

# PRISMATIC $F$ -CRYSTALS AND WACH MODULES

ABHINANDAN

ABSTRACT. We show a direct equivalence between the category of analytic/completed prismatic  $F$ -crystals on the prismatic site of a small (unramified at  $p$ ) base ring and the category of relative Wach modules from the theory of  $(\varphi, \Gamma)$ -modules. The result is obtained by showing that, over a Wach module, the data of the Galois action is equivalent to the prismatic descent data.

## 1. INTRODUCTION

Recent groundbreaking advances in integral  $p$ -adic Hodge theory have been brought on by the seminal works of Bhatt, Morrow and Scholze on  $A_{\text{inf}}$ -cohomology in [BMS18; BMS19], and of Bhatt and Scholze on prismatic cohomology in [BS22]. In the latter theory, the study of prismatic  $F$ -crystals has led to exciting applications towards the classification of  $p$ -divisible groups [AL23], and more generally, of all  $p$ -adic crystalline local systems over smooth  $p$ -adic formal schemes [BS23; DLMS24; GR24]. However, similar to the crystals appearing in the theory of crystalline cohomology of Grothendieck and Berthelot, the prismatic  $F$ -crystals are mysterious objects.

To unravel these objects, a common idea is to describe them in terms of certain equivalent and computable data, for example, in the crystalline cohomology theory, one usually replaces crystals with modules equipped with a flat connection as in [Ber74]. In the prismatic theory, several successful attempts have been made to understand prismatic crystals in terms of more explicit data in various settings, for example, crystals on the relative prismatic site in terms of generalised representations,  $q$ -de Rham complex and  $q$ -Higgs field in [MT20; Tsu24], crystals on the relative/absolute prismatic site in terms of twisted/absolute differential calculus in [GLQ22; GLQ23], Hodge-Tate crystals in terms of Higgs field in [Tia23] and prismatic (Laurent)  $F$ -crystals in terms of étale  $(\varphi, \Gamma)$ -modules and similar objects in [Wu21; DL21; MW21].

The aim of this article is to describe analytic/completed prismatic  $F$ -crystals of [DLMS24; GR24] in terms of more explicit data. More precisely, our main result provides a direct equivalence between the category of prismatic  $F$ -crystals over the absolute prismatic (ringed) site of a small (unramified at  $p$ ) base ring and the category of relative Wach modules (certain  $(\varphi, \Gamma)$ -modules) introduced in [Abh23b] (see Theorem 1.1). Morally, our result is obtained by showing that the data of the Galois action (i.e.  $\Gamma$ -action) on a Wach module is equivalent to the data of a prismatic stratification on it (see Theorem 1.3). Note that the proof of the preceding equivalence is highly non-trivial and constitutes the heart of this article. Moreover, our approach is different and independent of all previous methods and (categorical equivalence) results in [BS23; DLMS24; GR24] and [Wu21; DL21; MW21].

Let us note that in the case that the base ring is a complete discrete valuation ring with perfect residue field, Wach modules were studied in [Fon90], [Wac96; Wac97], [Col99] and [Ber04]. In this case, using our methods, we also show that the classical Wach modules from [Wac96; Ber04] descend to a smaller ring, beyond the Fontaine-Laffaille case treated in [Wac97] (see Theorem 1.5). Furthermore, let us note that the theory of Wach modules (in the context of  $(\varphi, \Gamma)$ -modules) and its relationship with crystalline representations in different settings was studied in [Abh21; Abh23a; Abh23b].

Besides being an explicit description of prismatic  $F$ -crystals, the usefulness of Wach modules stems from its applications towards the computation of  $p$ -adic vanishing cycles via syntomic complexes. Indeed, in order to generalise the computation of  $p$ -adic vanishing cycles via crystalline syntomic complexes of [CN17], to the case of crystalline coefficients, in [Abh23c] we used the theory of Wach modules from [Abh21] as an important ingredient. However, the results obtained in [Abh21] and [Abh23c] were restrictive. Furthermore, note that in [BMS19], the authors defined a prismatic syntomic complex

for smooth  $p$ -adic formal schemes and compared it to the complex of  $p$ -adic vanishing cycles integrally. Beyond the smooth case, similar comparison results have also been obtained in [AMMN22] and [BM23], where the latter uses the theory of prismatic cohomology. The preceding results were obtained for the case of constant coefficients and it is natural to ask the following: is it possible to generalise [BMS19, Theorem 10.1] to arbitrary crystalline coefficients, i.e. can one define a prismatic syntomic complex with coefficients in a prismatic  $F$ -crystal and compare it to the complex of  $p$ -adic vanishing cycles? In our approach to providing an answer to the preceding question, the prismatic interpretation of Wach modules from the current paper and the relationship between Wach modules and crystalline representations from [Abh23b] will serve as crucial inputs.

### 1.1. A categorical equivalence.

Let  $p$  be a fixed prime,  $\kappa$  a perfect field of characteristic  $p$  and  $O_F := W(\kappa)$ , the ring of  $p$ -typical Witt vectors with coefficients in  $\kappa$  and equipped with the natural Witt vector Frobenius endomorphism. Let  $R$  denote the  $p$ -adic completion of an étale algebra over the  $p$ -adically complete Laurent polynomial ring  $O_F\langle X_1^{\pm 1}, \dots, X_d^{\pm 1} \rangle$  such that its special fiber  $\mathrm{Spec}(R/pR)$  is connected. We take  $X := \mathrm{Spf} R$  to be an affine  $p$ -adic formal scheme and consider its absolute prismatic ringed site  $(X_{\Delta}, \mathcal{O}_{\Delta})$  in the sense of [BS22] (see Section 2.1). Let  $\mathrm{Vect}^{\mathrm{an}, \varphi}(X_{\Delta})$  denote the category of analytic prismatic  $F$ -crystals from [GR24] (note that one could also work with the equivalent notion of completed prismatic  $F$ -crystals from [DLMS24], see Subsection 2.3 for definitions and explanations).

Let  $R_{\infty}$  denote the  $R$ -algebra obtained by adjoining to  $R$  all  $p$ -power roots of unity and  $p$ -power roots of  $X_i$ , for all  $1 \leq i \leq d$ . Then  $R_{\infty}[1/p]$  is Galois over  $R[1/p]$  with the Galois group  $\Gamma_R := \mathrm{Gal}(R_{\infty}[1/p]/R[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^{\times}$ . Let  $A_{\mathrm{inf}}(R_{\infty}) := W(R_{\infty}^b)$ , where  $R_{\infty}^b$  denotes the tilt of  $R_{\infty}$  (see Subsection 1.6). Note that  $A_{\mathrm{inf}}(R_{\infty})$  is equipped with a Witt vector Frobenius endomorphism  $\varphi$  and a continuous action of  $\Gamma_R$  (see Subsection 3.1). Moreover, we have a subring  $A_R \subset A_{\mathrm{inf}}(R_{\infty})$  which is stable under the action of  $(\varphi, \Gamma_R)$  on the latter (see Subsection 3.1.1); we equip  $A_R$  with the induced structures. Let  $\varepsilon := (1, \zeta_p, \dots)$  denote a compatible system of  $p$ -power roots of unity in  $R_{\infty}^b$  and let  $q := [\varepsilon]$  denote its Teichmüller lift in  $A_{\mathrm{inf}}(R_{\infty})$ ; set  $\mu := q - 1$  and  $[p]_q := (q^p - 1)/(q - 1)$  in  $A_{\mathrm{inf}}(R_{\infty})$ . We denote the category of Wach modules over  $A_R$  (see Definition 4.1) as  $(\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q}$ .

Now, let us note that the pair  $(A_R, [p]_q)$  is a prism, an object of  $X_{\Delta}$  (see Lemma 3.10) and a cover of the final object of the topos  $\mathrm{Shv}(X_{\Delta})$  (see Lemma 3.11). Moreover, the action of  $\Gamma_R$  on  $A_R$  induces automorphisms of  $(A_R, [p]_q)$  in  $X_{\Delta}$  (see Lemma 3.12). Evaluating an analytic prismatic  $F$ -crystal on the prism  $(A_R, [p]_q)$  gives a Wach module (see Proposition 5.1), and we have a well-defined evaluation functor,

$$\begin{aligned} \mathrm{ev}_{A_R}^{\Delta} : \mathrm{Vect}^{\mathrm{an}, \varphi}(X_{\Delta}) &\longrightarrow (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q} \\ \mathcal{F} &\longmapsto \mathcal{F}(A_R, [p]_q). \end{aligned} \tag{1.1}$$

Our main result is the following claim:

**Theorem 1.1** (Theorem 5.5). *The evaluation functor in (1.1) induces an equivalence of categories  $\mathrm{ev}_{A_R}^{\Delta} : \mathrm{Vect}^{\mathrm{an}, \varphi}(X_{\Delta}) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q}$ .*

Our proof of Theorem 1.1 is direct, in particular, we do not assume the equivalence between crystalline  $\mathbb{Z}_p$ -representations and Wach modules over  $A_R$  from [Abh23b], or the categorical equivalence results from [BS23; GR24; DLMS24]. As mentioned earlier, we show that “the Galois action on a Wach module, i.e. the action of  $\Gamma_R$ , is equivalent to a prismatic stratification on it”. Our approach is inspired by the work of [MT20], where the authors study coefficients for relative prismatic cohomology. However, the results obtained and the techniques employed in our proofs are vastly different from loc. cit. This stems largely from the fact that our respective settings are quite different: we work over the absolute prismatic site of  $R$  and consider analytic prismatic  $F$ -crystals, whereas in loc. cit., the authors work over the relative prismatic site and consider prismatic  $F$ -crystals of vector bundles.

**1.2. Crystals as modules with stratification.** In order to prove Theorem 1.1, our first course of action is to bring the source of the functor  $\mathrm{ev}_{A_R}^\Delta$  in (1.1) on an equal footing with the target of that functor. This is achieved by interpreting analytic prismatic  $F$ -crystals as certain modules with stratification.

More precisely, let us note that  $(A_R, [p]_q)$  is a cover of the final object of the topos  $\mathrm{Shv}(X_\Delta)$  (see Lemma 3.11). We set  $A_R(\bullet)$  to be the cosimplicial ring obtained by taking the prismatic Čech nerve  $(A_R(\bullet), I(\bullet))$  of  $(A_R, [p]_q)$  in  $X_\Delta$ . Then, it is possible to describe the  $n^{\mathrm{th}}$ -term of  $A_R(\bullet)$  in the site  $X_\Delta$  (see Construction 3.13), and in case  $n = 1, 2$ , we can describe  $A_R(n)$  very explicitly (see the discussion after Lemma 3.14). Now, let  $\mathrm{Strat}^{\mathrm{an}, \varphi}(A_R(\bullet))$  denote the category of “analytic”  $\varphi$ -modules over  $A_R$  equipped with a stratification with respect to  $A_R(\bullet)$  (see Definition 5.6). Then by the general theory of crystals, evaluation of analytic  $F$ -crystals on the simplicial object  $(A_R(\bullet), I(\bullet))$  induces the following natural equivalence of categories (see the discussion after Definition 5.6).

$$\mathrm{ev}_{A_R(\bullet)}^\Delta : \mathrm{Vect}^{\mathrm{an}, \varphi}(X_\Delta, \mathcal{O}_\Delta) \xrightarrow{\sim} \mathrm{Strat}^{\mathrm{an}, \varphi}(A_R(\bullet)). \quad (1.2)$$

**1.3. Prismatic stratifications and Galois action on Wach modules.** Our next course of action is to relate  $\varphi$ -modules over  $A_R$  equipped with a stratification to Wach modules over  $A_R$ . So let  $(N, \varepsilon)$  denote an object of the category  $\mathrm{Strat}^{\mathrm{an}, \varphi}(A_R(\bullet))$ , where  $N$  is a finitely generated  $A_R$ -module satisfying certain conditions and  $\varepsilon$  is a stratification on  $N$  with respect to  $A_R(\bullet)$ . Moreover, let us note that the action of  $\Gamma_R$  on  $(A_R, [p]_q)$  induces a natural action of  $\Gamma_R^{\times(n+1)}$  on  $A_R(n)$ , for each  $n \in \mathbb{N}$ . Then, by using the action of  $\Gamma_R^2$  on  $A_R(1)$  and the stratification  $\varepsilon$ , we equip  $N$  with a continuous action of  $\Gamma_R$  satisfying the properties of a Wach module over  $A_R$  from Definition 4.1 (see Construction 5.7). In particular, we have a well-defined natural functor,

$$\mathrm{ev}_{A_R}^{\mathrm{Strat}} : \mathrm{Strat}^{\mathrm{an}, \varphi}(A_R(\bullet)) \longrightarrow (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q}, \quad (1.3)$$

and we show the following:

**Proposition 1.2** (Proposition 5.8). *The following diagram is commutative up to canonical isomorphisms:*

$$\begin{array}{ccc} \mathrm{Vect}^{\mathrm{an}, \varphi}(X_\Delta) & \xrightarrow[\sim]{\mathrm{ev}_{A_R(\bullet)}^\Delta (1.2)} & \mathrm{Strat}^{\mathrm{an}, \varphi}(A_R(\bullet)) \\ & \searrow \mathrm{ev}_{A_R}^\Delta (1.1) & \downarrow \mathrm{ev}_{A_R}^{\mathrm{Strat}} (1.3) \\ & & (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q}. \end{array}$$

From the diagram in Proposition 1.2, we see that in order to show that (1.1) induces a categorical equivalence, it is enough to show that (1.3) induces a categorical equivalence. Therefore, we show the following:

**Theorem 1.3** (Theorem 5.9). *The functor in (1.3) induces a natural equivalence of categories*

$$\mathrm{ev}_{A_R}^{\mathrm{Strat}} : \mathrm{Strat}^{\mathrm{an}, \varphi}(A_R(\bullet)) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q}.$$

In order to prove Theorem 1.3, we define a quasi-inverse to the functor in (1.3). More precisely, given a Wach module  $N$  over  $A_R$ , we use the action of  $\Gamma_R$  on  $N$  to build a stratification on  $N$  with respect to  $A_R(\bullet)$  and obtain a natural quasi-inverse to the functor in (1.3),

$$\mathrm{Strat}_{A_R(\bullet)} : (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q} \longrightarrow \mathrm{Strat}^{\mathrm{an}, \varphi}(A_R(\bullet)). \quad (1.4)$$

The construction of the stratification on a Wach module, using the action of  $\Gamma_R$  on it, is the *main technical heart* of the paper. Our construction is given via a “3-step” argument. The three steps

correspond to the three subgroups of different nature making up the Galois group  $\Gamma_R$ . More precisely, recall that  $\Gamma_R \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times$ . Furthermore,  $\mathbb{Z}_p^\times$  fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \mathbb{Z}_p^\times \longrightarrow \Gamma_{\text{tor}} \longrightarrow 1,$$

where, for  $p \geq 3$ , we have that  $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$  and for  $p = 2$ , we have that  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$ . Moreover, for  $p \geq 3$ , we have that  $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$ , and for  $p = 2$ , we have that  $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$ , as groups. Then the three steps of our argument correspond to the groups  $\mathbb{Z}_p(1)^d$ ,  $\Gamma_{\text{tor}}$  and  $\Gamma_0$ . The case  $p = 2$  is different from the cases  $p \geq 3$  and requires entirely different and quite technical arguments. While the arguments involving  $\mathbb{Z}_p(1)^d$  and  $\Gamma_0$  are “prismatic/crystalline” in nature and feature  $q$ -de Rham complexes, the arguments involving the group  $\Gamma_{\text{tor}}$  feature some techniques commonly used in Iwasawa theory (see Appendix A.2). We refer the reader to Subsection 5.2 for precise details on the construction of the functor in (1.4).

**1.4. Some descent results for Wach modules.** The most important input for the construction of the functor in (1.4) is the following comparison result (notations are explained after the statement):

**Theorem 1.4** (Theorem 4.5). *Let  $N$  be a Wach module over  $A_R$  and consider the  $R$ -module  $M := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\Gamma_R}$ , equipped with the tensor product Frobenius. Then  $M$  is a finitely generated  $p$ -torsion free  $R$ -module and we have a natural  $(\varphi, \Gamma_R)$ -equivariant isomorphism*

$$A_R(1)/p_1(\mu) \otimes_{p_1, R} M \xrightarrow{\sim} A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N.$$

Moreover, the preceding isomorphism induces a  $\varphi$ -equivariant (after inverting  $p$ ) isomorphism of  $R$ -modules  $M \xrightarrow{\sim} N/\mu N$ .

In Theorem 1.4, note that  $(A_R(1), [p]_q)$  denotes the self product of  $(A_R, [p]_q)$  in  $X_\Delta$  and  $p_1, p_2 : A_R \rightarrow A_R(1)$  are the two projection maps to the two components (see Construction 3.13).

The statement in Theorem 1.4 is a descent statement for the action of  $\Gamma_R$  and involves the ring  $A_R(1)/p_1(\mu)$ . Therefore, in order to prove the statement, one requires an understanding of the action of  $\Gamma_R$  on the objects appearing in the statement and an explicit description of the ring  $A_R(1)/p_1(\mu)$ . The latter is achieved in Proposition 3.25, where we use some computations of Tsuji on prismatic envelopes and divided power rings (see Appendix B). Next, similar to the discussion in the previous subsection, the descent for the action of  $\Gamma_R$  is quite technical. Our strategy is to prove the statement via a “3-step descent” argument. The case  $p = 2$  is different from the cases  $p \geq 3$  and requires entirely different and quite technical arguments. While the arguments involving  $\mathbb{Z}_p(1)^d$  and  $\Gamma_0$  are “crystalline” in nature and feature de Rham complexes, the arguments involving the group  $\Gamma_{\text{tor}}$  feature some techniques commonly used in Iwasawa theory. We refer the reader to Subsections 4.2 and 4.3 for precise details on the descent isomorphism.

In the case  $R = O_S$ , for  $p \geq 3$ , as a consequence of our descent arguments for the action of  $\Gamma_{\text{tor}} = \mathbb{F}_p^\times$ , we obtain a descent result for classical Wach modules. Let  $N$  be a Wach module over  $A_F = O_F[[\mu]]$ , i.e. a finite free  $A_F$ -module equipped with an  $A_F$ -linear Frobenius isomorphism  $\varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$  and a semilinear and continuous action of  $\Gamma_F$  commuting with Frobenius and such that the action of  $\Gamma_F$  is trivial on  $N/\mu N$ . Let us set,

$$\mu_0 := -p + \sum_{a \in \mathbb{F}_p} (1 + \mu)^{[a]} \quad \text{and} \quad \tilde{p} := \mu_0 + p,$$

as elements of  $A_F^{\mathbb{F}_p^\times}$ . Then, from Lemma 3.4, we have a  $(\varphi, \Gamma_0)$ -equivariant isomorphism of rings  $O_F[[\mu_0]] \xrightarrow{\sim} A_F^{\mathbb{F}_p^\times}$  and we show the following:

**Theorem 1.5** (Proposition 4.19). *Let  $N$  be a Wach module over  $A_F$ . Then  $N_0 := N^{\mathbb{F}_p^\times}$  is a finite free  $O_F[[\mu_0]]$ -module, equipped with a continuous and semilinear action of  $\Gamma_0$  such that the action of  $\Gamma_0$  is trivial on  $N_0/\mu_0 N_0 \xrightarrow{\sim} N/\mu N$ , and we have a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $A_F$ -modules  $A_F \otimes_{O_F[[\mu_0]]} N_0 \xrightarrow{\sim} N$ . Moreover,  $N_0$  is equipped with an  $O_F[[\mu_0]]$ -linear isomorphism  $\varphi^*(N_0)[1/\tilde{p}] \xrightarrow{\sim} N_0[1/\tilde{p}]$ , compatible with the action of  $\Gamma_0$ .*

**1.5. Relation to previous works.** The theory of prismatic  $F$ -crystals was introduced in [BS23] and its generalisation to analytic/completed prismatic  $F$ -crystals was studied in [DLMS24; GR24]. The theory of classical Wach modules was introduced and developed in [Fon90; Wac96; Col99; Ber04] and its generalisation to the relative case was introduced and developed in [Abh21; Abh23a; Abh23b]. Now, let us consider the following diagram:

$$\begin{array}{ccc}
\mathrm{Vect}^{\varphi, \mathrm{an}}(X_{\Delta}) & \xleftarrow{\mathcal{E} \mapsto \mathcal{E}[1/\mathcal{I}_{\Delta}] \otimes_{\mathcal{O}[1/\mathcal{I}_{\Delta}]} \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}} & \mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}) \\
\downarrow \mathrm{ev}_{A_R}^{\Delta} \text{ (1.1)} \wr & \begin{array}{c} \searrow T_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{an}} \\ \sim \\ \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_R) \end{array} & \begin{array}{c} \swarrow T_{\acute{\mathrm{e}}\mathrm{t}} \\ \sim \\ \mathrm{Rep}_{\mathbb{Z}_p}(G_R) \end{array} \\
& \begin{array}{c} \swarrow \mathbf{N}_R \\ \sim \\ (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q} \end{array} & \longleftrightarrow & \begin{array}{c} \searrow \mathbf{D}_R \\ \sim \\ (\varphi, \Gamma_R)\text{-Mod}_{B_R}^{\acute{\mathrm{e}}\mathrm{t}} \end{array} \\
& \xleftarrow{N \mapsto N \otimes_{A_R} B_R} & & \xrightarrow{\wr \mathrm{ev}_{B_R}^{\Delta}}
\end{array} \tag{1.5}$$

In top row of the diagram (1.5), the top right corner denotes the category of Laurent  $F$ -crystals over  $X_{\Delta}$  (see [BS23] or Definition 1.5); in the middle row,  $G_R$  denotes the étale fundamental group of  $R[1/p]$  (see Subsection 1.6), the right-hand-term denotes the category of  $\mathbb{Z}_p$ -representations of  $G_R$  and the left-hand-term denotes the category of lattices inside  $p$ -adic crystalline representations of  $G_R$  (see [Bri08]); in the bottom row we have  $B_R := A_R[1/\mu]_p^{\wedge}$  equipped with an induced action of  $(\varphi, \Gamma_R)$  and the bottom right corner denotes the category of étale  $(\varphi, \Gamma_R)$ -modules over  $B_R$  (see [And06]). The top horizontal arrow is a natural embedding from [GR24], the middle horizontal arrow is easily seen to be a natural embedding and the bottom horizontal arrow is a natural embedding from [Abh23b, Proposition 3.15]. From the first to the second row, the slanted arrow  $T_{\acute{\mathrm{e}}\mathrm{t}}$  is the natural étale realisation functor and an equivalence from [BS23] (see Lemma 2.13), and the slanted arrow  $T_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{an}}$  is an equivalence from [GR24] (see Definition 2.24 and Remark 2.26); the upper square commutes using loc. cit. From the middle row to the bottom row, the slanted arrow  $\mathbf{D}_R$  is the natural étale  $(\varphi, \Gamma)$ -module functor and an equivalence from [And06], and the slanted arrow  $\mathbf{N}_R$  is the natural Wach module functor and an equivalence from [Abh23b, Theorem 1.7]; commutativity of the bottom square follows from the compatibility between the results of [Abh23b] and [And06]. Next, in the diagram (1.5), the leftmost and the rightmost vertical arrows are evaluation functors, i.e. evaluation of an analytic (resp. Laurent) prismatic  $F$ -crystal over the prism  $(A_R, [p]_q)$ ; the left vertical arrow is an equivalence from Theorem 1.1. Note that the left-hand-side triangle commutes by comparing the explicit formulas for the slanted arrow  $T_{\acute{\mathrm{e}}\mathrm{t}}^{\mathrm{an}}$  in Definition 2.24 and the composition of the left vertical arrow  $\mathrm{ev}_{A_R}^{\Delta}$  in (1.1) with the quasi-inverse of the slanted arrow  $\mathbf{N}_R$  described in [Abh23b, Theorem 1.7], and similarly, for the right-hand-side triangle. In particular, we obtain that the right vertical arrow is an equivalence (also see [Wu21; MW21]) and the outer square commutes by the definition of the arrows. Hence, it follows that the results of this article are compatible with previous constructions.

