsilon^{\otimes r}$, where $\epsilon^{\otimes r}$ denote a $\mathbb{Q}_p$-basis of $\mathbb{Q}_p(r)$. The from Lemma 4.8 there exist $y', z' \in \mathbf{N}_F(V)$ such that

$$\tfrac{x'}{\mu^r} \otimes \epsilon^{\otimes r} = \nabla_q\big(\tfrac{y'}{\mu^{r-1}} \otimes \epsilon^{\otimes r}\big) - (1-[p]_q\varphi)(z' \otimes \epsilon^{\otimes r}).$$

Letting $z = z' \otimes \epsilon^{\otimes r}$ and $y = \frac{y'}{\mu^{r-1}} \otimes \epsilon^{\otimes r}$, we get that $x = \nabla_q(y) - (1-[p]_q\varphi)z$ with $y \in \mathbf{N}_F(V(r))$ and $z \in \mathbf{N}_F(V)(r) \subset \mathbf{N}_F(V(r))$. However, note that $[p]_q\varphi(z) = x + z + \nabla_q(y) \in \mathbf{N}_F(V(r))$, in particular, $z \in \mathrm{Fil}^{-1}\mathbf{N}_F(V(r))$. Hence, we get the claim. ■

Remark 4.7. For any $x \in \mathbf{N}_F(T)$ there exists some $y \in \mathbf{N}_F(T)$ such that $(1-[p]_q\varphi)y = x$ because the series $(1+[p]_q\varphi+([p]_q\varphi)^2+\cdots)$ converges as series of operators on $\mathbf{N}_F(T)$ since $\prod_{k=0}^{n}\varphi^k([p]_q) \in (p,\mu)^{n+1}$ for all $n \in \mathbb{N}$. In particular, $([p]_q\varphi)^n$ is (p,μ)-adically nilpotent and we can take $y = (1+[p]_q\varphi+([p]_q\varphi)^2+\cdots)x \in \mathbf{N}_F(T)$. Similar claim is also true for $\mathbf{N}_F(V)$.

Let $\epsilon^{\otimes r}$ denote a $\mathbb{Q}_p$-basis of $\mathbb{Q}_p(r)$. Then the following result was used in Proposition 4.6:

Lemma 4.8. *Let $x \in \mathbf{N}_F(V)$ then for $1 \leq k \leq r$ there exist $y, z \in \mathbf{N}_F(V)$ such that*

$$\tfrac{x}{\mu^k} \otimes \epsilon^{\otimes r} = \nabla_q\big(\tfrac{y}{\mu^{k-1}} \otimes \epsilon^{\otimes r}\big) - (1-[p]_q\varphi)(z \otimes \epsilon^{\otimes r}).$$

Proof. Note that $g = \gamma^{p-1}$ is a topological generator of $\mathrm{Gal}(F_\infty/F(\zeta_p))$, in particular, we have that $\chi(g)-1 \in p\mathbb{Z}_p$. Moreover, we can write $g-1 = \gamma^{p-1}-1 = (\gamma-1)(1+\gamma^2+\cdots+\gamma^{p-2})$ and up to multiplying by some power of p we may assume that $x \in \mathbf{N}_F(T)$. Therefore, it is enough to show that for $x \in \mathbf{N}_F(T)$ there exist $y, z \in \mathbf{N}_F(V)$ such that

$$\tfrac{x}{\mu^k} \otimes \epsilon^{\otimes r} = \tfrac{g-1}{\mu}\big(\tfrac{y}{\mu^{k-1}} \otimes \epsilon^{\otimes r}\big) - (1-[p]_q\varphi)(z \otimes \epsilon^{\otimes r}). \tag{4.3}$$

Let $(g-1)x = \mu x_1$ for some $x_1 \in \mathbf{N}_F(T)$ and we will prove the claim by induction on k. So let $k=1$ and consider the following

$$\tfrac{g-1}{\mu}\big(\tfrac{x}{\chi(g)^r-1} \otimes \epsilon^{\otimes r}\big) = \big(\tfrac{x}{\mu} + \tfrac{\chi(g)^r x_1}{\chi(g)^r-1}\big) \otimes \epsilon^{\otimes r} = \big(\tfrac{x}{\mu} + (1-[p]_q\varphi)z_1\big) \otimes \epsilon^{\otimes r},$$

where $z_1 \in \mathbf{N}_F(V)$ following Remark 4.7. Upon rearranging the terms we see that (4.3) holds for $k=1$. Now we write $u = (\chi(g)\mu)/g(\mu) \in 1+p\mu\mathbf{A}_F^+$, let $1 < k \leq r$ and assume (4.3) holds for $k-1$. Then we have

$$\begin{aligned}
\tfrac{g-1}{\mu}\big(\tfrac{x}{\mu^{k-1}(\chi(g)^{r-k+1}-1)} \otimes \epsilon^{\otimes r}\big) &= \tfrac{u^{k-1}\chi(g)^{r-k+1}-1}{\mu^k(\chi(g)^{r-k+1}-1)} x \otimes \epsilon^{\otimes r} + \tfrac{u^{k-1}\chi(g)^{r-k+1}}{\mu^{k-1}(\chi(g)^{r-k+1}-1)} x_1 \otimes \epsilon^{\otimes r}\\
&= \big(\tfrac{x}{\mu^k} + \tfrac{x_k}{\mu^{k-1}}\big) \otimes \epsilon^{\otimes r}\\
&= \tfrac{x}{\mu^k} \otimes \epsilon^{\otimes r} + \tfrac{g-1}{\mu}\big(\tfrac{y_k}{\mu^{k-2}} \otimes \epsilon^{\otimes r}\big) - (1-[p]_q\varphi)(z_k \otimes \epsilon^{\otimes r}),
\end{aligned}$$

for some $x_k, y_k, z_k \in \mathbf{N}_F(V)$ and the last equality follows from induction hypothesis. By rearranging the terms we get that (4.3) also holds for any $1 < k \leq r$. This concludes our proof. ∎

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