Syntomic cohomology and reciprocity laws

Abhinandan, Denis Benois, Nicola Mazzari

Institut de Mathématiques de Bordeaux, Université de Bordeaux

Abstract

The aim of this project is to realize an explicit version of Fontaine-Messing period map between syntomic and étale cohomology with nontrivial coefficients and apply the results to Bloch and Kato's exponential map.

Introduction

Let X denote the system of solutions of the equation

$$y^2 = x^3 - 5x + 8$$

over the complex numbers along with a point at infinity. This algebro-geometric object is known as an *elliptic curve* whose $\mathbb{C}$-rational points form an abelian group.

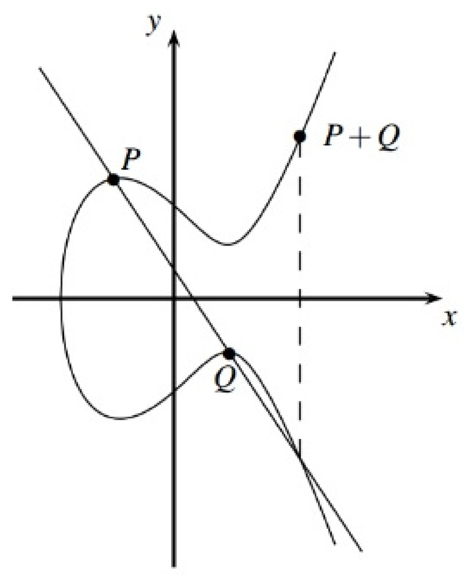

Figure: Elliptic curve addition law

For such a curve one can define the de Rham cohomology groups which are complex vector spaces,

$$H^0_{\mathrm{dR}}(X/\mathbb{C}) = \mathbb{C}, \quad H^1_{\mathrm{dR}}(X/\mathbb{C}) = \mathbb{C} \cdot \frac{dx}{y} + \mathbb{C} \cdot \frac{xdx}{y}.$$

Geometrically speaking, $X(\mathbb{C})$ is a *torus*. To such an object, one can attach singular cohomology groups which are finitely generated $\mathbb{Z}$-modules

$$H^0_{\mathrm{sing}}(X, \mathbb{Z}) = \mathbb{Z}, \quad H^1_{\mathrm{sing}}(X, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}.$$

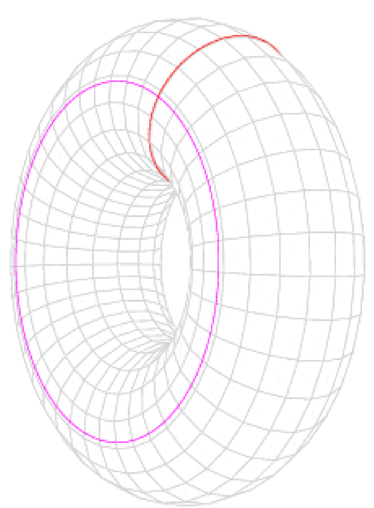

Figure: Torus: basis for H^1_{sing}

An important question in geometry is to compare such invariants attached to geometric objects.

Theorem 1 (de Rham, Grothendieck)

Let X be a smooth projective algebraic variety over $\mathbb{C}$. Then for each $i \geq 0$, there exists canonical isomorphisms

$$H^i_{\mathrm{sing}}(X(\mathbb{C}), \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{C} \simeq H^i_{\mathrm{dR}}(X/\mathbb{C}).$$

p-adic comparison theorem

Let p be a prime number, $\mathbb{Z}_p$ the p-adic completion of $\mathbb{Z}$, $\mathbb{Q}_p$ its field of fractions and $\bar{\mathbb{Q}}_p$ a fixed algebraic closure of $\mathbb{Q}_p$. Let E be an elliptic curve defined over $\mathbb{Q}_p$. Since $E(\bar{\mathbb{Q}}_p)$ is an abelian group, for each $n \in \mathbb{N}$, we have maps

$$p^n : E(\bar{\mathbb{Q}}_p) \longrightarrow E(\bar{\mathbb{Q}}_p).$$

The collection of the system of kernel of these maps for each $n \geq 0$ is denoted V_pE, the *p-adic Tate module*, which is a 2-dimensional $\mathbb{Q}_p$-vector space admitting a continuous action of the absolute Galois group $\mathscr{G} = \mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$.

(a) Evariste Galois

(b) Alexander Grothendieck

For E, one can also define p-adic étale cohomology groups, which are finite dimensional $\mathbb{Q}_p$-vector spaces

$$H^0_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq \mathbb{Q}_p, \quad H^1_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \simeq (V_pE)^*.$$

In the general case of a smooth proper scheme over $\mathbb{Q}_p$, Grothendieck defined étale cohomology groups which are p-adic representations of $\mathscr{G}$, as well as, de Rham cohomology groups which are finite dimensional filtered $\mathbb{Q}_p$-vector spaces, and wondered about the connection between these invariants.