As mentioned earlier, our proof of Theorem 1.1 is direct and we do not assume the equivalence between crystalline  $\mathbb{Z}_p$ -representations and Wach modules over  $A_R$  from [Abh23b], or the categorical equivalence results from [BS23; GR24; DLMS24] and [Wu21; MW21; DL21]). Furthermore, our approach is inspired by the work of [MT20], however, the results obtained and the techniques employed in our proofs are different: we work over the absolute prismatic site of  $R$  and consider analytic prismatic  $F$ -crystals, whereas in loc. cit., the authors work over the relative prismatic site and consider prismatic  $F$ -crystals of vector bundles, in particular, in this article we work with a non-commutative group  $\Gamma_R$  which has an ‘‘arithmetic’’ part (see  $\Gamma_F$  in Subsection 1.6), in contrast with the ‘‘geometric’’  $\Gamma$  considered in loc. cit.

**1.6. Setup and notations.** In this subsection we will describe the setup for this paper and fix some notations which are essentially the same as in [Abh23b, Subsection 1.4]. We will work under the convention that  $0 \in \mathbb{N}$ , the set of natural numbers.

Let  $p$  be a fixed prime number,  $\kappa$  a perfect field of characteristic  $p$ , let  $O_F := W(\kappa)$  denote the ring of  $p$ -typical Witt vectors with coefficients in  $\kappa$  and  $F := O_F[1/p]$ , the fraction field of  $O_F$ . In

particular,  $F$  is an unramified extension of  $\mathbb{Q}_p$  with ring of integers  $O_F$ . Let  $\overline{F}$  denote a fixed algebraic closure of  $F$  so that its residue field, denoted as  $\overline{\kappa}$ , is an algebraic closure of  $\kappa$ . Further, we denote by  $G_F := \text{Gal}(\overline{F}/F)$ , the absolute Galois group of  $F$ .

Let  $Z = (Z_1, \dots, Z_s)$  denote a set of indeterminates and  $\mathbf{k} = (k_1, \dots, k_s)$  in  $\mathbb{N}^s$  be a multi-index and set  $Z^{\mathbf{k}} := Z_1^{k_1} \cdots Z_s^{k_s}$ . We will write  $\mathbf{k} \rightarrow +\infty$  to denote  $\sum k_i \rightarrow +\infty$ . For a topological algebra  $S$ , define

$$S\langle Z \rangle := \left\{ \sum_{\mathbf{k} \in \mathbb{N}^s} a_{\mathbf{k}} Z^{\mathbf{k}}, \text{ where } a_{\mathbf{k}} \in S \text{ and } a_{\mathbf{k}} \rightarrow 0 \text{ as } \mathbf{k} \rightarrow +\infty \right\}.$$

Fix  $d \in \mathbb{N}$  and let  $X_1, X_2, \dots, X_d$  be some indeterminates. Let  $R$  be the  $p$ -adic completion of an étale algebra over  $R^\square = O_F\langle X_1^{\pm 1}, \dots, X_d^{\pm 1} \rangle$  with non-empty and connected special fiber. We fix an algebraic closure  $\overline{\text{Frac}(R)}$  of  $\text{Frac}(R)$  containing  $\overline{F}$ . Let  $\overline{R}$  denote the union of finite  $R$ -subalgebras  $R' \subset \overline{\text{Frac}(R)}$ , such that  $R'[1/p]$  is étale over  $R[1/p]$ . Let  $\overline{\eta}$  denote the fixed geometric point of the generic fiber  $\text{Spec} R[1/p]$  (defined by  $\overline{\text{Frac}(R)}$ ) and let  $G_R$  denote the étale fundamental group  $\pi_1^{\text{ét}}(\text{Spec} R[1/p], \overline{\eta})$ . We can write this étale fundamental group as the Galois group (of the fraction field of  $\overline{R}[1/p]$  over the fraction field of  $R[1/p]$ ), i.e.

$$G_R = \pi_1^{\text{ét}}(\text{Spec}(R[1/p]), \overline{\eta}) = \text{Gal}(\overline{R}[1/p]/R[1/p]).$$

Next, set  $F_\infty := F(\mu_{p^\infty})$ ,  $R_\infty := \cup_{i=1}^d R[\mu_{p^\infty}, X_i^{1/p^\infty}] \subset \overline{R}$  and we will consider the following groups (see [Abh21, Subsections 2 & 3] and [Abh23a, Section 2]),

$$\begin{aligned} H_R &:= \text{Gal}(\overline{R}[1/p]/R_\infty[1/p]), \quad H_F := \text{Gal}(\overline{F}/F_\infty) \\ \Gamma_R &:= G_R/H_R = \text{Gal}(R_\infty[1/p]/R[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times, \quad \Gamma_F := \text{Gal}(F_\infty/F) \xrightarrow{\sim} \mathbb{Z}_p^\times \\ \Gamma'_R &:= \text{Gal}(R_\infty[1/p]/R(\mu_{p^\infty})[1/p]) \xrightarrow{\sim} \mathbb{Z}_p(1)^d, \quad \text{Gal}(R(\mu_{p^\infty})[1/p]/R[1/p]) = \Gamma_R/\Gamma'_R \xrightarrow{\sim} \Gamma_F. \end{aligned}$$

Note that the isomorphism  $\chi : \Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$  is given via the  $p$ -adic cyclotomic character, and therefore,  $\Gamma_F$  fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\text{tor}} \longrightarrow 1, \quad (1.6)$$

where, for  $p \geq 3$ , we have  $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$  and for  $p = 2$ , we have  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$ . Moreover, for  $p \geq 3$ , we have  $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$ , and the projection map in (1.6) admits a section  $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$ , where the second map is given as  $a \mapsto [a]$ , the Teichmüller lift of  $a$ . Furthermore, note that for  $p = 2$ , we have  $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$ , as groups.

Let  $\varphi : R^\square \rightarrow R^\square$  denote a morphism extending the natural Witt vector Frobenius on  $O_F$  by setting  $\varphi(X_i) = X_i^p$ , for all  $1 \leq i \leq d$ . The endomorphism  $\varphi$  of  $R^\square$  is flat by [Bri08, Lemma 7.1.5] and faithfully flat since  $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ , for any maximal ideal  $\mathfrak{m} \subset R^\square$ . Moreover, it is finite of degree  $p^d$  using Nakayama Lemma and the fact that  $\varphi$  modulo  $p$  is evidently of degree  $p^d$ . Recall that the  $O_F$ -algebra  $R$  is given as the  $p$ -adic completion of an étale algebra  $R^\square$ , therefore, the Frobenius endomorphism  $\varphi$  on  $R^\square$  admits a unique extension  $\varphi : R \rightarrow R$  such that the induced map  $\varphi : R/p \rightarrow R/p$  is the absolute Frobenius  $x \mapsto x^p$  (see [CN17, Proposition 2.1]). Similar to above, we again note that the endomorphism  $\varphi : R \rightarrow R$  is finite and faithfully flat of degree  $p^d$ . Next, for  $k \in \mathbb{N}$ , let  $\Omega_R^k$  denote the  $p$ -adic completion of module of  $k$ -differentials of  $R$  relative to  $\mathbb{Z}$ . Then, we have that  $\Omega_R^1 = \oplus_{i=1}^d R d \log X_i$  and  $\Omega_R^k = \wedge_R^k \Omega_R^1$ .

Finally, let  $A = R_\infty$  or  $\overline{R}$ . Then the tilt of  $A$  is defined as  $A^b = \lim_{\varphi} A/p$  and the tilt of  $A[1/p]$  is defined as  $A[1/p]^b = A^b[1/p^b]$ , where  $p^b = (1, p^{1/p}, \dots)$  in  $A^b$  (see [Fon77, Chapitre V, Subsection 1.4]). Finally, let  $B$  be a  $\mathbb{Z}_p$ -algebra equipped with a Frobenius endomorphism  $\varphi$  lifting the absolute Frobenius on  $B/p$ , then for any  $B$ -module  $M$ , we set  $\varphi^*(M) := B \otimes_{\varphi, B} M$ .

**1.7. Outline of the paper.** In Section 2 we recall some basic definitions and results from the prismatic theory. In particular, in Subsection 2.1, we recall the definition of prismatic site and prismatic crystals from [BS22; BS23], and in Subsections 2.2 and 2.3 we recall the notion of prismatic  $F$ -crystals and its variations from [BS23; DLMS24; GR24], as well as, describe the étale realisation functors. Section 3 is devoted to the study of the prism  $(A_R, [p]_q)$  in detail. In Subsection 3.1, we

describe the ring  $A_R$  and study the structural properties of some of its subrings. In Subsection 3.2, we show that  $(A_R, [p]_q)$  is an object of  $(\mathrm{Spf} R)_{\Delta}$ , explicitly compute first few terms of its prismatic Čech nerve  $(A_R(\bullet), I(\bullet))$  in  $(\mathrm{Spf} R)_{\Delta}$ , and then study the action of  $\Gamma_R^{\times(n+1)}$  on  $A_R(n)$ . Then in Subsection 3.3 we provide an explicit description of the ring  $A_R(1)/p_1(\mu)$ , where  $p_1 : A \rightarrow A(1)$  is the first projection map, and study the action of  $1 \times \Gamma_R \subset \Gamma_R^2$  on it. Finally, in Subsection 3.4 we study the action of  $\Gamma_R^2$  on  $A_R(1)$  and carry out the “3-step” argument, described after Theorem 1.3, for constant coefficients, i.e. the ring  $A_R$ , in order to compute various rings that will appear at various stages of the proof of Theorem 1.3 in Section 5.

The goal of Section 4 is to state and prove Theorem 1.4. We begin by recalling the definition of Wach modules and some of its properties from [Abh23b] and in Subsection 1.4 we describe the  $q$ -connection on a Wach module and its scalar extensions originating from the natural action of  $\Gamma_R$ . Then, in Subsection 4.2, we carry out the first part of the proof of Theorem 1.4, i.e. descent for the action of  $\Gamma'_R$ , and in Subsection 4.2 we show the descent for the action of  $\Gamma_F \xrightarrow{\sim} \Gamma_0 \rtimes \Gamma_{\mathrm{tor}}$ . Note that the Subsection 4.2 is divided into two parts: the first part deals with the case  $p \geq 3$  where we also prove Theorem 1.5, and the second part deals with  $p = 2$  which requires completely different arguments. Finally, in Subsection 4.4 we put everything together to prove Theorem 1.4.

In Section 5, we state and prove Theorem 1.1. We begin by describing the functor  $\mathrm{ev}_{A_R}^{\Delta}$  and then in Subsection 5.1 we describe the relation between analytic/completed prismatic  $F$ -crystals and modules with stratification, as well as, prove Proposition 1.2. Subsection 5.2 is dedicated to the construction of stratification on Wach modules using the action of  $\Gamma_R$  via the “3-step” argument. This completes the proof of Theorem 1.1.

In Appendix A.1, we recall some standard definitions that have been used throughout in the text. Then in Appendix A.2 we describe the structure of modules admitting a continuous action of  $\mathbb{Z}_p^{\times}$  by recalling some standard constructions of Iwasawa [Iwa59]. Appendix B has been adapted from some notes of Tsuji and in that section we study the structure of certain  $\delta$ -rings and their reduction modulo  $\mu$ , which is a crucial input for determining the structure of  $A_R(1)/p_1(\mu)$  in Subsection 3.3.

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## 2. PRISMATIC SITE AND PRISMATIC $F$ -CRYSTALS

In this section, we will recall some fundamental definitions and results on prismatic site and prismatic  $F$ -crystals from [BS22; BS23] and analytic/completed prismatic  $F$ -crystals from [DLMS24] and [GR24]. For some standard definitions used in this section, we refer the reader to Appendix A.1. We start with the following:

**Definition 2.1.** A  $\delta$ -ring is a pair  $(A, \delta)$  where  $A$  is a commutative ring and  $\delta : A \rightarrow A$  is a map of sets with  $\delta(0) = \delta(1) = 0$  and satisfying:

$$\begin{aligned} \delta(xy) &= x^p \delta(y) + y^p \delta(x) + p \delta(x) \delta(y), \\ \delta(x+y) &= \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}. \end{aligned} \tag{2.1}$$

**Remark 2.2.** Given a  $\delta$ -ring  $(A, \delta)$  define  $\varphi : A \rightarrow A$  by the formula  $\varphi(x) = x^p + p\delta(x)$ , for any  $x$  in  $A$ . This determines a lift of the absolute Frobenius on  $A/pA$ . Conversely, if  $A$  is  $p$ -torsion-free then any lift  $\varphi : A \rightarrow A$  of the absolute Frobenius on  $A/pA$  determines a unique  $\delta$ -structure on  $A$ .

**Definition 2.3.** An element  $d$  of a  $\delta$ -ring  $A$  is distinguished if  $\delta(d)$  is a unit.

**2.1. Prismatic site and crystals.** Let  $A$  be a  $\delta$ -ring,  $I \subset A$  an ideal; we will refer to  $(A, I)$  as a  $\delta$ -pair.

**Definition 2.4** (Prism, [BS22, Definition 3.2]). A  $\delta$ -pair  $(A, I)$  is called a *prism* if  $I \subset A$  defines a Cartier divisor on  $\mathrm{Spec}(A)$  such that  $A$  is derived  $(p, I)$ -complete and  $p \in I + \varphi(I)A$ . The category of all prisms is the corresponding full subcategory of all  $\delta$ -pairs. The prism  $(A, I)$  is called perfect if  $A$  is a perfect  $\delta$ -ring, i.e.  $\varphi : A \rightarrow A$  is bijective. Finally,  $(A, I)$  is bounded if  $A/I$  has bounded  $p^\infty$ -torsion. A map  $(A, I) \rightarrow (B, J)$  of prisms is (faithfully) flat if the map  $A \rightarrow B$  is  $(p, I)$ -completely (faithfully) flat.

**Remark 2.5.** If  $(A, I)$  is a bounded prism then  $A$  is classically  $(p, I)$ -complete (see [BS22, Lemma 3.7]). For a map of prisms  $(A, I) \rightarrow (B, J)$  we have an isomorphism  $I \otimes_A B \xrightarrow{\sim} J$ , in particular,  $IB = J$  (see [BS22, Lemma 3.5]).

**Lemma 2.6** (Absolute prismatic site, [BS22, Corollary 3.12]). *Let  $\mathrm{Spf}(\mathbb{Z}_p)_\Delta$  denote the category opposite to that of the category of all bounded prisms  $(A, I)$  and endow it with a topology for which covers are determined by faithfully flat maps of prisms. The  $\mathrm{Spf}(\mathbb{Z}_p)_\Delta$  forms a site. Moreover, the functor  $\mathcal{O}_\Delta : \mathrm{Spf}(\mathbb{Z}_p)_\Delta \rightarrow \mathrm{Rings}$  (resp.  $\mathcal{I}_\Delta$ ) defined via  $(A, I) \mapsto A$  (resp.  $(A, I) \mapsto I$ ) forms a sheaf for this topology.*

**Definition 2.7** (Absolute prismatic site of  $X$ , [BS23, Definition 2.3]). Let  $X$  be a  $p$ -adic formal scheme. Define the *absolute prismatic site* of  $X$  denoted  $X_\Delta$  to be the category opposite to that of bounded prisms  $(A, I)$  which are equipped with a map  $\mathrm{Spf}(A/I) \rightarrow X$  and endow  $X_\Delta$  with the topology induced by the flat topology on prisms. We write  $\mathcal{O}_\Delta$  for the structure sheaf and denote the ideal sheaf of the Hodge-Tate divisor by  $\mathcal{I}_\Delta \subset \mathcal{O}_\Delta$ . Denote by  $\mathrm{Shv}(X_\Delta)$  the  $\infty$ -category of sheaves on  $X_\Delta$ .

**Proposition 2.8** ([BS23, Proposition 2.7]). *Let  $X$  be a  $p$ -adic formal scheme and let  $\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta)$  denote the category of vector bundles of  $\mathcal{O}_\Delta$ -modules. Then there is a natural equivalence*

$$\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta) \xrightarrow{\sim} \lim_{(A, I) \in X_\Delta} \mathrm{Vect}(A).$$

Moreover, a similar statement holds after replacing  $\mathcal{O}_\Delta$  with  $\mathcal{O}_\Delta[1/\mathcal{I}_\Delta]_p^\wedge$ .

Let  $X$  be a quasisyntomic  $p$ -adic formal scheme then one can describe crystals on  $(X_\Delta, \mathcal{O}_\Delta)$  using the quasisyntomic site of  $X$ .

**Definition 2.9** (Quasisyntomic site of  $X$ , [BS23, Definition 2.9]). Define the *quasisyntomic site* of  $X$  denoted  $X_{\mathrm{qsyn}}$  to be the category opposite to that of quasisyntomic maps  $\mathrm{Spf}(R) \rightarrow X$  and endow  $X_{\mathrm{qsyn}}$  with the topology induced by the quasisyntomic topology (see [BMS19, Definition 4.10 and Variant 4.35]). Define the full subcategory  $X_{\mathrm{qrsp}} \subset X_{\mathrm{qsyn}}$  spanned by objects  $\mathrm{Spf}(R) \rightarrow X$  with  $R$  quasisyntomic semiperfectoid (see [BMS19, Definition 4.20 and Variant 4.35]).

**Remark 2.10.** By [BMS19, Proposition 4.31 and Variant 4.35], restriction of sheaves induces an equivalence  $\mathrm{Shv}(X_{\mathrm{qsyn}}) \xrightarrow{\sim} \mathrm{Shv}(X_{\mathrm{qrsp}})$ . The functor defined by sending any  $(\mathrm{Spf}(R) \rightarrow X)$  in  $X_{\mathrm{qrsp}}$  to the initial prism  $\Delta_R$  (see [BS22, Proposition 7.2]) gives a sheaf of rings  $\Delta_\bullet$  on  $X_{\mathrm{qsyn}}$ ; we write  $\mathcal{I}_{\Delta_\bullet} \subset \Delta_\bullet$  for the ideal sheaf of the Hodge-Tate divisor.

**Lemma 2.11** ([BS23, Proposition 2.14]). *Let  $X$  be a quasisyntomic  $p$ -adic formal scheme. Then there are natural equivalences  $\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta) \xrightarrow{\sim} \mathrm{Vect}(X_{\mathrm{qsyn}}, \Delta_\bullet) \xrightarrow{\sim} \lim_{R \in X_{\mathrm{qrsp}}} \mathrm{Vect}(\Delta_R)$ .*

**2.2. Laurent  $F$ -crystals.** Let  $X$  be a bounded  $p$ -adic formal scheme and let  $X_\eta$  denote the generic fiber of  $X$ , regarded as a presheaf on perfectoid spaces over  $\mathbb{Q}_p$ .



**Definition 2.12** (Laurent  $F$ -crystals, [BS23, Definition 3.2]). Define the category of Laurent  $F$ -crystals of vector bundles on  $(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)$  as

$$\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1} := \lim_{(A,I) \in X_\Delta} \mathrm{Vect}(A[1/I]^\wedge)^{\varphi=1},$$

i.e. any  $\mathcal{E}$  in  $\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)$  is equipped with an isomorphism  $\varphi_\mathcal{E} : \varphi^*\mathcal{E} \xrightarrow{\sim} \mathcal{E}$ .

**Lemma 2.13** ([BS23, Corollary 3.8]). *Let  $X$  be a bounded  $p$ -adic formal scheme with generic fiber  $X_\eta$ . Let  $\mathrm{Loc}(X_\eta, \mathbb{Z}_p)$  denote the category of étale  $\mathbb{Z}_p$ -local systems on  $X_\eta$ . Then there is a natural equivalence*

$$T_{\text{ét}} : \mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1} \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(X_\eta).$$

Passing to the associated isogeny categories we obtain

$$\mathrm{Vect}(X_\Delta, \mathcal{O}_\Delta[1/\mathcal{I}_\Delta]^\wedge)^{\varphi=1} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \xrightarrow{\sim} \mathrm{Loc}_{\mathbb{Z}_p}(X_\eta) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

**2.3. Prismatic  $F$ -crystals.** In this subsection we will recall the definition of prismatic  $F$ -crystals and its analytic/completed variants.

**2.3.1. Vector bundles.** Let  $X$  be a  $p$ -adic formal scheme and let  $(A, I)$  denote a prism with  $\varphi$  its Frobenius endomorphism.

**Definition 2.14** (Prismatic  $F$ -crystals, [BS23, Definition 4.1]). Define the category  $\mathrm{Vect}^\varphi(A)$  of prismatic  $F$ -crystals of vector bundles on  $A$  as follows: an object is a pair  $(M, \varphi_M)$  with  $M$  a finite projective  $A$ -module equipped with an  $A$ -linear isomorphism  $\varphi_M : (\varphi^*M)[1/I] \xrightarrow{\sim} M[1/I]$ . Morphisms between two objects are given as  $A$ -linear maps compatible with  $\varphi_M$ . Say that  $(M, \varphi_M)$  is effective if  $\varphi_M$  carries  $\varphi^*M$  into  $M$ .

Define the category  $\mathrm{Vect}^\varphi(X_\Delta, \mathcal{O}_\Delta)$  of prismatic  $F$ -crystals of vector bundles on  $X_\Delta$  as follows: an object is a pair  $(\mathcal{E}, \varphi_\mathcal{E})$  with  $\mathcal{E}$  a vector bundle on  $(X_\Delta, \mathcal{O}_\Delta)$  equipped with an isomorphism  $\varphi_\mathcal{E} : (\varphi^*\mathcal{E})[1/\mathcal{I}_\Delta] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_\Delta]$ . Morphisms between two objects are maps of vector bundles compatible with  $\varphi_\mathcal{E}$ . Say that  $(\mathcal{E}, \varphi_\mathcal{E})$  is effective if  $\varphi_\mathcal{E}$  carries  $\varphi^*\mathcal{E}$  into  $\mathcal{E}$ .

**Remark 2.15.** From Proposition 2.8 and Lemma 2.11 we have equivalences  $\mathrm{Vect}^\varphi(X_\Delta, \mathcal{O}_\Delta) \xrightarrow{\sim} \lim_{(A,I) \in X_\Delta} \mathrm{Vect}^\varphi(A) \xrightarrow{\sim} \lim_{R \in X_{\mathrm{qrsp}}} \mathrm{Vect}^\varphi(\Delta_R)$ .

**2.3.2. Completed  $F$ -crystals.** Let  $O_K$  be a finite ramified extension of  $O_F$  and take  $\pi$  in  $O_K$  to be a uniformizer and  $E(u)$  in  $O_F[u]$  to be its minimal polynomial.

**Definition 2.16** (Completed prismatic crystals, [DLMS24, Definition 3.11]). Let  $X$  be a smooth  $p$ -adic formal scheme over  $O_K$ . Define the category  $\mathrm{CR}^\wedge(X_\Delta)$  of *finitely generated completed crystals* of  $\mathcal{O}_\Delta$ -modules on  $X_\Delta$  as follows: an object is a sheaf  $\mathcal{E}$  of  $\mathcal{O}_\Delta$ -modules on  $X_\Delta$  satisfying the following:

- (1) For each  $(A, I) \in X_\Delta$ , the evaluation  $\mathcal{E}(A) := \mathcal{E}(A, I)$  of  $\mathcal{F}$  on  $(A, I)$  is a finitely generated and  $(p, I)$ -complete  $A$ -module.
- (2) For every morphism  $(A, I) \rightarrow (B, IB)$  in  $X_\Delta$  the natural map  $B \widehat{\otimes}_A \mathcal{E}(A) \rightarrow \mathcal{E}(B)$  is an isomorphism, where we set  $B \widehat{\otimes}_A \mathcal{E}(A) := \lim_n (B \otimes_A \mathcal{E}(A)) / (p, I)^n$ .

Morphisms between two objects are maps of  $\mathcal{O}_\Delta$ -modules compatible with completed base change.

**Remark 2.17.** The category of completed crystals on  $X_\Delta$ , denoted as  $\mathrm{CR}^\wedge(X_\Delta)$ , satisfies descent for Zariski and étale topologies on  $X$  (see [DLMS24, Lemma 3.39, Remark 3.40]).

A  $p$ -adically completed  $O_K$ -algebra  $S$  is called *small* if it is a  $p$ -adically completed étale over a  $p$ -adically complete Laurent polynomial  $O_K$ -algebra  $O_K\langle X_1^{\pm 1}, \dots, X_d^{\pm 1} \rangle$ , for some  $d \in \mathbb{N}$ . For such an  $S$  there exists a unique  $O_F$ -algebra  $R$  such that  $O_K \otimes_{O_F} R \xrightarrow{\sim} S$  and  $R$  admits a lift of the absolute Frobenius modulo  $p$ , which we take to be the one extending the Witt vector Frobenius on  $O_F$  and such that  $\varphi(X_i) = X_i^p$ ; denote the lift of Frobenius as  $\varphi : R \rightarrow R$ . Let  $\mathfrak{S} = R[[u]]$  equipped with a Frobenius endomorphism  $\varphi$  which extends the Frobenius on  $R$  by setting  $\varphi(u) = u^p$ . Then  $(\mathfrak{S}, E(u))$  is an object of  $(\mathrm{Spf} S)_{\Delta}$  and called the *Breuil-Kisin prism*.

**Definition 2.18** (Completed prismatic  $F$ -crystals, [DLMS24, Definition 3.16, Definition 3.42]). Let  $X := \mathrm{Spf} S$  for  $S$  a small  $O_K$ -algebra. Define the category  $\mathrm{CR}^{\wedge, \varphi}(X_{\Delta})$  of *completed  $F$ -crystals of  $\mathcal{O}_{\Delta}$ -modules* on  $X_{\Delta}$  as follows: an object is a pair  $(\mathcal{E}, \varphi_{\mathcal{E}})$  with  $\mathcal{E}$  a completed prismatic crystal on  $X_{\Delta}$  such that we have an isomorphism  $\varphi_{\mathcal{E}} : \varphi^* \mathcal{E}[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$ . Moreover, for the Breuil-Kisin prism  $(\mathfrak{S}, E)$  in  $X_{\Delta}$ , assume that the  $\mathfrak{S}$ -module  $\mathcal{E}(\mathfrak{S})$  is torsionfree,  $\mathcal{E}(\mathfrak{S})[1/p]$  (resp.  $\mathcal{E}(\mathfrak{S})[1/E]$ ) is finite projective over  $\mathfrak{S}[1/p]$  (resp.  $\mathfrak{S}[1/E]$ ) and  $\mathcal{E}(\mathfrak{S}) = \mathcal{E}(\mathfrak{S})[1/p] \cap \mathcal{E}(\mathfrak{S})[1/E] \subset \mathcal{E}(\mathfrak{S})[1/p, 1/E]$ . Morphisms between two objects are given as maps of  $\mathcal{O}_{\Delta}$ -modules compatible with  $\varphi_{\mathcal{E}}$ . Say that  $(\mathcal{E}, \varphi_{\mathcal{E}})$  is effective if  $\varphi_{\mathcal{E}}$  carries  $\varphi^* \mathcal{E}$  into  $\mathcal{E}$ .

Let  $X$  be a smooth  $p$ -adic formal scheme over  $O_K$ . Define the category  $\mathrm{CR}^{\wedge, \varphi}(X_{\Delta})$  of *completed  $F$ -crystals of  $\mathcal{O}_{\Delta}$ -modules* on  $X_{\Delta}$  as follows: an object is a pair  $(\mathcal{E}, \varphi_{\mathcal{E}})$  with  $\mathcal{E}$  a completed prismatic crystal on  $X_{\Delta}$  such that we have an isomorphism  $\varphi_{\mathcal{E}} : \varphi^* \mathcal{E}[1/\mathcal{I}_{\Delta}] \xrightarrow{\sim} \mathcal{E}[1/\mathcal{I}_{\Delta}]$ . Moreover, there exists an affine open covering  $X = \cup_{\lambda \in \Lambda} U_{\lambda}$  by affine  $p$ -adic formal schemes, where  $U_{\lambda} = \mathrm{Spf}(S_{\lambda})$  is connected and small over  $O_K$ , and such that  $(\mathcal{E}, \varphi_{\mathcal{E}})|_{U_{\lambda, \Delta}}$  is an object of  $\mathrm{CR}^{\wedge, \varphi}(U_{\lambda, \Delta})$ . Morphisms between two objects are given as maps of  $\mathcal{O}_{\Delta}$ -modules compatible with  $\varphi_{\mathcal{E}}$ . Say that  $(\mathcal{E}, \varphi_{\mathcal{E}})$  is effective if  $\varphi_{\mathcal{E}}$  carries  $\varphi^* \mathcal{E}$  into  $\mathcal{E}$ .

**2.3.3. Analytic  $F$ -crystals.** Let  $X$  be a  $p$ -adic formal scheme.

**Definition 2.19** (Analytic prismatic crystals, [GR24, Definition 3.2]). Define the category of *analytic prismatic crystals of vector bundles* over  $X$  as

$$\mathrm{Vect}^{\mathrm{an}}(X_{\Delta}) := \lim_{(A, I) \in X_{\Delta}} \mathrm{Vect}(\mathrm{Spec}(A) \setminus V(p, I)).$$

**Lemma 2.20** ([GR24, Lemma 3.4]). *Let  $X$  be a quasisyntomic  $p$ -adic formal scheme. Then, there is a natural equivalence  $\mathrm{Vect}^{\mathrm{an}}(X_{\Delta}) \xrightarrow{\sim} \mathrm{Vect}^{\mathrm{an}}(X_{\mathrm{qsyn}}) := \lim_{R \in X_{\mathrm{qrsp}}} \mathrm{Vect}(\mathrm{Spec}(\Delta_R) \setminus V(p, I))$ .*

Let  $(A, I)$  be a prism and let  $\varphi$  denote the Frobenius endomorphism on  $A$ . Then we have that  $\varphi(I) \subset (p, I)$ , since  $\varphi(x) = x^p + p\delta(x)$  is in  $(p, I)$ . In particular, the Frobenius endomorphism  $\varphi$  on  $A$  preserves the subscheme  $\mathrm{Spec}(A) \setminus V(p, I)$  and we will denote the induced endomorphism on  $\mathrm{Spec}(A) \setminus V(p, I)$  again by  $\varphi$ .

**Definition 2.21** (Analytic prismatic  $F$ -crystals, [GR24, Definition 3.6]). Define the category  $\mathrm{Vect}^{\mathrm{an}, \varphi}(A)$  as follows: an object is a pair  $(M, \varphi_M)$  with  $M$  a vector bundle on  $\mathrm{Spec}(A) \setminus V(p, I)$  equipped with an  $A$ -linear isomorphism  $\varphi_M : (\varphi^* M)[1/I] \xrightarrow{\sim} M[1/I]$ . Morphisms between two objects are given as maps of vector bundles compatible with  $\varphi_M$ . Say that  $(M, \varphi_M)$  is effective if  $\varphi_M$  carries  $\varphi^* M$  into  $M$ .

Define the category of analytic prismatic  $F$ -crystals on  $X_{\Delta}$  as follows:

$$\mathrm{Vect}^{\mathrm{an}, \varphi}(X_{\Delta}) := \lim_{(A, I) \in X_{\Delta}} \mathrm{Vect}^{\varphi}(\mathrm{Spec}(A) \setminus V(p, I)).$$

Say that  $(\mathcal{E}, \varphi_{\mathcal{E}})$  is effective if  $\varphi_{\mathcal{E}}$  carries  $\varphi^* \mathcal{E}$  into  $\mathcal{E}$ .

**Remark 2.22.** From Lemma 2.11 we have an equivalence  $\mathrm{Vect}^{\mathrm{an}, \varphi}(X_{\Delta}) \xrightarrow{\sim} \mathrm{Vect}^{\mathrm{an}, \varphi}(X_{\mathrm{qsyn}}) := \lim_{R \in X_{\mathrm{qrsp}}} \mathrm{Vect}^{\varphi}(\mathrm{Spec}(\Delta_R) \setminus V(p, I))$ .

**Remark 2.23.** Let  $X = \mathrm{Spf}(O_F)$ , then from [GR24, Propositions 3.9] note that the restriction functor induces equivalence of categories  $\mathrm{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \mathrm{Vect}^{\mathrm{an}, \varphi}(X_{\Delta})$ .

**2.3.4. Étale realisation functor.** Now we introduce the étale realisation functor from the category of (completed/analytic) prismatic  $F$ -crystals using Lemma 2.13. Let  $X$  be a  $p$ -adic formal scheme and  $(\mathcal{E}, \varphi_{\mathcal{E}})$  an object of  $\text{Vect}^{\text{an}, \varphi}(X_{\Delta})$ . Consider the open embedding  $\text{Spec}(A[1/I]) \subset \text{Spec}(A) \setminus V(p, I)$  for any prism  $(A, I)$  in  $X_{\Delta}$  and let  $\mathcal{E}[1/\mathcal{I}_{\Delta}]$  denote the pullback of  $\mathcal{E}$  along this embedding, i.e. it is an object of  $\text{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}])$ . Using the equivalence  $\text{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge}) \xrightarrow{\sim} \text{Loc}_{\mathbb{Z}_p}(X_{\eta})$  from Lemma 2.13, we have the following:

**Definition 2.24.** Define the étale realisation functor for completed  $F$ -crystals as,

$$\begin{aligned} T_{\text{ét}}^{\wedge} : \text{CR}^{\wedge, \varphi}(X_{\Delta}) &\longrightarrow \text{Loc}_{\mathbb{Z}_p}(X_{\eta}) \\ (\mathcal{E}, \varphi_{\mathcal{E}}) &\longmapsto (\mathcal{E}[1/\mathcal{I}_{\Delta}] \otimes_{\mathcal{O}[1/\mathcal{I}_{\Delta}]} \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge})^{\varphi=1}, \end{aligned}$$

Similarly, define the étale realisation functor for analytic  $F$ -crystals as,

$$\begin{aligned} T_{\text{ét}}^{\text{an}} : \text{Vect}^{\text{an}, \varphi}(X_{\Delta}) &\longrightarrow \text{Loc}_{\mathbb{Z}_p}(X_{\eta}) \\ (\mathcal{E}, \varphi_{\mathcal{E}}) &\longmapsto (\mathcal{E}[1/\mathcal{I}_{\Delta}] \otimes_{\mathcal{O}[1/\mathcal{I}_{\Delta}]} \mathcal{O}_{\Delta}[1/\mathcal{I}_{\Delta}]^{\wedge})^{\varphi=1} \end{aligned}$$

**Remark 2.25.** The étale realisation functor from  $\text{Vect}^{\varphi}(X_{\Delta}, \mathcal{O}_{\Delta})$  in [BS23, Construction 4.8] naturally factors through the functor  $T_{\text{ét}}^{\wedge}$  (see [DLMS24, Proposition 3.43]) and  $T_{\text{ét}}^{\text{an}}$  (see [GR24, Construction 3.10]) in Definition 2.24.

**Remark 2.26.** The essential images of  $T_{\text{ét}}^{\wedge}$  and  $T_{\text{ét}}^{\text{an}}$  coincide and we have natural equivalences of categories (see [DLMS24, Theorem 1.3] and [GR24, Theorem A])

$$\text{CR}^{\wedge, \varphi}(X_{\Delta}) \xrightarrow[\sim]{T_{\text{ét}}^{\wedge}} \text{Loc}_{\mathbb{Z}_p}^{\text{cris}}(X_{\eta}) \xleftarrow[\sim]{T_{\text{ét}}^{\text{an}}} \text{Vect}^{\text{an}, \varphi}(X_{\Delta}),$$

where  $\text{Loc}_{\mathbb{Z}_p}^{\text{cris}}(X_{\eta})$  is the category of  $\mathbb{Z}_p$ -local systems  $\mathbb{L}$  on  $X_{\eta}$  such that  $\mathbb{L} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  is crystalline.

### 3. THE PRISM $(A_R, [p]_q)$

In this section, we will define and study a prism  $(A_R, [p]_q)$ , which is of fundamental importance in stating and proving the results in Sections 4 and 5. We will use the setup and notations from Subsection 1.6.

**3.1. Some important rings.** Let  $R_{\infty}^{\flat}$  and  $\overline{R}^{\flat}$  denote the tilt of  $R_{\infty}$  and  $\overline{R}$ , respectively (see Subsection 1.6). Set  $A_{\text{inf}}(R_{\infty}) := W(R_{\infty}^{\flat})$  and  $A_{\text{inf}}(\overline{R}) := W(\overline{R}^{\flat})$  to be the ring of  $p$ -typical Witt vectors with coefficients in  $R_{\infty}^{\flat}$  and  $\overline{R}^{\flat}$ , respectively, and equipped with the Frobenius endomorphism on Witt vectors and continuous (for the weak topology)  $\Gamma_R$ -action and  $G_R$ -action, respectively (see [And06, Proposition 7.2]). Moreover, from loc. cit., we have that  $A_{\text{inf}}(R_{\infty}) = A_{\text{inf}}(\overline{R})^{H_R}$ . We fix  $\overline{\mu} := \varepsilon - 1$ , where  $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \dots)$  is in  $R_{\infty}^{\flat}$  and let  $q := [\varepsilon]$  be the Teichmüller lift of  $\varepsilon$  in  $A_{\text{inf}}(R_{\infty})$ . Let  $\mu := q - 1$  and  $\xi := \mu/\varphi^{-1}(\mu)$  be some fixed elements of  $A_{\text{inf}}(R_{\infty})$  and note that  $\mu = \overline{\mu} \bmod p$ . Then from the description of the weak topology on  $A_{\text{inf}}(R_{\infty})$  and  $A_{\text{inf}}(\overline{R})$  in loc. cit., it is easy to see that the weak topology on these rings is the same as the  $(p, \mu)$ -adic topology. Next, let us note that we have  $\varphi(q) = q^p$  and for any  $g$  in  $G_R$ , we have that  $g(q) = q^{\chi(g)}$ , where  $\chi$  is the  $p$ -adic cyclotomic character, in particular, we see that  $\varphi(\mu) = (1 + \mu)^p - 1$  and  $g(1 + \mu) = (1 + \mu)^{\chi(g)}$ . Furthermore, we have a  $G_R$ -equivariant surjection  $\theta : A_{\text{inf}}(\overline{R}) \rightarrow \mathbb{C}(\overline{R})$ , where  $\mathbb{C}(\overline{R}) := \widehat{\overline{R}}[1/p]$  and  $\text{Ker } \theta = \xi A_{\text{inf}}(\overline{R})$ . The map  $\theta$  further induces a  $\Gamma_R$ -equivariant surjection  $\theta : A_{\text{inf}}(R_{\infty}) \rightarrow \widehat{R}_{\infty}$ , with  $\text{Ker } \theta = \xi A_{\text{inf}}(R_{\infty})$ .

For  $1 \leq i \leq d$ , we fix  $X_i^{\flat} = (X_i, X_i^{1/p}, X_i^{1/p^2}, \dots)$  in  $R_{\infty}$  and we take  $\{\gamma_1, \dots, \gamma_d\}$  to be topological generators of  $\Gamma'_R$ , such that  $\gamma_j(X_i^{\flat}) = \varepsilon X_i^{\flat}$ , if  $i = j$  and 0 otherwise. Moreover, we may view  $\Gamma_F$  as a subgroup of  $\Gamma_R$  and for any  $g$  in  $\Gamma_F$ , we have  $g\gamma_i g^{-1} = \gamma_i^{\chi(g)}$ , for all  $1 \leq i \leq d$ . Let us also fix Teichmüller lifts  $[X_i^{\flat}]$  in  $A_{\text{inf}}(R_{\infty})$ .

**3.1.1. The ring  $A_R$ .** Let  $A_\square$  denote the  $(p, \mu)$ -adic completion of the ring  $O_F[[\mu]][[X_1^{\flat}]^{\pm 1}, \dots, [X_d^{\flat}]^{\pm 1}]$ . Note that we have a natural embedding  $A_\square \subset A_{\text{inf}}(R_\infty)$  and  $A_\square$  is stable under the induced Frobenius endomorphism  $\varphi$  and the action of  $\Gamma_R$  (see [Abh21, Section 3]); we equip  $A_\square$  with induced structures. Moreover, we have an embedding  $\iota : R^\square \rightarrow A_\square$ , via the map  $X_i \mapsto [X_i^{\flat}]$ , for  $1 \leq i \leq d$ , and it extends to an isomorphism of rings  $R^\square[[\mu]] \xrightarrow{\sim} A_\square$ . Equip  $R^\square[[\mu]]$  with a Frobenius endomorphism  $\varphi$  extending the Frobenius on  $R^\square$  by setting  $\varphi(\mu) = (1 + \mu)^p - 1$  and note that the map  $\varphi$  is finite and faithfully flat of degree  $p^{d+1}$ . Additionally, equip  $R^\square[[\mu]]$  with an  $R^\square$ -linear continuous action of  $\Gamma_F$  by setting  $g(1 + \mu) = (1 + \mu)^{\chi(g)}$ , for  $g \in \Gamma_F$ . Then the embedding  $\iota$  and the isomorphism  $R^\square[[\mu]] \xrightarrow{\sim} A_\square$  are Frobenius and  $\Gamma_F$ -equivariant.

Let  $A_R$  denote the  $(p, \mu)$ -adic completion of the unique extension of the embedding  $A_\square \rightarrow A_{\text{inf}}(R_\infty)$  along the  $p$ -adically completed étale map  $R^\square \rightarrow R$  (see [Abh21, Subsection 3.3.2]). Then we have a natural embedding  $A_R \subset A_{\text{inf}}(R_\infty)$  and note that  $A_R$  is stable under the Frobenius and  $\Gamma_R$ -action on the latter; we equip it with induced structures. Moreover, note that the embedding  $\iota : R^\square \rightarrow A_\square \subset A_R$  and the isomorphism  $R^\square[[\mu]] \xrightarrow{\sim} A_\square \subset A_R$  extend to a unique embedding  $\iota : R \rightarrow A_R$  and an isomorphism of rings  $R[[\mu]] \xrightarrow{\sim} A_R$ . Equip  $R[[\mu]]$  with a Frobenius endomorphism  $\varphi$  extending the Frobenius on  $R$  by setting  $\varphi(\mu) = (1 + \mu)^p - 1$  and note that the map  $\varphi$  is finite and faithfully flat of degree  $p^{d+1}$ . Additionally, equip  $R[[\mu]]$  with an  $R$ -linear continuous action of  $\Gamma_F$  by setting  $g(1 + \mu) = (1 + \mu)^{\chi(g)}$ , for  $g \in \Gamma_F$ . Then, it is clear that the embedding  $\iota$  and the isomorphism  $R[[\mu]] \xrightarrow{\sim} A_R$  are Frobenius and  $\Gamma_F$ -equivariant. In particular, the Frobenius endomorphism  $\varphi$  on  $A_R$  is finite and faithfully flat of degree  $p^{d+1}$  and we have  $\varphi^*(A_R) = A_R \otimes_{\varphi, A_R} A_R \xrightarrow{\sim} \bigoplus_{\alpha} \varphi(A_R) u_\alpha$ , where  $u_\alpha = (1 + \mu)^{\alpha_0} [X_1^{\flat}]^{\alpha_1} \dots [X_d^{\flat}]^{\alpha_d}$  and  $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_d)$  is a  $(d+1)$ -tuple with  $\alpha_i$  taking values in  $\{0, 1, \dots, p-1\}$  for each  $0 \leq i \leq d$ .

**Remark 3.1.** For  $R = O_F$ , we will denote the ring  $A_R$  by  $A_F$  which is equipped with a Frobenius endomorphism  $\varphi$  and a continuous action of  $\Gamma_F$ .

Next, let us fix the following elements inside  $A_F$ :

$$\begin{aligned} [p]_q &:= \frac{q^p - 1}{q - 1} = \frac{\varphi(\mu)}{\mu}, \\ \mu_0 &:= \sum_{a \in \mathbb{F}_p^\times} ((1 + \mu)^{[a]} - 1) = -p + \sum_{a \in \mathbb{F}_p} (1 + \mu)^{[a]}, \\ \tilde{p} &:= \mu_0 + p = \sum_{a \in \mathbb{F}_p} (1 + \mu)^{[a]}. \end{aligned} \tag{3.1}$$

**Remark 3.2.** Note that for  $p = 2$ , we have  $\mu_0 = \mu$  and  $[p]_q = \mu + 2 = \tilde{p}$ . However, such equalities do not hold for  $p \geq 3$ . Let us also remark that we denote the sum  $-p + \sum_{a \in \mathbb{F}_p} (1 + \mu)^{[a]}$  as  $\mu_0$  following [Fon94, Subsection 5.2.5], where Fontaine denotes our  $\mu$  by  $\pi$  and  $\mu_0$  by  $\pi_0$ . Moreover, the notation for  $\tilde{p} = \mu_0 + p$  comes from [BL22, Subsection 3.8].

**Lemma 3.3.** *The element  $\tilde{p}$  is the product of  $[p]_q$  with a unit in  $A_F$ , in particular, both  $\tilde{p}$  and  $[p]_q$  generate the same ideal inside  $A_F$ .*

*Proof.* Note that  $A_F/[p]_q = O_F[\zeta_p]$  and we have  $[a] = a \bmod p\mathbb{Z}_p$ . Since  $\zeta_p^p = 1$  and  $\tilde{p} = \sum_{a \in \mathbb{F}_p} (1 + \mu)^{[a]}$ , we see that  $\tilde{p} = \sum_{a \in \mathbb{F}_p} \zeta_p^{[a]} = \sum_{a=0}^{p-1} \zeta_p^a = 0 \bmod [p]_q$ . In particular,  $\tilde{p} = [p]_q x$ , for some  $x$  in  $A_F$ . Moreover, since  $[p]_q = p \bmod \mu$  and  $\mu_0 = 0 \bmod \mu$ , we conclude that  $x = 1 + \mu y$  for some  $y$  in  $A_F$ , i.e.  $x$  is a unit in  $A_F$ . This proves the claim.  $\blacksquare$

**3.1.2. The ring  $R[[\mu_0]]$ .** From Subsection 3.1.1, recall that the ring  $R[[\mu]]$  is equipped with a continuous and  $R$ -linear action of  $\Gamma_F$ . Moreover, note that  $\mu_0 = \sum_{a \in \mathbb{F}_p} (1 + \mu)^{[a]}$  is an element of  $R[[\mu]]$  and it is invariant under the action of the subgroup  $\mathbb{F}_p^\times$  of  $\Gamma_F$  (see the discussion after (1.6)). Furthermore, from Lemma 3.3, the element  $\tilde{p}$  is the product of  $[p]_q$  with a unit in  $R[[\mu]]$ .

Now, consider an injective morphism of  $R$ -algebras  $R[[z]] \rightarrow R[[\mu]]$  defined by sending  $z \mapsto \mu_0$ . Denote the image inside  $R[[\mu]]$  by  $R[[\mu_0]]$  and we will view the latter as the ring of formal power series over  $R$  in the variable  $\mu_0$ . Note that the canonical injective homomorphism of  $R$ -algebras  $R[[\mu_0]] \rightarrow R[[\mu]]$  is continuous for the  $(p, \mu_0)$ -adic topology on the former and  $(p, \mu_0) = (p, \mu)$ -adic

topology on the latter. Moreover, from the explicit description of  $\mu_0$  in (3.1), it follows that  $R[[\mu_0]]$  is stable under the Frobenius endomorphism and the continuous action of  $\Gamma_0$  on  $R[[\mu]]$ ; we equip  $R[[\mu_0]]$  with the induced Frobenius endomorphism and continuous action of  $\Gamma_0$ .

**Lemma 3.4.** *Taking invariants of  $R[[\mu]]$  under the action of  $\mathbb{F}_p^\times$  induces a  $(\varphi, \Gamma_0)$ -equivariant isomorphism of rings  $R[[\mu_0]] \xrightarrow{\sim} R[[\mu]]^{\mathbb{F}_p^\times}$ . Similarly, we have a  $(\varphi, \Gamma_0)$ -equivariant and  $R[[\mu_0]]$ -linear isomorphism  $\mu_0 R[[\mu_0]] \xrightarrow{\sim} (\mu R[[\mu]])^{\mathbb{F}_p^\times}$ .*

*Proof.* Note that the map in the claim is  $(\varphi, \Gamma_0)$ -equivariant by definition. Moreover, by Remark 3.2, the claim is trivial for  $p = 2$ . So, assume that  $p \geq 3$  and consider the following  $(\varphi, \Gamma_F)$ -equivariant diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & R[[\mu_0]] & \xrightarrow{\tilde{p}} & R[[\mu_0]] & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & R[[\mu]] & \xrightarrow{\tilde{p}} & R[[\mu]] & \longrightarrow & R[\zeta_p] & \longrightarrow & 0, \end{array} \quad (3.2)$$

where the vertical arrows are natural inclusions and the bottom row is exact since we have  $R[[\mu]]/[p]_q \xrightarrow{\sim} R[\zeta_p]$  and  $[p]_q R[[\mu]] = \tilde{p} R[[\mu]]$  using Lemma 3.3. Now, note that the top row is  $\mathbb{F}_p^\times$ -invariant and for the bottom row, consider the associated long exact sequence for the cohomology of  $\mathbb{F}_p^\times$ -action and observe that  $H^1(\mathbb{F}_p^\times, R[[\mu]]) = 0$ , since  $p - 1$  is invertible in  $\mathbb{Z}_p$ , and  $R[\zeta_p]^{\mathbb{F}_p^\times} = R$ . In particular, from the long exact sequence we obtain that  $R[[\mu_0]]/(\tilde{p}) \xrightarrow{\sim} R[[\mu]]^{\mathbb{F}_p^\times}/(\tilde{p})$ . Since,  $R[[\mu_0]]$  and  $R[[\mu]]^{\mathbb{F}_p^\times}$  are  $\tilde{p}$ -adically complete and  $\tilde{p}$ -torsion free, it follows that  $R[[\mu_0]] \xrightarrow{\sim} R[[\mu]]^{\mathbb{F}_p^\times}$ .