Period	Event
1960's	Grothendieck defined de Rham and étale cohomology groups for schemes.
1970's	Following Grothendieck's idea, P. Berthelot defined crystalline cohomology groups which are naturally comparable to de Rham cohomology groups.
1980's	In an attempt to answer Grothendieck's question of *mysterious functor*, comparing étale and de Rham cohomology groups, J.-M. Fontaine formulated *crystalline comparison conjecture*.
1990's	G. Faltings, T. Tsuji (following Fontaine, W. Messing, K. Kato) and W. Niziol gave three different proofs of the conjecture.
2010's	A. Beilinson gave a completely new proof of the conjecture.

Theorem 2 (Fontaine et al.)

Let X be a smooth proper scheme defined over $\mathbb{Z}_p$. Then for each $i \geq 0$, there exists canonical isomorphisms

$$H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{crys}} \simeq H^i_{\mathrm{crys}}(X/\mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{crys}},$$

compatible with the action of $\mathscr{G}$, the Frobenius and the filtration.

$\mathbb{B}_{\mathrm{crys}}$ denotes the crystalline period ring in p-adic Hodge theory defined by Fontaine.

Syntomic complex

Syntomic = flat + locally of finite presentation + local complete intersection.

Syntomic : B. Mazur

I'm thinking of "local complete intersection" as being a way of cutting out a (sub-) space from an ambient surrounding space; the fact that it is flat over the parameter space means that each such "cutting" as you move along the parameter space, is–more or less–cut out similarly. I'm also thinking of the word "syntomic" as built from the verb temnein (i.e., to cut) and the prefix "syn" which I take in the sense of "same" or "together".

Figure: Jean-Marc Fontaine

In an attempt to prove his conjecture, Fontaine formulated the theory of syntomic cohomology which could be compared to étale cohomology as well as crystalline cohomology. Let X be a smooth proper scheme over $\mathbb{Z}_p$, and let $Y = X \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ be its special fibre. For such an X, Fontaine defined syntomic cohomology groups $H^i_{\mathrm{syn}}(X_{\bar{\mathbb{Z}}_p}, r)_{\mathbb{Q}}$ as the cohomology of the *syntomic complex* in $D^+((Y \otimes \bar{\mathbb{F}}_p)^{\text{ét}}, \mathbb{Q}_p)$, and produced a period map

$$H^i_{\mathrm{syn}}(X_{\bar{\mathbb{Z}}_p}, r)_{\mathbb{Q}} \longrightarrow H^i_{\text{ét}}(X_{\bar{\mathbb{Q}}_p}, \mathbb{Q}_p(r)),$$

which is an isomorphism for $0 \leq i \leq r$. In the other direction, we have a natural map from syntomic to crystalline cohomology

$$H^i_{\mathrm{syn}}(X_{\bar{\mathbb{Z}}_p}, r)_{\mathbb{Q}} \longrightarrow H^i_{\mathrm{crys}}(X/W) \otimes_{\mathbb{Z}_p} \mathbb{B}_{\mathrm{crys}}.$$

From these statements, one can deduce Theorem 2.

Generalizing the methods developed by Faltings, we have a comparison theorem with coefficients.

Theorem 3 (Andreatta-Iovita)

Let X be a smooth proper scheme over $\mathbb{Z}_p$ and $\mathscr{L}$ a crystalline sheaf on $X^{\text{ét}}_{\bar{\mathbb{Q}}_p}$. For each $i \geq 0$, there exists canonical isomorphisms

$$H^i(X^{\text{ét}}_{\bar{\mathbb{Q}}_p}, \mathscr{L}) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{crys}} \simeq H^i_{\mathrm{crys}}(X, \mathbb{D}^{\mathrm{ar}}_{\mathrm{crys}}(\mathscr{L})) \otimes_{\mathbb{Q}_p} \mathbb{B}_{\mathrm{crys}}$$

compatible with the action of $\mathscr{G}$, the Frobenius and the filtration.

The goal of this project is to realize the period map in Theorem 3 using syntomic methods in [1] and apply the results to Bloch and Kato's exponential map.

References

[1] P.Colmez, W. Niziol. *Syntomic complexes and p-adic nearby cycles.* Inventiones mathematicae, 208(1):1-108, 2017.

[2] F. Andreatta, & A. Iovita, *Comparison isomorphisms for smooth formal schemes.* Journal of the Institute of Mathematics of Jussieu, 12(1): 77-151, 2013.