For the second claim, we consider the following  $(\varphi, \Gamma_F)$ -equivariant diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mu_0 R[[\mu_0]] & \longrightarrow & R[[\mu_0]] & \longrightarrow & R & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \parallel & & \\ 0 & \longrightarrow & \mu R[[\mu]] & \longrightarrow & R[[\mu]] & \longrightarrow & R & \longrightarrow & 0, \end{array}$$

where the vertical arrows are natural inclusions. Now, note that the top row is  $\mathbb{F}_p^\times$ -invariant and for the bottom row, consider the associated long exact sequence for the cohomology of  $\mathbb{F}_p^\times$ -action and observe that  $H^1(\mathbb{F}_p^\times, \mu R[[\mu]]) = 0$ , since  $p - 1$  is invertible in  $\mathbb{Z}_p$  and we have  $R^{\mathbb{F}_p^\times} = R$  and  $R[[\mu_0]] \xrightarrow{\sim} R[[\mu]]^{\mathbb{F}_p^\times}$  from the first part. Hence, it follows that the left vertical arrow induces a  $(\varphi, \Gamma_0)$ -equivariant isomorphism  $\mu_0 R[[\mu_0]] \xrightarrow{\sim} (\mu R[[\mu]])^{\mathbb{F}_p^\times}$ .  $\blacksquare$

**Remark 3.5.** Note that from (A.1), we have an  $\mathbb{F}_p^\times$ -decomposition as  $R[[\mu]] = \bigoplus_{i=0}^{p-2} R[[\mu]]_i$ . Moreover, from Lemma 3.4, it follows that we have  $R[[\mu_0]] \xrightarrow{\sim} R[[\mu]]^{\mathbb{F}_p^\times} = R[[\mu]]_0$  and the preceding decomposition is  $R[[\mu_0]]$ -linear. In particular, for each  $0 \leq i < p - 2$ , we have that  $R[[\mu]]_i$  is a  $(p, \mu_0)$ -adically complete module over  $R[[\mu_0]]$ , equipped with a continuous action of  $\Gamma_0$ .

**Lemma 3.6.** *Let  $g$  be any element of  $\Gamma_0$ , then  $(g - 1)\mu_0$  is an element of  $\tilde{p}\mu_0 O_F[[\mu_0]]$ .*

*Proof.* Recall that  $O_F[[\mu_0]] \subset A_F$  is stable under the action of  $\Gamma_0$ . So if  $g$  is any element of  $\Gamma_0$ , then  $(g - 1)\mu_0$  is an element of  $(\mu A_F)^{\mathbb{F}_p^\times} \xleftarrow{\sim} \mu_0 O_F[[\mu_0]]$  (see Lemma 3.4). So let us write  $(g - 1)\mu_0 = \mu_0 x$ . Now, let us consider the diagram in (3.2) with  $R = O_F$ , where the projection map  $O_F[[\mu_0]] \rightarrow O_F$  sends  $\mu_0$  to  $-p$ . In particular, we see that  $(g - 1)\mu_0 = -px \pmod{\tilde{p}O_F[[\mu_0]]}$ . Moreover, from the  $\Gamma_F$ -equivariance of the diagram, it follows that  $(g - 1)\mu_0 = 0 \pmod{\tilde{p}O_F[[\mu_0]]}$ . As  $O_F$  is  $p$ -torsion free, therefore, we conclude that  $x$  must be an element of  $\tilde{p}O_F[[\mu_0]]$ . Hence,  $(g - 1)\mu_0$  is an element of  $\tilde{p}\mu_0 O_F[[\mu_0]]$ .  $\blacksquare$

**Lemma 3.7.** *The element  $\mu_0$  is the product of  $\mu^{p-1}$  with a unit in  $A_F$ , in particular, both  $\mu_0$  and  $\mu^{p-1}$  generate the same ideal inside  $A_F$ .*

*Proof.* The idea of the proof is motivated from [Fon94, Proposition 5.2.6]. Let  $\mu_1 = \prod_{a \in \mathbb{F}_p^\times} ((1 + \mu)^{[a]} - 1)$  be an element of  $A_F$ . Then, by expanding the product, it is easy to see that  $\mu_1 = \mu^{p-1}u$  for some unit  $u$  in  $A_F$ . Moreover,  $\mu_1$  is invariant under the action of  $\mathbb{F}_p^\times$  on  $A_F$ . Therefore, we get that  $\mu_1$  is also an element of  $O_F[[\mu]]^{\mathbb{F}_p^\times} \xleftarrow{\sim} O_F[[\mu_0]]$  (see Lemma 3.4). So, let us write  $\mu_1 = \sum_{k \in \mathbb{N}} a_k \mu_0^k$ , where  $a_k$  is an element of  $O_F$  for each  $k \in \mathbb{N}$ . But since  $\mu_1 = 0 \pmod{\mu A_F}$  and  $\mu_0 = 0 \pmod{\mu A_F}$ , we conclude that  $a_0$  must be 0. In particular, in  $A_F$ , we have

$$\mu^{p-1}u = \mu_1 = \mu_0(a_1 + \sum_{k \geq 2} a_k \mu_0^{k-1}). \quad (3.3)$$

Since  $A_F$  is  $\mu_0$ -adically complete, it is enough to show that in (3.3), the element  $a_1$  is a unit in  $O_F$ . Note that from the expression of  $\mu_0$ , it is easy to see that we have  $\mu_0 = \mu^{p-1} \pmod{pA_F}$ . Therefore, reducing (3.3) modulo  $p$ , and using the preceding observation, we get that  $\mu_0 u = \mu_0(a_1 + \sum_{k \geq 2} a_k \mu_0^{k-1}) \pmod{pA_F}$ . Now note that  $A_F/p = \kappa[[\mu]]$  is  $\mu_0$ -torsion free. Therefore, from the preceding equation, we conclude that  $u = a_1 + \sum_{k \geq 2} a_k \mu^{(p-1)(k-1)} \pmod{pA_F}$ . In particular,  $a_1 = u - \sum_{k \geq 2} a_k \mu^{(p-1)(k-1)} \pmod{pA_F}$ , is a unit in  $A_F/p = \kappa[[\mu]]$ . Since  $a_1$  is an element of  $O_F$ , it follows that  $a_1 \pmod{p}$  is a unit in  $\kappa$  and therefore  $a_1$  is a unit in  $O_F$ . Hence, from (3.3) we obtain that  $\mu^{p-1}$  is the product of  $\mu_0$  with a unit in  $A_F$ . ■

**Remark 3.8.** From Lemma 3.7 and its proof, it follows that  $A_F$  is a free  $O_F[[\mu_0]]$ -module of rank  $p - 1$ . In particular, the natural ring homomorphism  $O_F[[\mu_0]] \rightarrow A_F$  is faithfully flat and finite of degree  $p - 1$ . Similarly, the natural ring homomorphism  $R[[\mu_0]] \rightarrow R[[\mu]]$  is faithfully flat and finite of degree  $p - 1$ .

**Lemma 3.9.** *Let us consider the ring  $A_F$  equipped with Frobenius endomorphism  $\varphi$  and its subring  $O_F[[\mu_0]]$  equipped with the induced Frobenius. Then we have the following:*

- (1) *The ideal  $\mu A_F \subset A_F$  is  $\delta$ -stable in the sense of [BS22, Example 2.10].*
- (2) *The ideal  $\mu_0 O_F[[\mu_0]] \subset O_F[[\mu_0]]$  is  $\delta$ -stable.*
- (3) *We can write  $\varphi(\mu_0) = u \mu_0 \tilde{p}^{p-1}$ , for some unit  $u$  in  $O_F[[\mu_0]]$ .*

*Proof.* To prove (1), note that we have  $\delta(\mu) = \frac{\varphi(\mu) - \mu^p}{p}$  belongs to  $\mu A_F$ . Then, using the product formula for the  $\delta$ -structure from (2.1), it follows that for any  $x$  in  $A_F$ , we have  $\delta(\mu x) = \mu^p \delta(x) + x^p \delta(\mu) + p \delta(\mu) \delta(x)$ , i.e.  $\delta(\mu x)$  belongs to  $\mu A_F$ . Next, by induction on  $m \in \mathbb{N}$  and using the preceding observation, it easily follows that  $\delta^m(\mu x)$  also belongs to  $\mu A_F$ .

To show (2), let us first note that since the action of  $\Gamma_F$  on  $A_F$  commutes with the Frobenius, therefore, the action of  $\Gamma_F$  also commutes with the  $\delta$ -map, i.e. for any  $x$  in  $A_F$  and  $g$  in  $\Gamma_F$ , we have  $g(\delta(x)) = \delta(g(x))$ . Now using (1), we have that for any  $y$  in  $O_F[[\mu_0]]$ , the element  $\mu_0 y$  belongs to  $\mu A_F$ . Therefore, for any  $m \in \mathbb{N}$ , we get that  $\delta^m(\mu_0 y)$  belongs to  $\mu A_F$ . But  $\delta^m(\mu_0 y)$  also belongs to  $O_F[[\mu_0]]$ , so we get that  $\delta^m(\mu_0 y)$  is an element of  $O_F[[\mu_0]] \cap \mu A_F = (\mu A_F)^{\mathbb{F}_p^\times} \xleftarrow{\sim} \mu_0 O_F[[\mu_0]]$  (see Lemma 3.4). In particular, the ideal  $\mu_0 O_F[[\mu_0]] \subset O_F[[\mu_0]]$  is  $\delta$ -stable.

The proof of (3) is similar to that of [Fon94, Proposition 5.2.6]. Using Lemma 3.3 and Lemma 3.7, let us write  $[p]_q = v \tilde{p}$  and  $\mu_0 = w \mu^{p-1}$ , for some units  $v, w \in A_F$ . Then, we have

$$\varphi(\mu_0) = \varphi(w \mu^{p-1}) = \varphi(w) \varphi(\mu)^{p-1} = \varphi(w) \mu^{p-1} [p]_q^{p-1} = \varphi(w) v^{p-1} \mu^{p-1} \tilde{p}^{p-1} = u \mu_0 \tilde{p}^{p-1},$$

where  $u = \varphi(w) w^{-1} v^{p-1}$  is a unit in  $A_F$ . Since,  $\mu_0$  and  $\tilde{p}$  belong to  $O_F[[\mu_0]]$ , it follows that  $u = \varphi(\mu_0) / (\mu_0 \tilde{p}^{p-1})$  is an element of  $O_F[[\mu]]^{\mathbb{F}_p^\times} \xleftarrow{\sim} O_F[[\mu_0]]$  (see Lemma 3.4). ■

**3.2. The prism  $(A_R, [p]_q)$ .** In this subsection, we will show that  $(A_R, [p]_q)$  is an object of the site  $(\text{Spf } R)_{\Delta}$ , covering the final object of the topos  $\text{Shv}((\text{Spf } R)_{\Delta})$ . Moreover, we will compute the first few terms of its prismatic Čech nerve and study the induced action of  $\Gamma_R$  on these terms.

**3.2.1. An object of  $(\mathrm{Spf} R)_\Delta$ .** We begin with the following important observation:

**Lemma 3.10.** *The pair  $(A_R, [p]_q)$  is a prism and an object of  $(\mathrm{Spf} R)_\Delta$ .*

*Proof.* Note that the ring  $A_R$  is  $p$ -torsion free and equipped with a Frobenius endomorphism  $\varphi$ . So from Remark 2.2 we get that  $A_R$  is a  $\delta$ -ring. Moreover, we have  $[p]_q = \mu^{p-1} + p\mu^{p-2} + \cdots + p$ , so  $A_R$  is  $(p, [p]_q)$ -adically =  $(p, \mu)$ -adically complete. Now, since  $\varphi(\mu) = [p]_q\mu$ , therefore, we can write  $p = \varphi([p]_q) - (\mu^{p-1}[p]_q^{p-1} + p\mu^{p-2}[p]_q^{p-2} + \cdots + p(p-1)\mu[p]_q/2)$ , in particular,  $p$  is an element of  $[p]_q A_R + \varphi([p]_q) A_R$ . So we obtain that  $(A_R, [p]_q)$  is a prism in the sense of Definition 2.4. Next, we have a  $\Gamma_R$ -equivariant surjective map  $A_\square^+ \rightarrow A_\square^+/[p]_q \xrightarrow{\sim} R^\square[\zeta_p, X_1^{1/p}, \dots, X_d^{1/p}]$ , where the isomorphism is obtained by sending  $\mu \mapsto \zeta_p - 1$  and  $[X_i^p] \mapsto X_i^{1/p}$ , for  $1 \leq i \leq d$ . The surjective map above extends uniquely along the  $p$ -adically completed étale map  $R^\square \rightarrow R$ , i.e. we have a  $\Gamma_R$ -equivariant surjective map

$$A_R \rightarrow A_R/[p]_q \xrightarrow{\sim} R[\zeta_p, X_1^{1/p}, \dots, X_d^{1/p}]. \quad (3.4)$$

Clearly, we have the structure map  $R \rightarrow R[\zeta_p, X_1^{1/p}, \dots, X_d^{1/p}]$  and  $A_R/[p]_q$  is  $p$ -torsion free. Hence,  $(A_R, [p]_q)$  satisfies the axioms of Definition 2.7 and it is an object of  $(\mathrm{Spf} R)_\Delta$ . ■

From Lemma 3.10 we have that  $(A_R, [p]_q)$  is regarded as a prism via  $A_R \rightarrow A_R/[p]_q \leftarrow R$ . Moreover, we have the following:

**Lemma 3.11.** *The object  $(A_R, [p]_q)$  is a cover of the final object of the topos  $\mathrm{Shv}((\mathrm{Spf} R)_\Delta)$ .*

*Proof.* Let us first note that  $\widehat{R}_\infty$  is a perfectoid  $R$ -algebra (see [BMS19, Definition 4.18]) and the pair  $(A_{\mathrm{inf}}(R_\infty), [p]_q)$  is a prism and an object of  $(\mathrm{Spf} R)_\Delta$  since  $A_{\mathrm{inf}}(R_\infty)/[p]_q \xrightarrow{\sim} \widehat{R}_\infty$ . Furthermore, the natural map  $A_R \rightarrow A_{\mathrm{inf}}(R_\infty)$  is compatible with the prism structure on both rings, in particular, we have a map  $(A_R, [p]_q) \rightarrow (A_{\mathrm{inf}}(R_\infty), [p]_q)$  of prisms in  $(\mathrm{Spf} R)_\Delta$ . So, to show that  $(A_R, [p]_q)$  is a cover of the final object of the topos  $\mathrm{Shv}((\mathrm{Spf} R)_\Delta)$ , it is enough to show that  $(A_{\mathrm{inf}}(R_\infty), [p]_q)$  is a cover of the final object of the topos  $\mathrm{Shv}((\mathrm{Spf} R)_\Delta)$ . Now, note that the natural map  $R \rightarrow \widehat{R}_\infty$  is faithfully flat and the  $p$ -complete Tor amplitude of the cotangent complex  $L_{\widehat{R}_\infty/R}$  in  $D(\widehat{R}_\infty)$  (see Appendix A.1), is in  $[-1, 0]$  because  $R$  is a  $p$ -complete smooth  $\mathbb{Z}_p$ -algebra and  $\widehat{R}_\infty$  is a perfectoid  $\mathbb{Z}_p$ -algebra, hence, quasisyntomic by [BMS19, Proposition 4.19]. Therefore, from [BS22, Proposition 7.11], we obtain that  $(A_{\mathrm{inf}}(R_\infty), [p]_q)$  is a cover of the final object of the topos  $\mathrm{Shv}((\mathrm{Spf} R)_\Delta)$ . This allows us to conclude. ■

**Lemma 3.12.** *Let  $g \in \Gamma_R$ , then the action of  $g$  on  $A_R$  induces an automorphism of  $(A_R, [p]_q)$  in  $(\mathrm{Spf} R)_\Delta$ . Moreover, we have  $(g-1)A_R \subset \mu A_R$ .*

*Proof.* As the action of  $\Gamma_R$  on  $A_R$  is continuous, it is enough to check the claim for the topological generators  $\{\gamma_1, \dots, \gamma_d\}$  of  $\Gamma'_R$  and each  $g \in \Gamma_F$ . Note that for  $1 \leq i \leq d$ , using the description of  $A_R$  from Subsection 3.1.1 and explicit computations, it easily follows that we have  $(\gamma_i - 1)A_R \subset \mu A_R$  and  $\gamma_i([p]_q A_R) \subset [p]_q A_R$ . For any  $g$  in  $\Gamma_F$ , note that we have  $(g-1)\mu = (1+\mu)^{\chi(g)} - 1 - \mu = \chi(g)\mu u - \mu$ , for some unit  $u$  in  $A_R$ , therefore, it follows that  $(g-1)A_R \subset \mu A_R$ . Similarly, note that  $g([p]_q) = g(\varphi(\mu)/\mu) = \varphi(\mu u)/(\mu u)$ , which belongs to  $[p]_q A_R$ . This proves both the claims. ■

**3.2.2. Prismatic Čech nerve of  $(A_R, [p]_q)$ .** In this subsection, we will provide a description of the simplicial object  $(A_R(\bullet), I(\bullet))$  in  $(\mathrm{Spf} R)_\Delta$ . Such a description would help in describing any prismatic  $F$ -crystal over  $(\mathrm{Spf} R)_\Delta$  in terms of an  $A_R$ -module with stratification.

**Construction 3.13.** Let  $A_R(\bullet)$  denote the cosimplicial ring obtained by taking the prismatic Čech nerve  $(A_R(\bullet), I(\bullet))$  of  $(A_R, [p]_q)$  in  $(\mathrm{Spf} R)_\Delta$ . Clearly, we have that  $A_R(0) = A_R$ . As products in  $(\mathrm{Spf} R)_\Delta$  are computed by prismatic envelopes (which exist in our case), we can describe  $A_R(n)$  a bit more explicitly. For  $n \in \mathbb{N}$ , let  $(A_R)^{\otimes(n+1)}$  denote the  $(n+1)$ -fold tensor product of  $A_R$  over  $O_F$  and we consider the  $A_R$ -algebra structure on  $(A_R)^{\otimes(n+1)}$  via the first component, i.e.  $a \mapsto a \otimes 1^{\otimes n}$ . Let

$(A_R)^{\widehat{\otimes}(n+1)}$  denote the  $(p, [p]_q)$ -adic completion of  $(A_R)^{\otimes(n+1)}$ , which can be given as a  $(p, [p]_q)$ -adic completed étale algebra over a polynomial algebra in finitely many variables over a power-series  $A_R$ -algebra in finitely many variables, in particular, we see that  $(A_R)^{\widehat{\otimes}(n+1)}$  is  $(p, [p]_q)$ -completely flat over  $(A_R, [p]_q)$ . In other words, for each  $m \in \mathbb{N}_{\geq 1}$ , we have that the natural map  $A_R/(p, [p]_q)^m \rightarrow (A_R)^{\widehat{\otimes}(n+1)}/(p, [p]_q \otimes 1)^m$  is flat. Since  $A_R$  is noetherian, from [Sta23, Tag 0912], it follows that the map  $A_R \rightarrow (A_R)^{\widehat{\otimes}(n+1)}$  is flat. Now, for each  $n \in \mathbb{N}$ , we have a surjection

$$(A_R)^{\widehat{\otimes}(n+1)} \twoheadrightarrow R[\zeta_p, X_1^{1/p}, \dots, X_d^{1/p}]^{\otimes(n+1)}, \quad (3.5)$$

where on the right hand term, the tensor product is taken over  $R$ . For  $1 \leq i \leq n+1$ , let  $n_i : A_R \rightarrow (A_R)^{\widehat{\otimes}(n+1)}$  denote the map, sending  $a \mapsto 1 \otimes \dots \otimes a \otimes \dots \otimes 1$ , in the  $i^{\text{th}}$  position. Then the kernel of (3.5) is given by the ideal

$$J(n) = (n_i([p]_q), n_j([X_s^b]^p) - n_k([X_s^b]^p) \text{ for } 1 \leq s \leq d, 1 \leq i, j, k \leq d, j \neq k) \subset (A_R)^{\widehat{\otimes}(n+1)}.$$

Note that from the proof of Lemma 3.10, we have that  $A_R/[p]_q \xrightarrow{\sim} R[\zeta_p, X_1^{1/p}, \dots, X_d^{1/p}]$  (see (3.4)). Therefore, it easily follows that the sequence  $\{p, n_i([p]_q), n_j([X_s^b]^p) - n_k([X_s^b]^p) \text{ for } 1 \leq s \leq d, 1 \leq i, j, k \leq d, j \neq k\}$  is regular on  $(A_R)^{\widehat{\otimes}(n+1)}$ . So, from [BS22, Proposition 3.13], we get that  $(A_R(n), n_1([p]_q))$  is the prismatic envelope of  $((A_R)^{\widehat{\otimes}(n+1)}, J(n))$  over the bounded prism  $(A_R, [p]_q)$ . By the universal property of prismatic envelopes, the maps  $n_i$  from above extend uniquely to  $n_i : A_R \rightarrow A_R(n)$ , for  $1 \leq i \leq n+1$ . Let us denote the cosimplicial ring  $A_R(\bullet)$  by the usual diagram

$$A_R(\bullet) : \quad \begin{array}{ccccc} & & p_2 & & p_{23} \\ & \curvearrowright & & \curvearrowleft & \\ & & \Delta & & \\ & \curvearrowleft & & \curvearrowright & \\ & & p_1 & & p_{12} \end{array} \quad \begin{array}{ccc} & & \\ & \curvearrowleft & \\ & & p_{13} \\ & \curvearrowright & \\ & & \end{array} \quad \begin{array}{c} A_R(0) \leftarrow A_R(1) \rightarrow A_R(2) \cdots \end{array}$$

where explicitly we write  $p_1 = 1_1 : A_R(0) \rightarrow A_R(1)$  for the map  $a \mapsto a \otimes 1$  and  $p_2 = 1_2 : A_R(0) \rightarrow A_R(1)$  for the map  $a \mapsto 1 \otimes a$ . Similarly, we define maps  $p_{12}, p_{13}$  and  $p_{23}$ . To avoid confusion, we will write  $r_j = 2_j : A_R \rightarrow A_R(2)$  for the map sending  $a$  to the  $j^{\text{th}}$  position for  $j = 1, 2, 3$ . ■

**Lemma 3.14.** *The pair  $(A_R(n), n_1([p]_q))$  is a prism and an object of  $(\text{Spf } R)_{\Delta}$ . Moreover, we have that  $(A_R(n), n_1([p]_q))$  is the  $n$ -fold self product of  $(A_R, [p]_q)$  over the final object of  $\text{Shv}((\text{Spf } R)_{\Delta})$ .*

*Proof.* From [BS22, Proposition 3.13] we have that  $(A_R(n), n_1([p]_q))$  is a prism and moreover, it is  $(p, [p]_q)$ -completely flat over  $(A_R, [p]_q)$ , in particular,  $(A_R(n), n_1([p]_q))$  is a bounded prism by [BS22, Lemma 3.7]. Furthermore, we have that  $A_R(n)/(n_1([p]_q)) \xrightarrow{\sim} R[\zeta_p, X_1^{1/p}, \dots, X_d^{1/p}]^{\otimes(n+1)} \leftarrow R$ , i.e.  $(A_R(n), n_1([p]_q))$  is an object of  $(\text{Spf } R)_{\Delta}$ . Finally, from [BS22, Proposition 3.13], self products are computed via prismatic envelopes (when they exist), so from Construction 3.13, it follows that  $(A_R(n), n_1([p]_q))$  is the  $n$ -fold self product of  $(A_R, [p]_q)$  over the final object of  $\text{Shv}((\text{Spf } R)_{\Delta})$ . ■

For later calculations it will be convenient to have an explicit presentation of  $A_R(1)$  and  $A_R(2)$ . Recall that the ideal  $J(1) = ([p]_q \otimes 1, 1 \otimes [p]_q, [X_i^b]^p \otimes 1 - 1 \otimes [X_i^b]^p \text{ for } 1 \leq i \leq d) \subset A_R \widehat{\otimes}_{O_F} A_R$ , is the kernel of the map in (3.5) for  $n = 1$ . Consider a free  $\delta$ -algebra over  $A_R \widehat{\otimes}_{O_F} A_R$  in  $d+1$  variables given as  $(A_R \widehat{\otimes}_{O_F} A_R) \left\{ \frac{1 \otimes [p]_q}{[p]_q \otimes 1}, \frac{[X_1^b]^p \otimes 1 - 1 \otimes [X_1^b]^p}{[p]_q \otimes 1}, \dots, \frac{[X_d^b]^p \otimes 1 - 1 \otimes [X_d^b]^p}{[p]_q \otimes 1} \right\}_{\delta}$ . Then, from [BS22, Proposition 3.13], we have

$$A_R(1) = (A_R \widehat{\otimes}_{O_F} A_R) \left\{ \frac{1 \otimes [p]_q}{[p]_q \otimes 1}, \frac{[X_1^b]^p \otimes 1 - 1 \otimes [X_1^b]^p}{[p]_q \otimes 1}, \dots, \frac{[X_d^b]^p \otimes 1 - 1 \otimes [X_d^b]^p}{[p]_q \otimes 1} \right\}_{(p, [p]_q \otimes 1)}^{\wedge}, \quad (3.6)$$

i.e.  $A_R(1)$  is the  $(p, [p]_q \otimes 1)$ -adic completion of the free  $\delta$ -algebra  $(A_R \widehat{\otimes}_{O_F} A_R) \left\{ \frac{J(1)}{[p]_q \otimes 1} \right\}_{\delta}$ . Similarly, using the ideal  $J(2) \subset (A_R)^{\widehat{\otimes}3}$ , from [BS22, Proposition 3.13], we have

$$A_R(2) = (A_R)^{\widehat{\otimes}3} \left\{ \frac{J(2)}{[p]_q \otimes 1} \right\}_{(p, [p]_q \otimes 1)}^{\wedge},$$



i.e.  $A_R(2)$  is the  $(p, [p]_q \otimes 1)$ -adic completion of the free  $\delta$ -algebra  $(A_R)^{\widehat{\otimes}^3} \left\{ \frac{J(2)}{[p]_q \otimes 1} \right\}_\delta$ .

**Lemma 3.15.** *The  $A_R$ -linear morphisms  $n_i : A_R \rightarrow A_R(n)$ , for  $1 \leq i \leq n+1$ , are faithfully flat. In particular, the morphisms  $p_i$  for  $i = 1, 2$  and  $r_j$  for  $j = 1, 2, 3$  are faithfully flat. Moreover, the element  $\frac{n_i([p]_q)}{n_1([p]_q)}$  is a unit in  $A_R(n)$ , for  $1 \leq i \leq n+1$ . In particular,  $\frac{1 \otimes [p]_q}{[p]_q \otimes 1}$  is a unit in  $A_R(1)$ .*

*Proof.* Note that from Lemma 3.14 and [BS22, Proposition 3.13], the map  $n_i : A_R \rightarrow A_R(n)$  is  $(p, [p]_q)$ -completely flat. In particular,  $n_i : A_R/(p, [p]_q)^m \rightarrow A_R(n)/(p, [p]_q \otimes 1)^m$ , is flat for each  $m \in \mathbb{N}_{\geq 1}$ . Since  $A_R$  is noetherian, from [Sta23, Tag 09I2], it follows that  $n_i$  is flat. Moreover, as we have  $\Delta \circ n_i = id$ , therefore for an  $A_R$ -module  $M$ , we see that  $A_R(n) \otimes_{n_i, A_R} M = 0$  if and only if  $N = 0$ . Hence, the morphisms  $n_i$  for  $1 \leq i \leq n+1$  are faithfully flat. Finally, it is easy to see that  $n_i([p]_q)$  is a distinguished element of  $A_R(n)$  and writing  $n_i([p]_q) = n_1([p]_q) \frac{n_i([p]_q)}{n_1([p]_q)}$ , from [BS22, Lemma 2.24], we get that  $\frac{n_i([p]_q)}{n_1([p]_q)}$  is a unit in  $A_R(n)$ .  $\blacksquare$

**Remark 3.16.** From now on we will denote the prism  $(A_R(1), [p]_q \otimes 1)$  simply as  $(A_R(1), [p]_q)$ . In light of Lemma 3.15 this simplification should not cause any confusion to the reader.

**3.2.3. Galois action on  $(A_R, [p]_q)$ .** Note that for  $n \in \mathbb{N}$ , the product  $\Gamma_R^{\times(n+1)}$  of  $n+1$  copies of  $\Gamma_R$ , naturally acts on the  $\delta$ -ring  $(A_R)^{\widehat{\otimes}^n}$  and the ring homomorphism  $(A_R)^{\widehat{\otimes}^n} \rightarrow R[\zeta_p, X_1^{1/p}, \dots, X_d^{1/p}]^{\otimes(n+1)}$  from (3.5) is easily checked to be  $\Gamma_R^{\times(n+1)}$ -equivariant. So, by the universal property of prismatic envelopes, the continuous action of  $\Gamma_R^{\times(n+1)}$  on  $(A_R)^{\widehat{\otimes}^n}$  naturally extends to a continuous action on  $A_R(n)$ . Moreover, since the action of  $\Gamma_R$  is identity on  $A_R/\mu$ , we claim the following:

**Proposition 3.17.** *The action of the  $i^{\text{th}}$ -component of  $\Gamma_R^{\times(n+1)}$  is trivial on  $A_R(n)/(n_i(\mu))$ .*

*Proof.* From the definition of the action of  $\Gamma_R^{\times(n+1)}$  on  $(A_R)^{\widehat{\otimes}^n}$ , it is easy to see that the  $i^{\text{th}}$ -component acts trivially on  $(A_R)^{\widehat{\otimes}^n}/(n_i(\mu))$ . Then, from Construction 3.13, we note that to get the claim, it is enough to show that for any  $x$  in the set of generators  $\{n_j([p]_q), n_k([X_s^b]^p) - n_l([X_s^b]^p)\}$ , for  $1 \leq s \leq d$ ,  $1 \leq j, k, l \leq d$  and  $k \neq l$ , of the ideal  $J(n) \subset (A_R)^{\widehat{\otimes}^n}$ , we have that

$$(g-1)\delta^m\left(\frac{x}{n_1([p]_q)}\right) \in n_i(\mu)A_R(n), \quad (3.7)$$

for each  $m \in \mathbb{N}$  and any  $g$  in the  $i^{\text{th}}$ -component of  $\Gamma_R^{\times(n+1)}$ . We can reduce this claim further as follows. Using Lemma 3.9 (1), let us first note that  $n_i(\mu)A_R(n)$  is a  $\delta$ -stable ideal of  $A_R(n)$  in the sense of [BS22, Example 2.10]. Then, by using Lemma 3.19 for  $A = A_R(n)$  and  $\alpha = n_i(\mu)$ , we see that to prove our main claim, it is enough to show (3.7) for  $m = 0$ , i.e.  $(g-1)\left(\frac{x}{n_1([p]_q)}\right)$  belongs to  $n_i(\mu)A_R(n)$ .

Let us first assume  $1 < i \leq n+1$  and since the action of  $\Gamma_R = i^{\text{th}}$ -component of  $\Gamma_R^{\times(n+1)}$ , on  $A_R(n)$  is continuous, it is enough to check the preceding claim for the topological generators  $\{\gamma_1, \dots, \gamma_d\}$  of  $\Gamma'_R$  and each  $g \in \Gamma_F$ . Now, take  $x = n_k([X_s^b]^p) - n_l([X_s^b]^p)$  in  $J(n)$ . Then, the claim is obvious for  $k \neq i$  and  $l \neq i$ . So, assume  $k = i$  and  $l \neq k$  and note that we have

$$(\gamma_s - 1)\left(\frac{x}{n_1([p]_q)}\right) = \frac{(1+n_i(\mu))^p n_i([X_s^b]^p) - n_l([X_s^b]^p)}{n_1([p]_q)} = \frac{n_i([p]_q) n_i(\mu) n_i([X_s^b]^p)}{n_1([p]_q)} \in n_i(\mu)A_R(n),$$

where we have used the fact that  $\frac{n_i([p]_q)}{n_1([p]_q)}$  is a unit in  $A_R(n)$  (see Lemma 3.15). Next, let  $x = n_i([p]_q)$ , then for any  $g$  in  $\Gamma_F$ , we have that

$$(g-1)\left(\frac{x}{n_1([p]_q)}\right) = \frac{n_i([p]_q) n_i(\mu) y}{n_1([p]_q)} \in n_i(\mu)A_R(n),$$

where the first equality follows from Lemma 3.18. Therefore, by using Lemma 3.19 for  $A = A_R(n)$  and  $\alpha = n_i(\mu)$ , it follows that we have shown (3.7) for all generators of  $J(n)$  and  $1 < i \leq n+1$ .

Let us now assume  $i = 1$  and again note that since the action of  $\Gamma_R = 1^{\text{st}}$ -component of  $\Gamma_R^{\times(n+1)}$ , on  $A_R(n)$  is continuous, therefore, it is enough to check the preceding claim for the topological generators

$\{\gamma_1, \dots, \gamma_d\}$  of  $\Gamma'_R$  and each  $g \in \Gamma_F$ . An argument similar to above, shows that for all  $x$  in  $J(n)$  and  $1 \leq s \leq d$ , we have that  $(\gamma_s - 1)(\frac{x}{n_1([p]_q)})$  belongs to  $n_1(\mu)A_R(n)$ . Now take any  $g \in \Gamma_F$  and  $x$  an element of  $J(n)$ . Then we have that

$$(g - 1)(\frac{x}{n_1([p]_q)}) = x(\frac{1}{g(n_1([p]_q))} - \frac{1}{n_1([p]_q)}) \in n_i(\mu)A_R(n),$$

using Lemma 3.18. Therefore, by using Lemma 3.19, it follows that we have shown (3.7) for all generators of  $J(n)$  and  $i = 1$ . This allows us to conclude.  $\blacksquare$

The following observation was used above:

**Lemma 3.18.** *Let  $g \in \Gamma_F$ , then  $(g - 1)[p]_q$  belongs to  $[p]_q\mu A_F$  and  $(g - 1)(\frac{1}{[p]_q})$  belongs to  $\frac{\mu}{[p]_q}A_F$ .*

*Proof.* Let us first observe that  $g(\mu) = \chi(g)u_g\mu$ , where  $\chi$  is the  $p$ -adic cyclotomic character, and  $u_g = 1 + x_g\mu$ , for some  $x_g \in A_F$ , is a unit in  $A_F$ . Then we have

$$(g - 1)[p]_q = (g - 1)\frac{\varphi(\mu)}{\mu} = \frac{\varphi(g(\mu))}{g(\mu)} - \frac{\varphi(\mu)}{\mu} = (\frac{\varphi(u_g)}{u_g} - 1)[p]_q \in [p]_q\mu A_F.$$

Next, observe that

$$(g - 1)(\frac{1}{[p]_q}) = (g - 1)\frac{\mu}{\varphi(\mu)} = \frac{g(\mu)}{\varphi(g(\mu))} - \frac{\mu}{\varphi(\mu)} = (\frac{u_g}{\varphi(u_g)} - 1)\frac{1}{[p]_q} \in \frac{\mu}{[p]_q}A_F.$$

This proves the lemma.  $\blacksquare$

We prove a general statement for  $\delta$ -rings admitting an action of  $\Gamma_R$ , which also be useful in Subsection 3.4.3.

**Lemma 3.19.** *Let  $A$  be a  $p$ -adically complete  $p$ -torsion free  $\delta$ -ring such that  $A$  admits a continuous and  $\varphi$ -equivariant action of  $\Gamma_R$  and let  $\alpha$  be an element of  $A$  such that the ideal  $(\alpha) \subset A$  is  $\delta$ -stable in the sense of [BS22, Example 2.10]. Assume that for some  $x \in A$  and  $g$  an element of  $\Gamma_R^{\times(n+1)}$ , we are given that  $(g - 1)x$  is in  $(\alpha)$ . Then for each  $m \in \mathbb{N}$ , we have that  $(g - 1)\delta^m(x)$  belongs to  $(\alpha)$  as well.*

*Proof.* Let us fix some  $x \in A$  and  $g$  an element of  $\Gamma_R^{\times(n+1)}$  such that  $(g - 1)x$  lies in  $(\alpha)$ . We will proceed by induction on  $m \in \mathbb{N}$ . So for some  $m \in \mathbb{N}$ , let us write  $(g - 1)\delta^m x = \alpha y$ . Then, observe that

$$\begin{aligned} (g - 1)(\delta^{m+1}x) &= (g - 1)\frac{\varphi(\delta^m x) - (\delta^m x)^p}{p} \\ &= \frac{1}{p}(\varphi((g - 1)\delta^m x) - (g - 1)(\delta^m x)^p) \\ &= \frac{1}{p}(\varphi(\alpha y) - ((g - 1)\delta^m x + \delta^m x)^p + (\delta^m x)^p) \\ &= \frac{1}{p}(\varphi(\alpha y) - \sum_{k=1}^p \binom{p}{k} (\alpha y)^k (\delta^m x)^{p-k}) \\ &= \delta(\alpha y) - \frac{1}{p} \sum_{k=1}^{p-1} \binom{p}{k} (\alpha y)^k (\delta^m x)^{p-k}. \end{aligned} \tag{3.8}$$

Using the product formula for the delta map from (2.1), it follows that  $\delta(\alpha y)$  belongs to  $(\alpha)$ . Hence,  $(g - 1)(\delta^{m+1}x)$  belongs to  $(\alpha)$ , thus proving the claim.  $\blacksquare$

**Remark 3.20.** For  $n \in \mathbb{N}$ , the  $\delta$ -ring  $(A_R)^{\widehat{\otimes} n}$  is equipped with a Frobenius endomorphism given as tensor product of Frobenius on each component. By compatibility of  $\delta$ -structures between  $(A_R)^{\widehat{\otimes} n}$  and  $A_R(n)$ , we deduce that the Frobenius endomorphism on  $A_R(n)$  is compatible with the one on  $(A_R)^{\widehat{\otimes} n}$ . Then, by the description of the Galois action on  $A_R(1)$ , it easily follows that the natural map  $p_i : A_R \rightarrow A_R(1)$  is  $(\varphi, \Gamma_R^2)$ -equivariant for  $i = 1, 2$ , where  $\Gamma_R^2$  acts on the source via projection on to the  $i^{\text{th}}$  coordinate. Similarly, the natural map  $r_j : A_R \rightarrow A_R(2)$  is  $(\varphi, \Gamma_R^3)$ -equivariant for  $j = 1, 2, 3$ , where  $\Gamma_R^3$  acts on the source via projection on to the  $j^{\text{th}}$  coordinate.

**3.3. The ring  $A_R(1)/p_1(\mu)$ .** From Remark 3.20 recall that we have a  $(\varphi, \Gamma_R^2)$ -equivariant maps  $p_i : A_R \rightarrow A_R(1)$  for  $i = 1, 2$ , where  $A_R$  is equipped with a  $\Gamma_R^2$ -action via projection on to the  $i^{\text{th}}$  coordinate. Moreover, we note that there is an induced action of  $\Gamma_R^2$  on  $A_R(1)/p_1(\mu)$ , where the action of the first component is identity. Additionally, from Subsection 3.1 recall that we have fixed topological generators  $\{\gamma_1, \dots, \gamma_d\}$  of  $\Gamma'_R$ . The goal of this subsection is to give an explicit description of the ring  $A_R(1)/p_1(\mu)$  and explain some of its properties. We start with some computations.

**3.3.1. Some divided power calculations.** The goal of this subsection is to show the following claim:

**Proposition 3.21.** *The following natural map is a  $\varphi$ -equivariant isomorphism of  $p$ -torsion free rings*

$$\beta : \Lambda_R := R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge \xrightarrow{\sim} R[[\mu]][(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge. \quad (3.9)$$

*Notation.* For any element  $x$ , we use  $x^{[k]}$  to denote  $x^k/(k!)$ , for all  $k \in \mathbb{N}$ .

*Proof.* Let  $B$  denote the divided power envelope of the  $R[\mu]$ -subalgebra  $R[\mu, \mu^{p-1}/p] \subset R[\mu][1/p]$ , with respect to the ideal  $(\mu^{p-1}/p) \subset R[\mu, \mu^{p-1}/p]$ , i.e.  $B = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]$ . Similarly, let  $B'$  denote the divided power envelope of the  $R[[\mu]]$ -subalgebra  $R[[\mu]][\mu^{p-1}/p] \subset R[[\mu]][1/p]$ , with respect to the ideal  $(\mu^{p-1}/p) \subset R[[\mu]][\mu^{p-1}/p]$ , i.e.  $B' = R[[\mu]][(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]$ . Note that the source and target of (3.9) are the  $p$ -adic completions of  $B$  and  $B'$ , respectively. Moreover, we have a natural morphism of rings  $B \rightarrow B'$  induced by the inclusion  $R[\mu][1/p] \subset R[[\mu]][1/p]$ . Evidently, the rings  $B$  and  $B'$  are  $p$ -torsion free, the natural map  $B \rightarrow B'$  is injective and to show that (3.9) is an isomorphism, it is enough to show that the natural map  $B \rightarrow B'$  induces an isomorphism of rings  $B/p \xrightarrow{\sim} B'/p$ . Since,  $\mu^{p-1} = p(\mu^{p-1}/p)$  in  $B$  and  $B'$ , therefore  $B/p = B/(\mu^{p-1}, p) = (B/\mu^{p-1})/p$  and  $B'/p = B'/(\mu^{p-1}, p) = (B'/\mu^{p-1})/p$ . Now, from Lemma 3.22, using the explicit description of  $B$  and  $B'$ , we have a natural commutative diagram

$$\begin{array}{ccc} ((R[\mu]/\mu^{p-1})[Y_0, Y_1, \dots]/(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k \in \mathbb{N}})/p & \xrightarrow{\sim} & B/p \\ \downarrow & & \downarrow \\ ((R[[\mu]]/\mu^{p-1})[Y_0, Y_1, \dots]/(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k \in \mathbb{N}})/p & \xrightarrow{\sim} & B'/p, \end{array}$$

where the vertical arrows are natural maps, the top horizontal arrow is reduction modulo  $p$  of the isomorphism in (3.10) and the bottom horizontal arrow is reduction modulo  $p$  of the isomorphism in (3.11). Since  $R[\mu]/\mu^{p-1} \xrightarrow{\sim} R[[\mu]]/\mu^{p-1}$ , it follows that the left vertical map is an isomorphism as well. So we obtain that  $B/p \xrightarrow{\sim} B'/p$  and hence (3.9) is an isomorphism of  $p$ -torsion free rings. Finally, note that by definition the map  $\beta$  in (3.9) is  $\varphi$ -equivariant. This concludes our proof.  $\blacksquare$

The following descriptions of  $B$  and  $B'$  were used in the proof of Proposition 3.21:

**Lemma 3.22.** *The following natural map is an isomorphism of rings*

$$R[\mu][Y_0, Y_1, \dots]/(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k \in \mathbb{N}} \xrightarrow{\sim} B, \quad (3.10)$$

where  $Y_k \mapsto (\mu^{p-1}/p)^{[p^k]}$ . Similarly, the following natural map is an isomorphism of rings

$$R[[\mu]][Y_0, Y_1, \dots]/(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k \in \mathbb{N}} \xrightarrow{\sim} B', \quad (3.11)$$

where  $Y_k \mapsto (\mu^{p-1}/p)^{[p^k]}$ .

*Proof.* We will only show that the map in (3.10) is bijective, the claim for (3.11) follows by a similar argument. Define an  $R[\mu]$ -linear map

$$R[\mu][Y_0, Y_1, \dots] \longrightarrow B, \quad Y_k \longmapsto \left(\frac{\mu^{p-1}}{p}\right)^{[p^k]}. \quad (3.12)$$

Let us first note that this map is surjective. Indeed, let  $n \in \mathbb{N}$  and write  $n = \sum_{i=0}^r n_i p^i$  in base  $p$ , with  $0 \leq n_i \leq p-1$ . Then we have,

$$\left(\frac{\mu^{p-1}}{p}\right)^{[n]} = \frac{1}{n!} \left(\frac{\mu^{p-1}}{p}\right)^{\sum_{i=0}^r n_i p^i} = \frac{1}{n!} \prod_{i=0}^r (p^i!)^{n_i} \prod_{i=0}^r \left(\frac{\mu^{p-1}}{p}\right)^{[p^i]}.$$

An easy computation shows that the  $p$ -adic valuation of  $\prod_{i=0}^r (p^i!)^{n_i}$  equals the  $p$ -adic valuation of  $n!$ . So, it follows that (3.12) is surjective and it is clear that the kernel of (3.12) is given by the ideal  $(pY_0 - \mu^{p-1}, pY_{k+1} - Y_k)_{k \in \mathbb{N}} \subset R[\mu][Y_0, Y_1, \dots]$ . This proves the claim.  $\blacksquare$

**Remark 3.23.** Let  $\Lambda_F := O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge \xrightarrow{\sim} O_F[[\mu]][(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$  and from Proposition 3.21 note that it is a  $p$ -torsion free algebra over  $A_F = O_F[[\mu]]$ . We equip  $\Lambda_F$  with a continuous  $(\varphi, \Gamma_F)$ -action by extending the  $(\varphi, \Gamma_F)$ -action on  $A_F$  and using the formulas  $\varphi(\mu^{p-1}) = [p]_q^{p-1} \mu^{p-1}$  and  $g(\mu^{p-1}) = \chi(g)^{p-1} \mu^{p-1} u$ , for some unit  $u$  in  $(A_F)^\times$ . Now recall that from Subsection 3.1.1 we have a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings  $\iota : R[[\mu]] \xrightarrow{\sim} A_R$ . This extends to a  $(\varphi, \Gamma_F)$ -equivariant isomorphism

$$\iota : R[[\mu]] \widehat{\otimes}_{O_F[[\mu]]} \Lambda_F \xrightarrow{\sim} A_R \widehat{\otimes}_{A_F} \Lambda_F, \quad (3.13)$$

where we take tensor product action of  $\varphi$  and  $\Gamma_F$  and completion is with respect to the  $p$ -adic topology. Furthermore, using the completed tensor product description, we get that the isomorphism (3.9) in Proposition 3.21 is a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings

$$\beta : \Lambda_R := R \widehat{\otimes}_{O_F} \Lambda_F \xrightarrow{\sim} R[[\mu]] \widehat{\otimes}_{O_F[[\mu]]} \Lambda_F. \quad (3.14)$$

For any  $k \in \mathbb{N}$ , let  $k = (p-1)f(k) + r(k)$ , with  $r(k), f(k) \in \mathbb{N}$  and  $0 \leq r(k) < p-1$ . Set  $\mu^{\{k\}} = \frac{\mu^k}{f(k)! p^{f(k)}}$ , and note that we have  $\Lambda_F = O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = O_F[\mu, \mu^{\{k\}}, k \in \mathbb{N}]_p^\wedge$ . In particular, for any  $x$  in  $\Lambda_F$  we have a unique presentation  $x = \sum_{k \in \mathbb{N}} a_k \mu^{\{k\}}$  in  $\Lambda_F$  with  $a_k$  in  $O_F$  for all  $k \in \mathbb{N}$  and  $p$ -adically  $a_k \rightarrow 0$  as  $k \rightarrow +\infty$ . We note an important observation which will be used throughout.

**Lemma 3.24.** *The element  $t = \log(1 + \mu) = \sum_{k \in \mathbb{N}} (-1)^k \frac{\mu^{k+1}}{k+1}$  converges in  $\mu\Lambda_F$ . Moreover, the elements  $t/\mu$ ,  $[p]_q/p$  and  $\tilde{p}/p$  are units in  $\Lambda_F$ .*

*Proof.* Formally, write  $t/\mu = \sum_{k \in \mathbb{N}} (-1)^k \frac{\mu^k}{k+1} = \sum_{k \in \mathbb{N}} a_k \mu^k$  and  $\mu/t = \frac{\mu}{\log(1+\mu)} = \sum_{k \in \mathbb{N}} b_k \mu^k$ , with  $v_p(a_k), v_p(b_k) \geq -\frac{k}{p-1}$ , for all  $k \in \mathbb{N}$ . From the discussion before the lemma, recall that  $\mu^{\{k\}} = \frac{\mu^k}{f(k)! p^{f(k)}}$ . Then we have

$$v_p\left(\frac{a_k}{f(k)! p^{f(k)}}\right) \geq -\frac{k}{p-1} + v_p(f(k)!) + f(k) > v_p(\lfloor \frac{k}{p-1} \rfloor!) - 1,$$

which goes to  $+\infty$  as  $k \rightarrow +\infty$ . Similar claim holds with  $a_k$  replaced by  $b_k$  above. Hence, we conclude that  $t/\mu$  and  $\mu/t$  converge in  $\Lambda_F$  as inverse to each other. Moreover, we have  $[p]_q = \varphi(\mu)/\mu = p\varphi(\mu/t)t/\mu$  in  $\Lambda_F$ , so  $[p]_q/p$  is a unit in  $\Lambda_F$ . Furthermore, using Lemma 3.3, we get that  $\tilde{p}/p$  is a unit in  $\Lambda_F$ .  $\blacksquare$

**3.3.2. Explicit description of  $A_R(1)/p_1(\mu)$ .** In the subsection, the  $A_R$ -algebra structure on  $A_R(1)/p_1(\mu)$  is given by the composition  $A_R \xrightarrow{p_2} A_R(1) \rightarrow A_R(1)/p_1(\mu)$  and the  $\Gamma_R$ -action on  $A_R(1)/p_1(\mu)$  means the continuous (for the  $p$ -adic topology) action of  $1 \times \Gamma_R$  on  $A_R(1)/p_1(\mu)$ . Let us consider the divided power  $A_R$ -algebra  $A_R[\prod_{k \in \mathbb{N}^{d+1}} (\mu^{p-1}/p)^{[k_0]} T_1^{[k_1]} \dots T_d^{[k_d]}]$ , where  $T_1, \dots, T_d$  are variables. We equip this ring with a continuous (for the  $p$ -adic topology) action of  $\Gamma_R$  by extending the  $\Gamma_R$ -action on  $A_R$  by setting  $\gamma_j(T_i) = \mu[X_i^b] + T_i$ , for  $j = i$  or  $T_i$  otherwise. Moreover, we have  $\varphi(\mu^{p-1}) = [p]_q^{p-1} \mu^{p-1}$ , so we can further equip this ring with a Frobenius endomorphism  $\varphi$ , extending the Frobenius on  $A_R$  by setting  $\varphi([X_i^b] - T_i) = ([X_i^b] - T_i)^p$ . Then we have the following description of  $A_R(1)/p_1(\mu)$ :

**Proposition 3.25.** *There exists a natural  $(\varphi, \Gamma_R)$ -equivariant isomorphism of  $p$ -adically complete and  $p$ -torsion free  $A_R$ -algebras*

$$\alpha : A_R(1)/p_1(\mu) \xrightarrow{\sim} A_R[\prod_{\mathbf{k} \in \mathbb{N}^{d+1}} (\mu^{p-1}/p)^{[k_0]} T_1^{[k_1]} \cdots T_d^{[k_d]}]_p^\wedge, \quad (3.15)$$

where  $p_2(\mu) \mapsto \mu$ ,  $p_2([X_i^b]) \mapsto [X_i^b]$  and  $p_1([X_i^b]) \mapsto [X_i^b] - T_i$ .

*Proof.* The map in (3.15) is evidently  $(\varphi, \Gamma_R)$ -equivariant and we need to show that it is bijective. To show the bijectivity of (3.15), we will use the results of Appendix B. Let  $A := A_F \widehat{\otimes}_{O_F} A_F$  denote the  $(p, p_1(\mu))$ -adic completion of  $A_F \otimes_{O_F} A_F$  equipped with the tensor product Frobenius, in particular,  $A$  is a  $p$ -torsion free  $\delta$ -ring. Moreover, note that  $\varphi^n(p_1(\mu))$  is a nonzerodivisor on  $A$ , for each  $n \in \mathbb{N}$ , and  $A/p_1(\mu)A \xrightarrow{\sim} A_F$  is  $p$ -torsion free. Therefore, by setting  $q$  to be  $p_1(\mu) + 1$ , we see that the ring  $A$  satisfies Assumption B.1. Next, let  $B := A_R \widehat{\otimes}_{O_F} A_R$  denote the  $(p, p_1(\mu))$ -adic completion of  $A_R \otimes_{O_F} A_R$  equipped with the tensor product Frobenius, in particular,  $B$  is a  $p$ -torsion free  $\delta$ -algebra over  $A$ . Note that  $\varphi^n(p_1(\mu))$  is a nonzerodivisor on  $B$ , for each  $n \in \mathbb{N}$ , and  $B/p_1(\mu)B \xrightarrow{\sim} R \widehat{\otimes}_{O_F} A_R$ , as the  $p$ -adic completion of  $R \otimes_{O_F} A_R$ , is  $p$ -torsion free. Set  $Y_0 := p_2([p]_q) - p_1([p]_q)$  and  $Y_i := p_2([X_i^b]^p) - p_1([X_i^b]^p)$ , for each  $1 \leq i \leq d$ , as elements in  $B$ . Then, it is clear that the sequence  $\{Y_1, \dots, Y_d\}$  is regular on  $B$  and  $B/pB$ . Moreover, from the discussion in Construction 3.13, recall that the sequence  $\{p, p_1([p]_q), p_2([p]_q), Y_1, \dots, Y_d\}$  is regular on  $B$  and since  $[p]_q = \mu^{p-1} \pmod{p}$ , therefore, it follows that the sequence  $\{p, p_1(\mu), p_2([p]_q), Y_1, \dots, Y_d\}$  is also regular on  $B$ . In particular, we see that  $\{Y_0, \dots, Y_d\}$  is regular on  $B/p_1([p]_q)B$  and on  $B/(p, p_1(\mu))B$ . Therefore, we obtain that the  $A$ -algebra  $B$  satisfies Assumption B.4.

Now, let  $I$  denote the set of natural numbers  $\{0, \dots, d\}$  and for each  $i$  in  $I$ , let  $y_i := Y_i/p_1([p]_q)$  be an element of  $B[1/p_1([p]_q)]$ . Set  $E_0$  to be the  $B$ -subalgebra of  $B[1/p_1([p]_q)]$  generated by  $\{y_i, i \in I\}$  (see the discussion before Lemma B.6) and  $E$  to be the  $\delta$ -subalgebra of  $E_0[1/p, 1/\varphi^n(p_1([p]_q)), n \in \mathbb{N}]$  generated over  $B$  by  $\{y_i, i \in I\}$  (see the discussion before (B.4)). From the discussion before (3.6), note that the ring  $A_R(1)$  is given as the  $(p, p_1([p]_q)) = (p, p_1(\mu))$ -adic completion of  $E$ , therefore, the ring  $A_R(1)/p_1(\mu)$  is given as the  $p$ -adic completion of  $E/p_1(\mu)E$ . Next, let  $\bar{A} := A/\mu A$  and  $\bar{B} := B/\mu B$  equipped with the induced Frobenius, in particular, an associated  $\delta$ -structure, since  $\bar{A}$  and  $\bar{B}$  are  $p$ -torsion free (see Remark 2.2). Let  $\bar{E}_0 := E_0/\mu E_0$  and for each  $i$  in  $I$ , denote by  $\bar{Y}_i$  (resp.  $\bar{y}_i$ ), the image of  $Y_i$  (resp.  $y_i$ ) in  $\bar{E}_0$ . Define  $\bar{E}$  to be the  $\delta$ -subalgebra of  $\bar{E}_0[1/p]$  generated by  $\{\bar{y}_i, i \in I\}$  over  $\bar{B}$  (see the discussion before (B.4)). Then, from (B.4), we have a natural surjective map  $E \rightarrow \bar{E}$  inducing an isomorphism of  $p$ -torsion free  $\delta$ -rings  $E/\mu E \xrightarrow{\sim} \bar{E}$ , by Proposition B.10.

Now, for each  $1 \leq i \leq d$ , set  $\tau_i := p_2([X_i^b]) - p_1([X_i^b])$  as an element of  $\bar{E}_0$ . Moreover, note that  $p_2(\mu)^{p-1} = \bar{Y}_0 + pa$ , for some  $a$  in  $p_2(\mu)\bar{E}_0$ , so we set  $\tau_0 := p_2(\mu)^{p-1}/p = \bar{y}_0 + a$  as an element of  $\bar{E}_0$ . Let us set  $\bar{E}' := \bar{E}_0[\tau_i^{[n]}, i \in I, n \in \mathbb{N}]$  as a divided power  $\bar{E}_0$ -subalgebra of  $\bar{E}_0[1/p]$ . We claim that  $\bar{E} = \bar{E}'$  in  $\bar{E}_0[1/p]$ . Indeed, to show the inclusion  $\bar{E}' \subset \bar{E}$ , note that for each  $1 \leq i \leq d$ , we have  $\tau_i^p = \bar{Y}_i - p\delta(\tau_i)$ , where  $\delta(\tau_i)$  is in  $\bar{E}_0$  by Lemma 3.26, in particular,  $\tau_i^p/p = \bar{y}_i - \delta(\tau_i)$  is in  $\bar{E}_0 \subset \bar{E}$ . Moreover, using the equalities above we have that  $\tau_0^p = \bar{y}_0^p + pb$ , for some  $b$  in  $\bar{E}_0$  since  $p_2(\mu)^p$  is in  $p\bar{E}_0$ , and by using Lemma 3.26 we have that  $\bar{y}_0^p = \varphi(\bar{y}_0) - p\delta(\bar{y}_0)$  is in  $p\bar{E}$ . Therefore, we get that  $\tau_0^p/p = b + \bar{y}_0^p/p$  is in  $\bar{E}$ . Hence, by using [BS22, Lemma 2.35], it follows that  $\bar{E}' \subset \bar{E}$ . Next, to show the inclusion  $\bar{E} \subset \bar{E}'$ , since we have that  $\bar{y}_i$  is in  $\bar{E}'$ , for each  $i \in I$ , therefore, it is enough to show that  $\varphi$  on  $\bar{E}[1/p]$  preserves  $\bar{E}'$  and the resulting endomorphism of  $\bar{E}'$  gives a  $\delta$ -structure, or equivalently, that  $\varphi$  restricts to a lift of Frobenius on  $\bar{E}'$ . Note that  $\varphi(\tau_0^{[n]}) = p_2([p]_q)^n \tau_0^{[n]}$  is in  $p\bar{E}'$ , and for each  $i \in I$ , we can write

$$\begin{aligned} \varphi(\tau_i^{[n]}) &= \varphi\left(\frac{\tau_i^{[n]}}{n!}\right) = \frac{(\tau_i^{[n]} + p\delta(\tau_i))^{[n]}}{n!} = \sum_{k=0}^n \frac{1}{n!} \binom{n}{k} \tau_i^{[k]} p^{n-k} \delta(\tau_i)^{n-k} \\ &= \sum_{k=0}^n \frac{(pk)! p^{n-k}}{n!} \binom{n}{k} \tau_i^{[pk]} \delta(\tau_i)^{n-k}. \end{aligned}$$

An easy computation shows that all coefficients in the last term of the equation above lie in  $p\mathbb{Z}_p$ . Therefore, it follows that  $\varphi$  preserves  $\bar{E}'$  and  $\varphi(\tau_i^{[n]}) = 0 \pmod{p\bar{E}'}$ , for each  $i \in I$  and  $n \in \mathbb{N}_{\geq 1}$ . Moreover, note that  $(\tau_i^{[n]})^p$  is also in  $p\bar{E}'$ , for each  $n \in \mathbb{N}_{\geq 1}$ . So, it follows that the endomorphism

$\varphi$  on  $\overline{E}'$  reduces modulo  $p$  to the absolute Frobenius on all generators, hence on all elements, of  $\overline{E}'$ . Therefore, we obtain that  $\overline{E}' = \overline{E} \subset \overline{E}_0[1/p]$ .

Finally, let us consider  $\overline{E}$  as an  $A_R$ -algebra via the map  $p_2$  and define a ring homomorphism between  $A_R$ -algebras as follows:

$$\overline{E} = \overline{E}' \longrightarrow A_R[\prod_{\mathbf{k} \in \mathbb{N}^{d+1}} (\mu^{p-1}/p)^{[k_0]} T_1^{[k_1]} \cdots T_d^{[k_d]}],$$

by sending  $p_2(\mu) \mapsto \mu$ ,  $\tau_0^n/n! \mapsto (\mu^{p-1}/p)^n/n!$ ,  $p_2([X_i^b]) \mapsto [X_i^b]$ ,  $p_1([X_i^b]) \mapsto [X_i^b] - T_i$  and  $\tau_i^n/n! \mapsto T_i^n/n!$ . Evidently, the map above is bijective and passing to its  $p$ -adic completion gives the isomorphism in (3.15). This finishes our proof.  $\blacksquare$

The following observation was used above:

**Lemma 3.26.** *For each  $i \in I$ , the element  $\varphi(\overline{y}_i)$  is in  $p\overline{E}_0$ , and for each  $1 \leq i \leq d$ , the element  $\delta(\tau_i)$  is in  $\overline{E}_0$ .*

*Proof.* For  $1 \leq i \leq d$ , we have that

$$\begin{aligned} \varphi(\overline{y}_i) &= \frac{\varphi(p_2([X_i^b])^p - p_1([X_i^b])^p)}{p} \\ &= \frac{(p_2([X_i^b])^p - p_1([X_i^b])^p + p_1([X_i^b])^p - p_1([X_i^b])^p)^2}{p} \\ &= \frac{(p_2([X_i^b])^p - p_1([X_i^b])^p)^2}{p} + \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} (p_2([X_i^b])^p - p_1([X_i^b])^p)^k p_1([X_i^b])^{p-k} \\ &= p^{p-1} \overline{y}_i^p + \sum_{k=1}^{p-1} p^{k-1} \binom{p}{k} \overline{y}_i^k p_1([X_i^b])^{p-k}, \end{aligned}$$

is an element of  $p\overline{E}_0$ . Moreover, for  $i = 0$ , we have that

$$\begin{aligned} \varphi(\overline{y}_0) &= \frac{p_2(q)^{p^2-1}}{p(p_2(q)^{p-1})} - 1 \\ &= \frac{(p_2(q)^{p-1+1})^{p-1}}{p(p_2(q)^{p-1})} - 1 \\ &= \frac{(p_2(q)^{p-1})^{p-1}}{p} + \sum_{k=2}^{p-1} \frac{1}{p} \binom{p}{k} (p_2(q)^p - 1)^{k-1} \\ &= p^{p-2} (p_2(q) - 1)^{p-1} (\overline{y}_0 + 1)^{p-1} + \sum_{k=2}^{p-1} p^{k-1} \binom{p}{k} (p_2(q) - 1)^{k-1} (\overline{y}_0 + 1)^{k-1}. \end{aligned}$$

For  $p \geq 3$ , it is clear that  $\varphi(\overline{y}_0)$  is in  $p\overline{E}_0$ , whereas for  $p = 2$ , note that  $p_2(q) - 1 = p_2(q) + 1 - 2 = p_2([p]_q) - 2 = 2\overline{y}_0$ , i.e.  $\varphi(\overline{y}_0)$  is in  $p\overline{E}_0$ . This shows the first claim. Next, for  $1 \leq i \leq d$ , note that

$$\begin{aligned} \delta(\tau_i) &= \delta(p_2([X_i^b]) - p_1([X_i^b])) \\ &= \frac{p_2([X_i^b])^p - p_1([X_i^b])^p - (p_2([X_i^b]) - p_1([X_i^b]))^p}{p} \\ &= \frac{-p_1([X_i^b])^p - (-p_1([X_i^b]))^p}{p} - \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} p_2([X_i^b])^k (-p_1([X_i^b]))^{p-k}, \end{aligned}$$

is evidently in  $\overline{E}_0$ . This proves the second claim.  $\blacksquare$

**Remark 3.27.** In this remark we will use  $\widehat{\otimes}$  to denote the  $p$ -adic completion of the usual tensor product. Let  $\Lambda_F = O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$  and note that it is a  $p$ -torsion free algebra over  $A_F = O_F[[\mu]]$  (see Proposition 3.21 for  $R = O_F$ ). Note that the target ring in the isomorphism (3.15) of Proposition 3.25 is  $p$ -torsion free and by definition the ring  $A_R[\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}]$  is  $p$ -torsion free as well. Therefore, by checking modulo  $p$ , it is easy to see that we have a natural  $(\varphi, \Gamma_R)$ -equivariant isomorphism of rings,

$$A_R[\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}] \widehat{\otimes}_{A_F} \Lambda_F \xrightarrow{\sim} A_R[\prod_{\mathbf{k} \in \mathbb{N}^{d+1}} (\mu^{p-1}/p)^{[k_0]} T_1^{[k_1]} \cdots T_d^{[k_d]}]_p^\wedge, \quad (3.16)$$

where we take the diagonal action of  $\varphi$  and  $\Gamma_R$  on the source. In particular, using (3.15) and (3.16) it follows that  $A_R(1)/p_1(\mu)$  is the  $p$ -adic completion of a PD-polynomial algebra over  $A_R \widehat{\otimes}_{A_F} \Lambda_F$ .

Next, by an argument similar to (3.16), we note that the isomorphism in (3.13) readily extends to a  $(\varphi, \Gamma_F)$ -equivariant isomorphism

$$\iota : R[[\mu]] [\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}] \widehat{\otimes}_{\mathcal{O}_F[[\mu]]} \Lambda_F \xrightarrow{\sim} A_R [\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}] \widehat{\otimes}_{A_F} \Lambda_F.$$

Furthermore, again arguing similar to (3.16), we note that the isomorphism in (3.14) extends to a  $(\varphi, \Gamma_F)$ -equivariant isomorphism

$$\beta : R [\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}] \widehat{\otimes}_{\mathcal{O}_F} \Lambda_F \xrightarrow{\sim} R[[\mu]] [\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}] \widehat{\otimes}_{\mathcal{O}_F[[\mu]]} \Lambda_F.$$

Composing these with the isomorphism in Proposition 3.25, we obtain a  $(\varphi, \Gamma_F)$ -equivariant isomorphism

$$\beta^{-1} \circ \iota^{-1} \circ \alpha : A_R(1)/p_1(\mu) \xrightarrow{\sim} R [\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}] \widehat{\otimes}_{\mathcal{O}_F} \Lambda_F =: C_R. \quad (3.17)$$

Let  $P_R := R [\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}]_p^\wedge$  and note that the target ring of the isomorphism (3.17) can also be written as  $C_R = P_R \widehat{\otimes}_R \Lambda_R = P_R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = \Lambda_R [\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}]_p^\wedge$ . In particular,  $C_R$  is the  $p$ -adic completion of a PD-polynomial algebra over  $\Lambda_R$ . Now for  $k \in \mathbb{N}$ , let  $k = (p-1)f(k) + r(k)$ , with  $r(k), f(k) \in \mathbb{N}$  and  $0 \leq r(k) < p-1$ . Set  $\mu^{\{k\}} = \frac{\mu^k}{f(k)! p^{f(k)}}$ , then we have  $C_R = P_R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = P_R[\mu, \mu^{\{k\}}, k \in \mathbb{N}]_p^\wedge$ . In particular, from the isomorphism (3.17), it follows that for any  $x$  in  $A_R(1)/p_1(\mu)$  we have a unique presentation  $(\beta^{-1} \circ \iota^{-1} \circ \alpha)(x) = \sum_{k \in \mathbb{N}} a_k \mu^{\{k\}}$  in  $C_R$ , with  $a_k$  in  $P_R$  for all  $k \in \mathbb{N}$  and since  $A_R(1)/p_1(\mu) \xrightarrow{\sim} C_R$  are  $p$ -adically separated, we have that  $a_k \rightarrow 0$  as  $k \rightarrow +\infty$ .

**Remark 3.28.** The target ring of the isomorphism  $\alpha$  in Proposition 3.25 admits a natural  $\Gamma_R$ -equivariant embedding into the crystalline period ring  $\mathcal{O}_{A_{\text{cris}}}(R_\infty)$ . Indeed, first of all note that the embedding  $A_R \rightarrow A_{\text{inf}}(R_\infty)$  from Subsection 3.1 extends to a natural  $(\varphi, \Gamma_R)$ -equivariant embedding  $A_R \widehat{\otimes}_{A_F} \Lambda_F \rightarrow A_{\text{inf}}(R_\infty) \widehat{\otimes}_{A_F} \Lambda_F \xrightarrow{\sim} A_{\text{cris}}(R_\infty)$ , where the latter isomorphism follows in a manner similar to the proof of [Bri08, Proposition 6.2.14]. Then, from the isomorphism (3.15) in Proposition 3.25 and from the description of  $\mathcal{O}_{A_{\text{cris}}}(R_\infty)$  as the  $p$ -adic completion of a PD-polynomial algebra over  $A_{\text{cris}}(R_\infty)$  (see [Bri08, Chapitre 6] and [Abh21, Subsection 2.2]), i.e.  $\mathcal{O}_{A_{\text{cris}}}(R_\infty) \xrightarrow{\sim} A_{\text{cris}}(R_\infty) [\prod_{\mathbf{k} \in \mathbb{N}^d} T_1^{[k_1]} \cdots T_d^{[k_d]}]_p^\wedge$  it follows that we have a  $(\varphi, \Gamma_R)$ -equivariant injective map  $A_R(1)/p_1(\mu) \rightarrow \mathcal{O}_{A_{\text{cris}}}(R_\infty)$ .

**Remark 3.29.** Recall that there are two  $A_R$ -algebra structures on  $A_R(1)$  written as  $p_i : A_R \rightarrow A_R(1)$  for  $i = 1, 2$  (see Construction 3.13). Composing it with the injective map  $\iota : R \rightarrow A_R$  (see Subsection 3.1.1), gives morphism of rings  $R \xrightarrow{\iota} A_R \xrightarrow{p_i} A_R(1)$ . Reducing  $p_1$  modulo  $\mu$  induces an  $R$ -algebra structure on  $A_R(1)/p_1(\mu)$  via the composition  $R \xrightarrow{\sim} A_R/\mu \xrightarrow{p_1} A_R(1)/p_1(\mu)$ , which we again denote by  $p_1$  and where the first isomorphism is induced by the map  $\iota$ . In particular, we have  $(p_1 \circ \alpha)(X_i) = [X_i^b] - T_i$ , for all  $1 \leq i \leq d$ . Moreover, we will denote the composition  $R \xrightarrow{\iota} A_R \xrightarrow{p_2} A_R(1)$  just by  $p_2$ . Note that both  $p_1$  and  $p_2$  described above are  $\varphi$ -equivariant.

**Lemma 3.30.** *The morphisms  $p_i : R \rightarrow A_R(1)/p_1(\mu)$  for  $i = 1, 2$  are faithfully flat.*

*Proof.* From Lemma 3.15 the map  $p_1 : A_R \rightarrow A_R(1)$  is faithfully flat, therefore the map  $p_1 : R \xrightarrow{\sim} A_R/\mu \rightarrow A_R(1)/p_1(\mu)$  is faithfully flat. Now recall that we have  $A_\square^\pm \xrightarrow{\sim} R^\square[[\mu]]$  (see Subsection 3.1.1) and consider the following isomorphism of rings

$$f : A_\square^\pm [\prod_{\mathbf{k} \in \mathbb{N}^{d+1}} (\mu^{p-1}/p)^{[k_0]} T_1^{[k_1]} \cdots T_d^{[k_d]}]_p^\wedge \xrightarrow{\sim} A_\square^\pm [\prod_{\mathbf{k} \in \mathbb{N}^{d+1}} (\mu^{p-1}/p)^{[k_0]} T_1^{[k_1]} \cdots T_d^{[k_d]}]_p^\wedge,$$

where  $[X_i]^b \mapsto [X_i^b] + T_i$ ,  $T_i \mapsto T_i$  and  $\mu \mapsto \mu$ . Extending this isomorphism along the  $p$ -adically completed étale map  $R^\square \rightarrow R$  (see Subsection 3.1.1) gives an automorphism  $f$  of the  $p$ -adically complete PD-algebra  $A_R [\prod_{\mathbf{k} \in \mathbb{N}^{d+1}} (\mu^{p-1}/p)^{[k_0]} T_1^{[k_1]} \cdots T_d^{[k_d]}]_p^\wedge$  and we consider the commutative diagram:

$$\begin{array}{ccc} R & \xrightarrow{p_1} & A_R(1)/p_1(\mu) \\ & \searrow p_2 & \downarrow \wr \alpha^{-1} \circ f \circ \alpha \\ & & A_R(1)/p_1(\mu). \end{array} \quad (3.18)$$

Since  $p_1$  is faithfully flat, from the diagram (3.18), it follows that  $p_2$  is faithfully flat as well.  $\blacksquare$

**3.3.3.  $q$ -connection on  $A_R(1)/p_1(\mu)$ .** In this subsection, we will interpret the action of the geometric part of  $\Gamma_R$ , i.e.  $\Gamma'_R$  as a  $q$ -connection on  $A_R(1)/p_1(\mu)$ . We start by recalling the definition of  $q$ -connection from [MT20, Definition 2.1] and [Abh23b, Subsection 5.1].

Let  $D$  be commutative ring and consider a  $D$ -algebra  $A$  equipped with  $d$  commuting  $D$ -algebra automorphisms  $\gamma_1, \dots, \gamma_d$ , i.e. an action of  $\mathbb{Z}^d$ . Moreover, fix an element  $q$  in  $D$  such that  $q - 1$  is a nonzerodivisor of  $D$  and  $\gamma_i = 1 \pmod{(q - 1)A}$ , for all  $1 \leq i \leq d$ .

**Definition 3.31.** Let  $q\Omega_{A/D}^\bullet := \bigoplus_{k=0}^d q\Omega_{A/D}^k$  be a differential graded  $D$ -algebra defined as:

- $q\Omega_{A/D}^0 = A$  and  $q\Omega_{A/D}^1$  is a free left  $A$ -module on formal basis elements  $d\log(U_i)$ .
- The right  $A$ -module structure on  $q\Omega_{A/D}^1$  is twisted by the rule  $d\log(U_i) \cdot f = \gamma_i(f) d\log(U_i)$ .
- $d\log(U_i) d\log(U_j) = -d\log(U_j) d\log(U_i)$  if  $i \neq j$  and 0 if  $i = j$ .
- The following map, where  $I_k = \{\mathbf{i} = (i_1, \dots, i_k) \in \mathbb{N}^k \text{ such that } 1 \leq i_1 < \dots < i_k \leq d\}$ , is an isomorphism of  $A$ -modules

$$\bigoplus_{\mathbf{i} \in I_k} A \xrightarrow{\sim} q\Omega_{A/D}^k$$

$$(f_{\mathbf{i}}) \mapsto \sum_{\mathbf{i} \in I_k} f_{\mathbf{i}} d\log(U_{i_1}) \cdots d\log(U_{i_k}).$$

- The 0<sup>th</sup> differential  $d_q : A \rightarrow q\Omega_{A/D}^1$  is given as  $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f) - f}{q-1} d\log(U_i)$ .
- The elements  $d\log(U_i) \in q\Omega_{A/D}^1$  are cocycles for all  $1 \leq i \leq d$ .

The data  $d_q : A \rightarrow q\Omega_{A/D}^1$  forms a differential ring over  $D$ , i.e.  $q\Omega_{A/D}^1$  is an  $D$ -bimodule and  $d_q$  is  $D$ -linear satisfying the Leibniz rule  $d_q(fg) = d_q(f)g + fd_q(g)$ .

**Remark 3.32.** In Definition 3.31, we will denote the operator  $\frac{\gamma_i - 1}{q-1} : A \rightarrow A$  by  $\nabla_{q,i}^{\log}$  and refer to these as the  $q$ -differential operators in logarithmic coordinates. The operators  $\nabla_{q,i} := U_i \nabla_{q,i}^{\log}$ , will be referred to as the  $q$ -differential operators in non-logarithmic coordinates.

**Remark 3.33.** In most of the cases, we will fix units  $U_1, \dots, U_d \in A^\times$  such that  $\gamma_i(U_j) = qU_j$  if  $i = j$  or  $U_j$  if  $i \neq j$ .

**Example 3.34.** Take  $D$  to be  $A_F = O_F[[\mu]]$ , and  $A$  to be  $A_R$  equipped with the  $A_F$ -linear action of  $\Gamma_R$  and let  $\{\gamma_1, \dots, \gamma_d\}$  be the topological generators of  $\Gamma'_R$  (see Subsection 3.1). Then setting  $q = 1 + \mu$  and  $U_i = [X_i^b]$ , for  $1 \leq i \leq d$ , we have  $\gamma_i = 1 \pmod{\mu A_R}$  for all  $1 \leq i \leq d$ . In particular,  $A_R$  satisfies the hypotheses of Definition 3.31. Moreover, in this case,  $q\Omega_{A_R/A_F}^1$  identifies with  $\Omega_{A_R/A_F}^1$ , i.e. the  $(p, \mu)$ -adic completion of the module of Kähler differentials of  $A_R$  with respect to  $A_F$ . Furthermore, note that we have an isomorphism of rings  $A_R/\mu \xrightarrow{\sim} R$ , so from [MT20, Remarks 2.4 and 2.10], reduction modulo  $q - 1$  of the differential ring  $d_q : A_R \rightarrow \Omega_{A_R/A_F}^1$ , is the usual continuous de Rham differential  $d : R \rightarrow \Omega_R^1$ .

**Example 3.35.** Take  $D$  to be  $\Lambda_F = O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}_p^\wedge]$  equipped with an action of  $(\varphi, \Gamma_F)$  as in Remark 3.23 and take  $A$  to be  $A^{\text{PD}} := A_R[(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}_p^\wedge] = A_R \widehat{\otimes}_{A_F} \Lambda_F$ , equipped with the natural and continuous tensor product  $(\varphi, \Gamma_R)$ -action. From Remark 3.23, note that the structure map  $\Lambda_F \rightarrow A^{\text{PD}}$  is  $(\varphi, \Gamma_F)$ -equivariant and from (3.13) note that we also have a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings  $\iota : \Lambda_R \xrightarrow{\sim} A^{\text{PD}}$ . Moreover, the  $\Lambda_F$ -algebra  $A^{\text{PD}}$  is equipped with a  $\Lambda_F$ -linear continuous (for the  $p$ -adic topology) action of  $\Gamma'_R \subset \Gamma_R$  and we have  $\{\gamma_1, \dots, \gamma_d\}$  as topological generators of  $\Gamma'_R$  (see Subsection 3.1). Then setting  $q = 1 + \mu$  and  $U_i = [X_i^b]$ , for  $1 \leq i \leq d$ , it follows that  $\gamma_i = 1 \pmod{\mu A^{\text{PD}}}$ , for all  $1 \leq i \leq d$ . In particular,  $A^{\text{PD}}$  satisfies the hypotheses of Definition 3.31. Furthermore, in this case,  $q\Omega_{A^{\text{PD}}/\Lambda_F}^1$  identifies with  $\Omega_{A^{\text{PD}}/\Lambda_F}^1$ , i.e. the  $(p, \mu)$ -adic =  $(p, [p]_q)$ -adic =  $p$ -adic (see Lemma 3.24) completion of the module of Kähler differentials of  $A^{\text{PD}}$  with



respect to  $\Lambda_F$ . From Definition 3.31 the  $q$ -connection on  $A^{\text{PD}}$ , denoted  $\nabla_q : A^{\text{PD}} \rightarrow q\Omega_{A^{\text{PD}}/\Lambda_F}^1$ , is given as  $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)} d\log([X_i^{\flat}])$ . Moreover, the  $q$ -connection  $\nabla_q$  on  $A^{\text{PD}}$  is  $p$ -adically quasi-nilpotent because we have  $\frac{\gamma_i-1}{p_2(\mu)[X_i^{\flat}]}([X_i^{\flat}]) = 1$ , and it is flat because  $\gamma_i$  commute with each other. Furthermore, using the  $q$ -connection,  $A^{\text{PD}}$  can be equipped with a quasi-nilpotent flat connection  $\nabla$  as in Proposition 3.37, which coincides with the universal  $\Lambda_F$ -linear continuous de Rham differential  $d : A^{\text{PD}} \rightarrow \Omega_{A^{\text{PD}}/\Lambda_F}^1$ . Then using Proposition 3.37, it follows that we have  $(A^{\text{PD}})^{\Gamma_R} = (A^{\text{PD}})^{\nabla_q=0} \xrightarrow{\sim} (A^{\text{PD}})^{\nabla=0} = (A^{\text{PD}})^{d=0}$ , where the first equality follows since the action of  $\Gamma_R$  on  $A^{\text{PD}}$  is continuous. Moreover, recall that from Subsection 3.1, for any  $g$  in  $\Gamma_F$ , we have  $g\gamma_i g^{-1} = \gamma_i^{\chi(f)}$ , for all  $1 \leq i \leq d$ . Hence, from Remark 3.38, the isomorphism  $\Lambda_F \xrightarrow{\sim} (A^{\text{PD}})^{\nabla_q=0} = (A^{\text{PD}})^{\Gamma_R}$ , induced by the structure map, is  $(\varphi, \Gamma_F)$ -equivariant as well.

**Example 3.36.** Take  $D$  to be  $\Lambda_R = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^{\wedge}$  equipped with an action of  $(\varphi, \Gamma_F)$  as in Remark 3.23 and take  $A$  to be  $\bar{A}(1) := A_R(1)/p_1(\mu)$  as a  $\Lambda_R$ -algebra via the morphism of rings  $\iota_{\Lambda} : \Lambda_R \rightarrow \bar{A}(1)$  defined by extending the  $R$ -algebra structure  $p_1 : R \rightarrow \bar{A}(1)$  in Remark 3.29 via  $\mu \mapsto p_2(\mu)$  and  $(\mu^{p-1}/p)^{[k]} \mapsto (p_2(\mu)^{p-1}/p)^{[k]}$ . Using Lemma 3.30 and the explicit description of  $\bar{A}(1)$  in Proposition 3.25 and Remark 3.27, it follows that the map  $\iota_{\Lambda}$  is injective and  $(\varphi, \Gamma_F)$ -equivariant. Moreover, the  $\Lambda_R$ -algebra  $\bar{A}(1)$  is equipped with a  $\Lambda_R$ -linear (via  $\iota_{\Lambda}$ ) continuous (for the  $p$ -adic topology) action of  $\Gamma'_R \subset 1 \times \Gamma_R$  and we have  $\{\gamma_1, \dots, \gamma_d\}$  as topological generators of  $\Gamma'_R$  (see Subsection 3.1). Then, by setting  $q = 1 + \mu$  and  $U_i = [X_i^{\flat}]$ , for  $1 \leq i \leq d$ , from the explicit description of  $\bar{A}(1)$  in Proposition 3.25, we have that  $\gamma_i = 1 \bmod p_2(\mu)\bar{A}(1)$ , for all  $1 \leq i \leq d$ . In particular,  $\bar{A}(1)$  satisfies the hypotheses of Definition 3.31. Furthermore, in this case,  $q\Omega_{\bar{A}(1)/\Lambda_R}^1$  identifies with  $\Omega_{\bar{A}(1)/\Lambda_R}^1$ , i.e. the  $(p, \mu)$ -adic =  $(p, [p]_q)$ -adic =  $p$ -adic (see Lemma 3.24) completion of the module of Kähler differentials of  $\bar{A}(1)$  with respect to  $\Lambda_R$ . From Definition 3.31 the  $q$ -connection on  $\bar{A}(1)$ , denoted  $\nabla_q : \bar{A}(1) \rightarrow q\Omega_{\bar{A}(1)/\Lambda_R}^1$ , is given as  $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)} d\log([X_i^{\flat}])$ . Moreover, the  $q$ -connection  $\nabla_q$  on  $\bar{A}(1)$  is  $p$ -adically quasi-nilpotent because we have  $\frac{\gamma_i-1}{p_2(\mu)[X_i^{\flat}]}([X_i^{\flat}]) = 1$ , and it is flat because  $\gamma_i$  commute with each other. Furthermore, using the  $q$ -connection,  $\bar{A}(1)$  can be equipped with a  $p$ -adically quasi-nilpotent flat connection  $\nabla$  as in Proposition 3.37 which coincides with the universal  $\Lambda_R$ -linear continuous de Rham differential  $d : \bar{A}(1) \rightarrow \Omega_{\bar{A}(1)/\Lambda_R}^1$ . Then using Proposition 3.37, it follows that we have  $\bar{A}(1)^{\nabla_q=0} = \bar{A}(1)^{\Gamma'_R} \xrightarrow{\sim} \bar{A}(1)^{\nabla=0} = \bar{A}(1)^{d=0}$ , where the isomorphism follows since the action of  $\Gamma_R$  on  $\bar{A}(1)$  is continuous. Moreover, from Subsection 3.1, recall that for any  $g$  in  $\Gamma_F$ , we have  $g\gamma_i g^{-1} = \gamma_i^{\chi(f)}$ , for all  $1 \leq i \leq d$ . Hence, the isomorphism  $\Lambda_R \xrightarrow{\sim} \bar{A}(1)^{\Gamma'_R} = \bar{A}(1)^{\nabla_q=0} \xrightarrow{\sim} \bar{A}(1)^{\nabla=0}$ , induced by  $\iota_{\Lambda}$  in Remark 3.38, is moreover  $(\varphi, \Gamma_F)$ -equivariant. In particular, it follows that  $R \xrightarrow{\sim} \Lambda_R^{\Gamma_F} \xrightarrow{\sim} (\bar{A}(1)^{\nabla_q=0})^{\Gamma_F} \xrightarrow{\sim} \bar{A}(1)^{\Gamma_R}$ , induced by  $p_1 : R \rightarrow \bar{A}(1)$ .

From Lemma 3.24, we have that  $t = \log(1 + \mu)$  converges in  $\mu\Lambda_F \subset \mu\Lambda_R$  and  $t/\mu$  is a unit.

**Proposition 3.37.** *Let  $D = \Lambda_R$  (resp.  $\Lambda_F$ ) and  $A = A_R(1)/p_1(\mu)$  (resp.  $A^{\text{PD}}$ ). Then for  $1 \leq i \leq d$ , the series of operators  $\nabla_i^{\log} = \frac{\log \gamma_i}{t} = \frac{1}{t} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i-1)^k}{k+1}$  converge  $p$ -adically on  $A$ . This defines a  $D$ -linear  $p$ -adically quasi-nilpotent flat connection on  $A$ , denoted  $\nabla : A \rightarrow \Omega_{A/D}^1$  and given as  $f \mapsto \sum_{i=1}^d \nabla_i^{\log}(f) d\log([X_i^{\flat}])$ . The connection  $\nabla$  coincides with the universal  $D$ -linear continuous de Rham differential operator  $d : A \rightarrow \Omega_{A/D}^1$ . Moreover, the data of the connection  $\nabla$  on  $A$  is equivalent to the data of the  $q$ -connection  $\nabla_q$  described in Example 3.36 (resp. Example 3.35), i.e. one may be recovered from the other. Furthermore, the  $q$ -de Rham complex  $q\Omega_{A/D}^{\bullet}$  is naturally quasi-isomorphic to the de Rham complex  $\Omega_{A/D}^{\bullet}$ . In particular, we have  $A^{\nabla_q=0} \xrightarrow{\sim} A^{\nabla=0}$ .*

*Proof.* The first two claims, i.e. the convergence of the operators  $\nabla_i^{\log}$  on  $A$  and the fact that the operator  $\nabla$  defines a  $D$ -linear  $p$ -adically quasi-nilpotent flat connection on  $A$  and the claim of equivalence between the connection  $\nabla$  and quasi-isomorphism of complexes, will be shown in Proposition 4.13 for finite  $A$ -modules admitting a continuous action of  $\Gamma_R$  (trivial modulo  $\mu$ ), in particular,  $A$  itself.

Moreover, in the proof of Proposition 4.13, we will also show that the connection  $\nabla$  coincides with the universal  $D$ -linear continuous de Rham differential operator  $d : A \rightarrow \Omega_{A/D}^1$ . Note that the proof of Proposition 4.13 is independent of the subsequent claims proved in this section. This allows us to conclude.  $\blacksquare$

**Remark 3.38.** Let  $\overline{A}(1) = A_R(1)/p_1(\mu)$  as in Proposition 3.37. From the isomorphism (3.17) in Remark 3.27 recall that we have a ring  $C_R \xleftarrow{\sim} \overline{A}(1)$ . Moreover, from Remark 3.27, we have that  $C_R$  is the  $p$ -adic completion of a PD-polynomial algebra over  $\Lambda_R$  and the structure map  $\Lambda_R \rightarrow C_R$  coincides with the composition  $\Lambda_R \xrightarrow{\iota_\Lambda} \overline{A}(1) \xrightarrow{(3.17)} C_R$ . Now note that  $C_R$  is equipped with the universal  $\Lambda_R$ -linear continuous de Rham differential  $d : C_R \rightarrow \Omega_{C_R/\Lambda_R}^1$ , which can be given as  $f \mapsto \sum_{i=1}^d \partial_i^{\log}(f) d \log(X_i)$ , where  $\partial_i^{\log} = X_i \partial_i$  and  $\partial_i : C_R \rightarrow C_R$  is the unique  $\Lambda_R$ -linear continuous differential operator such that  $\partial_i(X_j) = 1$  if  $i = j$  and 0 otherwise. So, the isomorphism in (3.17) is  $\Lambda_R$ -linear, and it induces an isomorphism between complexes of  $\overline{A}(1)$ -modules  $\Omega_{\overline{A}(1)/\Lambda_R}^\bullet \xrightarrow{\sim} \Omega_{C_R/\Lambda_R}^\bullet = C_R \otimes_{P_R} \Omega_{P_R/R}^1$ , where the right hand side is an  $\overline{A}(1)$ -module via the isomorphism (3.17). Now, let  $f$  be any element of  $C_R$ , then from Remark 3.27 we have a unique presentation  $f = \sum_{k \in \mathbb{N}} a_k \mu^{\{k\}}$ , with  $a_k$  in  $P_R$  for all  $k \in \mathbb{N}$  and  $p$ -adically  $a_k \rightarrow 0$  as  $k \rightarrow +\infty$ . Then, it follows that  $f \in C_R^{d=0}$  if and only if  $a_k$  in  $R$ , i.e.  $f$  is in  $\Lambda_R$ . In particular, we get that  $\Lambda_R \xrightarrow{\sim} C_R^{d=0}$ , in particular,  $\Lambda_R \xrightarrow{\sim} \overline{A}(1)^{d=0}$  via  $\iota_\Lambda$ . Using the same argument as above by replacing  $\Lambda_R$  by  $\Lambda_F$ ,  $\overline{A}(1)$  by  $A^{\text{PD}}$ ,  $C_R$  by  $\Lambda_R$  and  $P_R$  by  $O_F$ , we obtain that  $\Lambda_F \xrightarrow{\sim} (A^{\text{PD}})^{d=0}$ .

**Remark 3.39.** Let us consider the ring  $A^{\text{PD}}$  from Example 3.35, equipped with a continuous action of  $(\varphi, \Gamma_R)$ . From (3.13) in Remark 3.23, note that we have a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings  $\iota : \Lambda_R \xrightarrow{\sim} A^{\text{PD}}$ . Moreover, we have a  $(\varphi, \Gamma_R)$ -equivariant embedding  $A^{\text{PD}} \rightarrow A_{\text{cris}}(R_\infty)$  from Remark 3.28. Furthermore, the injective structure map  $p_2 : A_R \rightarrow A_R(1)/p_1(\mu) = \overline{A}(1)$  from Remark 3.29 extends to an injective ring homomorphism  $p_2 : A^{\text{PD}} \rightarrow \overline{A}(1)$  and the target is the  $p$ -adic completion of a PD-polynomial algebra over the source (see Remark 3.27). Now, consider the universal  $A^{\text{PD}}$ -linear continuous de Rham differential  $d : \overline{A}(1) \rightarrow \Omega_{\overline{A}(1)/A^{\text{PD}}}^1$ , which can be given as  $f \mapsto \sum_{i=1}^d \partial_i^{\log}(f) dY_i$ , where  $Y_i = p_1(X_i)$  (see Remark 3.29),  $\partial_i^{\log} = Y_i \partial_i$  and  $\partial_i : \overline{A}(1) \rightarrow \overline{A}(1)$  is the unique  $A^{\text{PD}}$ -linear continuous differential operator such that  $\partial_i(Y_j) = 1$  if  $i = j$  and 0 otherwise. Then, it easily follows that the  $(\varphi, \Gamma_R)$ -equivariant injective map  $\overline{A}(1) \rightarrow \mathcal{O}A_{\text{cris}}(R_\infty)$  from Remark 3.28 is  $A^{\text{PD}}$ -linear and compatible with the aforementioned  $A^{\text{PD}}$ -linear differential operator on  $\overline{A}(1)$  and the unique  $A_{\text{cris}}(R_\infty)$ -linear connection on  $\mathcal{O}A_{\text{cris}}(R_\infty)$  from [Bri08, Section 6.2].

**Remark 3.40.** Let  $\overline{A}(1) = A_R(1)/p_1(\mu)$  and  $A^{\text{PD}}$  as in Example 3.35. Note that we have a natural injective  $(\varphi, \Gamma_R)$ -equivariant homomorphism of rings  $p_2 : A^{\text{PD}} \rightarrow \overline{A}(1)$  (see Remark 3.39). Then using the Leibniz rule for the connection on  $\overline{A}(1)$  (see the proof of Proposition 4.13) it follows that the respective connections on  $\overline{A}(1)$  and  $A^{\text{PD}}$  are compatible and we have  $\Omega_{\overline{A}(1)/\Lambda_R}^1 = \overline{A}(1) \otimes_{A^{\text{PD}}} \Omega_{A^{\text{PD}}/\Lambda_F}^1$ .

**3.4. Galois action on  $A_R(1)$ .** Note that we have the  $\varphi$ -equivariant multiplication map  $\Delta : A_R(1) \rightarrow A_R$ . Moreover, recall that in Remark 3.20, we described the action of  $\Gamma_R^2$  on  $A_R(1)$ . In this subsection, our goal is to prove the following claim:

**Proposition 3.41.** *The  $\varphi$ -equivariant homomorphism  $\Delta : A_R(1) \rightarrow A_R$  restricts to a  $\varphi$ -equivariant isomorphism  $A_R(1)^{1 \times \Gamma_R} \xrightarrow{\sim} A_R$ . Moreover, the preceding isomorphism is compatible with the action of  $\Gamma_R$  on each side.*

In order to prove Proposition 3.41, we will study the action of  $1 \times \Gamma_R$  on  $A_R(1)$  in three steps, namely, the geometric part, i.e.  $\Gamma'_R$  below, the torsion part, i.e.  $\mathbb{F}_p^\times$  below, and the (free) arithmetic part, i.e.  $\Gamma_0$  below. From Subsection 1.6, recall that  $\Gamma_R$  fits into the following exact sequence:

$$1 \longrightarrow \Gamma'_R \longrightarrow \Gamma_R \longrightarrow \Gamma_F \longrightarrow 1.$$

Furthermore, from (1.6), recall that  $\Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$ , via the  $p$ -adic cyclotomic character, and it fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\text{tor}} \longrightarrow 1,$$

where, for  $p \geq 3$ , we have  $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$  and for  $p = 2$ , we have  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$ . Moreover, for  $p \geq 3$ , we have that  $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$  and the projection map  $\Gamma_F \rightarrow \Gamma_{\text{tor}}$ , admits a section  $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$ , where the second map is given as  $a \mapsto [a]$ , the Teichmüller lift of  $a$ . Finally, for  $p = 2$ , we have  $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$ , as groups.

Let us note that the results for the action of the geometric part of  $\Gamma_R$  in Subsection 3.4.1, are applicable for all primes  $p$ . However, for  $p = 2$ , since  $\mathbb{F}_p^\times$  is the trivial group, in Subsections 3.4.2 and 3.4.3 we assume that  $p \geq 3$ . For  $p = 2$ , the arithmetic action of  $\Gamma_R$  will be handled in Subsection 3.4.4.

**Remark 3.42.** Note that it is possible to prove Proposition 3.41, by studying the action of  $1 \times \Gamma_R$  at once and using Galois cohomology arguments, instead of the “3-step” argument presented here. However, the methods used in the “3-step” proof are applicable to the case of Wach modules as well (see Subsection 5.2), whereas the Galois cohomology arguments do not seem to generalise.

**3.4.1. The action of  $\Gamma'_R$ .** In this subsection, our first goal is to show the following claim:

**Lemma 3.43.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant sequence is exact:*

$$0 \longrightarrow (A_R(1)/p_1(\mu))^{1 \times \Gamma'_R} \xrightarrow{p_1(\mu)^n} (A_R(1)/p_1(\mu)^{n+1})^{1 \times \Gamma'_R} \longrightarrow (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \longrightarrow 0. \quad (3.19)$$

Before proving the Lemma 3.43, we provide a more explicit description of the ring  $A_R(1)^{1 \times \Gamma'_R}$  equipped with the induced  $(\varphi, \Gamma_R \times \Gamma_F)$ -action (introduced in the proof of Lemma 3.43). We start with the following:

**Construction 3.44.** From Construction 3.13, let us consider the  $(\varphi, \Gamma_R^2)$ -equivariant surjective map in (3.5), for  $n = 1$ . Taking invariants of this map, under the action of  $1 \times \Gamma'_R$ , we obtain a surjective map

$$A_R \widehat{\otimes}_{O_F} A_F \rightarrow R[\zeta_p, X_1^{1/p}, \dots, X_d^{1/p}] \otimes_{O_F} O_F[\zeta_p]. \quad (3.20)$$

Using the description of the kernel of the map in (3.5), it follows that the kernel of the surjective map (3.20), is generated by the ideal  $J = (1 \otimes [p]_q, [p]_q \otimes 1) \subset A_R \widehat{\otimes}_{O_F} A_F$ . Clearly, the sequence  $\{p, 1 \otimes [p]_q, [p]_q \otimes 1\}$  is regular on  $A_R \widehat{\otimes}_{O_F} A_F$ , so from [BS22, Proposition 3.13], we see that  $(\tilde{\Lambda}_R, [p]_q \otimes 1)$  is the prismatic envelope of  $(A_R \widehat{\otimes}_{O_F} A_F, J)$  over the bounded prism  $(A_R, [p]_q)$ . More explicitly, similar to the description of  $A_R(1)$  before Lemma 3.15, let us consider a free  $\delta$ -algebra over  $A_R \widehat{\otimes}_{O_F} A_F$  in one variable given as  $(A_R \widehat{\otimes}_{O_F} A_F) \left\{ \frac{1 \otimes [p]_q}{[p]_q \otimes 1} \right\}_\delta$ . Then, from [BS22, Proposition 3.13], we have that

$$\tilde{\Lambda}_R = (A_R \widehat{\otimes}_{O_F} A_F) \left\{ \frac{1 \otimes [p]_q}{[p]_q \otimes 1} \right\}_{(p, [p]_q \otimes 1)}^\wedge,$$

i.e.  $\tilde{\Lambda}_R$  is the  $(p, [p]_q \otimes 1)$ -adic completion of the free  $\delta$ -algebra  $(A_R \widehat{\otimes}_{O_F} A_F) \left\{ \frac{J}{[p]_q \otimes 1} \right\}_\delta$ . Moreover, from Lemma 3.3, recall that  $\tilde{p} = \mu_0 + p$  is the product of  $[p]_q$  with a unit in  $A_F$ . Therefore, the ring  $\tilde{\Lambda}_R$  has another presentation given as

$$\tilde{\Lambda}_R = (A_R \widehat{\otimes}_{O_F} A_F) \left\{ \frac{1 \otimes \tilde{p}}{\tilde{p} \otimes 1} \right\}_{(p, \tilde{p} \otimes 1)}^\wedge. \quad (3.21)$$

Using notations similar to Lemma 3.15, we will denote the two projection maps as  $p_1 : A_R \rightarrow \tilde{\Lambda}_R$  and  $p_2 : A_F \rightarrow \tilde{\Lambda}_R$ . Then, similar to Lemma 3.15, it is easy to see that both the preceding maps are faithfully flat, in particular,  $\tilde{\Lambda}_R$  is  $p_1(\mu)$ -torsion free.

Next, note that we have a natural  $(\varphi, \Gamma_R)$ -equivariant injective map  $A_F \rightarrow A_R$ , where the action of  $\Gamma_R$  factors through  $\Gamma_F \subset \Gamma_R$ . So, by the universal property of prismatic envelopes the continuous action of  $\Gamma_R \times \Gamma_F$  on  $A_R \widehat{\otimes}_{O_F} A_F$  naturally extends to a continuous action on  $\tilde{\Lambda}_R$  and since the morphism in (3.20) is compatible with the morphism in (3.5), therefore, we obtain the following natural  $(\varphi, \Gamma_R \times \Gamma_R)$ -equivariant map

$$\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R \longrightarrow A_R(1), \quad (3.22)$$

where the action of  $\Gamma_R \times \Gamma_R$  on the source of  $\iota_{\tilde{\Lambda}}$  factors through  $\Gamma_R \times \Gamma_F \subset \Gamma_R \times \Gamma_R$ . ■

**Remark 3.45.** Using an argument similar to the proof of Proposition 3.25, it can be shown that we have a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings

$$\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge, \quad (3.23)$$

where  $p_2(\mu) \mapsto \mu$  and  $p_1([X_i]^b) \mapsto X_i$  for  $1 \leq i \leq d$  (see Proposition 3.21 for equivalent descriptions of  $\Lambda_R$ ).

**Remark 3.46.** In Construction 3.44, by switching the two components in (3.20) and carrying out essentially the same steps, we can define another ring  $E := (A_F \widehat{\otimes}_{O_F} A_R) \left\{ \frac{1 \otimes [p]_q}{[p]_q \otimes 1} \right\}_{(p, [p]_q \otimes 1)}^\wedge$ , as the prismatic envelope of  $(A_F \widehat{\otimes}_{O_F} A_R, J)$  over the bounded prism  $(A_F, [p]_q)$ , where  $J = (1 \otimes [p]_q, [p]_q \otimes 1) \subset A_F \widehat{\otimes}_{O_F} A_R$ , i.e.  $E$  is the  $(p, [p]_q \otimes 1)$ -adic completion of the free  $\delta$ -algebra  $(A_R \widehat{\otimes}_{O_F} A_F) \left\{ \frac{J}{[p]_q \otimes 1} \right\}_\delta$ . Now, if we use notations similar to Lemma 3.15 to denote the two projection maps as  $p_1 : A_F \rightarrow E$  and  $p_2 : A_R \rightarrow E$ . Then, similar to Lemma 3.15, it is easy to see that both the preceding maps are faithfully flat. Moreover, by the universal property of prismatic envelopes, we see that the continuous action of  $\Gamma_F \times \Gamma_R$  on  $A_F \widehat{\otimes}_{O_F} A_R$  naturally extends to a continuous action on  $E$ . Furthermore, using an argument similar to the proof of Proposition 3.25, it can be shown that we have a natural  $(\varphi, \Gamma_R)$ -equivariant isomorphism of rings  $E/p_1(\mu) \xrightarrow{\sim} A^{\text{PD}} = A_R[(\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ , where  $p_2(\mu) \mapsto \mu$  and  $p_2([X_i]^b) \mapsto [X_i^b]$  for  $1 \leq i \leq d$  (see Example 3.35 for the definition of  $A^{\text{PD}}$ ).

**Lemma 3.47.** For each  $n \geq 1$ , the natural map in (3.22) induces a  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism

$$\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R/p_1(\mu)^n \xrightarrow{\sim} (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma'_R}. \quad (3.24)$$

In particular, (3.22) induces a  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism  $\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R \xrightarrow{\sim} A_R(1)^{1 \times \Gamma'_R}$ .

*Proof.* As the action of  $\Gamma_R \times \Gamma_R$  on  $\tilde{\Lambda}_R$  factors through  $\Gamma_R \times \Gamma_F$ , we see that reducing (3.22) modulo  $p_1(\mu)^n$  and taking  $(1 \times \Gamma'_R)$ -invariants, we obtain the  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant map in (3.24). It remains to show that it is an isomorphism. For  $n = 1$ , we consider the following  $(\varphi, \Gamma_F)$ -equivariant diagram,

$$\begin{array}{ccc} \tilde{\Lambda}_R/p_1(\mu) & \xrightarrow{(3.24)} & (A_R(1)/p_1(\mu))^{1 \times \Gamma'_R} \\ \downarrow \wr (3.23) & \nearrow \iota_{\tilde{\Lambda}} \sim & \\ \Lambda_R & & \end{array}$$

where the isomorphism  $\iota_{\tilde{\Lambda}}$  is from Example 3.36. It is easy to see that the diagram above commutes. Therefore, the top horizontal map of the diagram is an isomorphism, i.e.  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} (A_R(1)/p_1(\mu))^{1 \times \Gamma'_R}$ . Now as both  $A_R(1)$  and  $\tilde{\Lambda}_R$  are  $p_1(\mu)$ -torsion free (see Lemma 3.15 and Construction 3.44), we consider the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\Lambda}_R/p_1(\mu) & \xrightarrow{p_1(\mu)^n} & \tilde{\Lambda}_R/p_1(\mu)^{n+1} & \longrightarrow & \tilde{\Lambda}_R/p_1(\mu)^n \longrightarrow 0 \\ & & \downarrow \wr (3.24) & & \downarrow (3.24) & & \downarrow (3.24) \\ 0 & \longrightarrow & (A_R(1)/p_1(\mu))^{1 \times \Gamma'_R} & \xrightarrow{p_1(\mu)^n} & (A_R(1)/p_1(\mu)^{n+1})^{1 \times \Gamma'_R} & \longrightarrow & (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \longrightarrow 0, \end{array}$$

where the second exact sequence is from (3.19). Using the diagram, an easy induction on  $n \geq 1$ , gives the natural  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism induced from the map (3.22) as  $\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R/p_1(\mu)^n \xrightarrow{\sim} (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma'_R}$ , i.e. the isomorphism in (3.24). Finally, since both  $\tilde{\Lambda}_R$  and  $A_R(1)$  are  $p_1(\mu)$ -adically complete, therefore, by taking the limit over  $n \geq 1$  and noting that limit commutes with right adjoint functors, in particular, with taking  $(1 \times \Gamma'_R)$ -invariants, we obtain the  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism  $\iota_{\tilde{\Lambda}} : \tilde{\Lambda}_R \xrightarrow{\sim} A_R(1)^{1 \times \Gamma'_R}$ .  $\blacksquare$

Let us note an important observation for the action of  $\Gamma_R \times \Gamma_F$  on  $\tilde{\Lambda}_R$ .

**Lemma 3.48.** *The action of  $1 \times \Gamma_F$  is trivial on  $\tilde{\Lambda}_R/p_2(\mu)$  and the action of  $\Gamma_R \times 1$  is trivial on  $\tilde{\Lambda}_R/p_1(\mu)$ .*

*Proof.* The proof of the claim is similar to that of Proposition 3.17. For the first part, let us note that the action of  $1 \times \Gamma_F$  is trivial modulo  $p_2(\mu)$  on  $A_R \widehat{\otimes}_{O_F} O_F[[\mu]]$ . Then, from the explicit description of  $\tilde{\Lambda}_R$  in Construction 3.44, it is enough to show that for any  $m \in \mathbb{N}$  and  $g$  in  $1 \times \Gamma_F$ , we have that

$$(g-1)\delta^m \left( \frac{1 \otimes [p]_q}{[p]_q \otimes 1} \right) \in p_2(\mu) \tilde{\Lambda}_R.$$

Now, using Lemma 3.9 (2), let us note that  $p_2(\mu) \tilde{\Lambda}_R$  is a  $\delta$ -stable ideal of  $\tilde{\Lambda}_R$ , in the sense of [BS22, Example 2.10]. Then, using Lemma 3.19 for  $A = \tilde{\Lambda}_R$  and  $\alpha = p_2(\mu)$ , we see that to prove our claim, it is enough to show that  $(g-1) \left( \frac{1 \otimes [p]_q}{[p]_q \otimes 1} \right)$  belongs to  $p_2(\mu) \tilde{\Lambda}_R$ , which follows from Lemma 3.18. In particular, we have shown that the action of  $1 \times \Gamma_F$  is trivial on  $\tilde{\Lambda}_R/p_2(\mu)$ . For the second claim, note that the action of  $\Gamma_R \times 1$  is trivial modulo  $p_1(\mu)$  on  $A_R \widehat{\otimes}_{O_F} O_F[[\mu]]$ . Now, from the explicit description of  $\tilde{\Lambda}_R$  in Construction 3.44, it is enough to show that for any  $m \in \mathbb{N}$  and  $g$  in  $\Gamma_R \times 1$ , we have that

$$(g-1)\delta^m \left( \frac{1 \otimes [p]_q}{[p]_q \otimes 1} \right) \in p_1(\mu) \tilde{\Lambda}_R.$$

Similar to the first part, using Lemma 3.9 (1), we note that  $p_1(\mu) \tilde{\Lambda}_R$  is a  $\delta$ -stable ideal of  $\tilde{\Lambda}_R$ . Then, using Lemma 3.19 for  $A = \tilde{\Lambda}_R$  and  $\alpha = p_1(\mu)$ , we see that to prove our claim, it is enough to show that  $(g-1) \left( \frac{1 \otimes [p]_q}{[p]_q \otimes 1} \right)$  belongs to  $p_1(\mu) \tilde{\Lambda}_R$ , which again follows from Lemma 3.18. Hence, the action of  $\Gamma_R \times 1$  is trivial on  $\tilde{\Lambda}_R/p_1(\mu)$ .  $\blacksquare$

**Remark 3.49.** From Lemma 3.48, note that the action of  $1 \times \Gamma_F$  is trivial on  $\tilde{\Lambda}_R/p_2(\mu)$  and the element  $p_1(\mu)$  is invariant under this action. Therefore, it follows that for any  $g$  in  $1 \times \Gamma_F$  and any  $x$  in  $\tilde{\Lambda}_R/p_1(\mu)^n$  we have that  $(g-1)x$  is an element of  $p_2(\mu) \tilde{\Lambda}_{R,0}/p_1(\mu)^n$ . In particular, for  $n = 1$ , using the isomorphism  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$  from (3.23), we get that for any  $g$  in  $\Gamma_F$  and any  $x$  in  $\Lambda_R$  the element  $(g-1)x$  belongs to  $\mu \Lambda_R$ .

We are now ready to prove the main claim:

*Proof of Lemma 3.43.* To lighten notations, let us set  $A(1) := A_R(1)$  and  $\bar{A}(1) := A_R(1)/p_1(\mu)$ . Instead of working with the action of  $1 \times \Gamma'_R$ , we will work with the  $q$ -connection arising from this action. More precisely, in the notation of Definition 3.31, take  $D$  to be  $\tilde{\Lambda}_R \xrightarrow{\sim} A(1)^{1 \times \Gamma'_R}$  (see Lemma 3.47), and  $A$  to be  $A(1)$  equipped with a  $\tilde{\Lambda}_R$ -linear action of  $1 \times \Gamma'_R$  and let  $\{\gamma_1, \dots, \gamma_d\}$  be the topological generators of  $\Gamma'_R$  (see Subsection 3.1). Then setting  $q = 1 + p_2(\mu)$  and  $U_i = p_2([X_i^b])$  for  $1 \leq i \leq d$ , we have that  $\gamma_i = 1 \pmod{p_2(\mu)A_R(1)}$  for all  $1 \leq i \leq d$  (see Proposition 3.17). In particular,  $A(1)$  satisfies the hypotheses of Definition 3.31. Moreover, in this case,  $q\Omega_{A(1)/\tilde{\Lambda}_R}^1$  identifies with  $\Omega_{A(1)/\tilde{\Lambda}_R}^1$ , and given as the  $(p, \mu)$ -adic completion of the module of Kähler differentials of  $A(1)$  with respect to  $\tilde{\Lambda}_R$ . From Definition 3.31, the  $q$ -connection on  $A(1)$ , denoted  $\nabla_q : A(1) \rightarrow q\Omega_{A(1)/\tilde{\Lambda}_R}^1$ , is given as  $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)} d \log(p_2([X_i^b]))$ . Moreover, the  $q$ -connection  $\nabla_q$  on  $A(1)$  is  $(p, p_1(\mu))$ -adically quasi-nilpotent because we have  $\frac{\gamma_i-1}{p_2(\mu)p_2([X_i^b])} (p_2([X_i^b])) = 1$ , and it is flat because  $\gamma_i$  commute with each other. Let us also note that  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R \xrightarrow{\sim} (A(1)/p_1(\mu))^{1 \times \Gamma'_R}$  (see (3.23), Example 3.36 and the proof of Lemma 3.47). Now consider the following exact sequence of  $q$ -de Rham complexes:

$$0 \longrightarrow q\Omega_{\bar{A}(1)/\Lambda_R}^\bullet \xrightarrow{p_1(\mu)^n} A(1)/p_1(\mu)^{n+1} \otimes_{A(1)} q\Omega_{A(1)/\tilde{\Lambda}_R}^\bullet \longrightarrow A(1)/p_1(\mu)^n \otimes_{A(1)} q\Omega_{A(1)/\tilde{\Lambda}_R}^\bullet \longrightarrow 0.$$

Since the action of  $1 \times \Gamma'_R$  is continuous for the  $(p, p_1(\mu))$ -adic topology on  $A(1)$ , therefore, we get that  $(A(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} = (A(1)/p_1(\mu)^n)^{\nabla_q=0}$ . In particular, showing that (3.19) is exact, is equivalent to showing that  $H^1(q\Omega_{\bar{A}(1)/\Lambda_R}^\bullet) = 0$ . Now, from Proposition 3.37, recall that the  $q$ -de Rham complex  $q\Omega_{\bar{A}(1)/\Lambda_R}^\bullet$  is naturally quasi-isomorphic to the de Rham complex  $\Omega_{\bar{A}(1)/\Lambda_R}^\bullet$ . Then, from the explicit description of  $\bar{A}(1)$  in Proposition 3.25, it is clear that the de Rham complex  $\Omega_{\bar{A}(1)/\Lambda_R}^\bullet$  is acyclic in positive degree, in particular,  $H^1(q\Omega_{\bar{A}(1)/\Lambda_R}^\bullet) = H^1(\Omega_{\bar{A}(1)/\Lambda_R}^\bullet) = 0$ . Hence, it follows that (3.19) is exact.  $\blacksquare$

For computations to be carried out in Subsection 5.2, we will consider a “diagonal map for the geometric variables” from  $A_R(1)$  to  $\tilde{\Lambda}_R$ . More precisely, consider the following surjective  $A_R$ -linear (via  $p_1$ ) and  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant map

$$\Delta' : A_R(1) \longrightarrow \tilde{\Lambda}_R,$$

where  $p_2(\mu) \mapsto p_2(\mu)$  and  $p_2([X_i^b]) \mapsto p_1([X_i^b])$  for  $1 \leq i \leq d$ . Then it is easy to verify that the composition  $\Delta' \circ \iota_{\tilde{\Lambda}}$  (see (3.22)), is the identity on  $\tilde{\Lambda}_R$ . As  $p_1(\mu)$  is invariant under the action of  $1 \times \Gamma'_R$ , we see that the reduction of  $\Delta'$  modulo  $p_1(\mu)^n$  is  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant, for each  $n \geq 1$ . Now taking invariants of the source under the action of  $1 \times \Gamma'_R$ , we obtain the following  $(\varphi, \Gamma_F)$ -equivariant morphism:

$$\Delta' : (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \longrightarrow \tilde{\Lambda}_R/p_1(\mu)^n. \quad (3.25)$$

Then, we claim the following:

**Lemma 3.50.** *The  $(\varphi, \Gamma_F)$ -equivariant map in (3.25) is an isomorphism. Moreover, (3.25) induces a  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $\Delta' : A_R(1)^{1 \times \Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R$*

*Proof.* Let us first consider the following  $(\varphi, \Gamma_F)$ -equivariant diagram

$$\Lambda_R \xrightarrow[\sim]{\iota_{\Lambda}} (A_R(1)/p_1(\mu))^{1 \times \Gamma'_R} \xrightarrow{(3.25)} \tilde{\Lambda}_R/p_1(\mu) \xrightarrow[\sim]{(3.23)} \Lambda_R.$$

Using the isomorphism (3.24) in Lemma 3.47, it is easy to verify that the composition above is the identity on  $\Lambda_R$ . Therefore, it follows that (3.25) is an isomorphism for  $n = 1$ . Now as both  $A_R(1)$  and  $\tilde{\Lambda}_R$  are  $p_1(\mu)$ -torsion free (see Lemma 3.15 and Construction 3.44), we consider the following diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & (A_R(1)/p_1(\mu))^{1 \times \Gamma'_R} & \xrightarrow{p_1(\mu)^n} & (A_R(1)/p_1(\mu)^{n+1})^{1 \times \Gamma'_R} & \longrightarrow & (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \longrightarrow 0 \\ & & \downarrow (3.25) & & \downarrow (3.25) & & \downarrow (3.25) \\ 0 & \longrightarrow & \tilde{\Lambda}_R/p_1(\mu) & \xrightarrow{p_1(\mu)^n} & \tilde{\Lambda}_R/p_1(\mu)^{n+1} & \longrightarrow & \tilde{\Lambda}_R/p_1(\mu)^n \longrightarrow 0, \end{array}$$

where the first exact sequence is from (3.19). Using the diagram, an easy induction on  $n \geq 1$ , gives the natural  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism  $\Delta' : (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R/p_1(\mu)^n$  in (3.25). Finally, since both  $A_R(1)$  and  $\tilde{\Lambda}_R$  are  $p_1(\mu)$ -adically complete, therefore, by taking the limit over  $n \geq 1$  and noting that limit commutes right adjoint functors, in particular, with taking  $(1 \times \Gamma'_R)$ -invariants, we obtain the  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $\Delta' : A_R(1)^{1 \times \Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R$ .  $\blacksquare$

**3.4.2. The action of  $\mathbb{F}_p^\times$ .** In this subsection, we will assume that  $p \geq 3$  and consider the invariants of the exact sequence (3.19) (more precisely, its image under the inverse of the map  $\iota_{\tilde{\Lambda}}$  of (3.24)), for the action of  $1 \times \mathbb{F}_p^\times$ .

**Lemma 3.51.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant sequence is exact:*

$$0 \longrightarrow (\tilde{\Lambda}_R/p_1(\mu))^{1 \times \mathbb{F}_p^\times} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1 \times \mathbb{F}_p^\times} \longrightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \mathbb{F}_p^\times} \longrightarrow 0. \quad (3.26)$$

*Proof.* Using the discussion in Construction 3.44 and the isomorphism (3.24) in Lemma 3.47, the exact sequence (3.19) in Lemma 3.43 can be written as the following  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant exact sequence:

$$0 \longrightarrow \tilde{\Lambda}_R/p_1(\mu) \xrightarrow{p_1(\mu)^n} \tilde{\Lambda}_R/p_1(\mu)^{n+1} \longrightarrow \tilde{\Lambda}_R/p_1(\mu)^n \longrightarrow 0.$$

By considering the associated long exact sequence for the cohomology of  $(1 \times \mathbb{F}_p^\times)$ -action and noting that  $H^1(1 \times \mathbb{F}_p^\times, \Lambda_R) = 0$ , since  $p - 1$  is invertible in  $\mathbb{Z}_p$ , we obtain that the sequence in (3.26) is exact.  $\blacksquare$

Next, let us describe the rings  $(\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \mathbb{F}_p^\times}$ , more explicitly.

**Construction 3.52.** Note that from (A.1), the action of  $1 \times \mathbb{F}_p^\times$  induces a decomposition  $\tilde{\Lambda}_R = \bigoplus_{i=0}^{p-2} (\tilde{\Lambda}_R)_i$ . Let us set  $\tilde{\Lambda}_{R,0} := (\tilde{\Lambda}_R)_0 = (\tilde{\Lambda}_R)^{1 \times \mathbb{F}_p^\times}$ , complete for the  $p$ -adic topology and equipped with induced Frobenius and a continuous action of  $\Gamma_R \times \Gamma_0$ , from the corresponding structures on  $\tilde{\Lambda}_R$ . Using the explicit presentation of  $\tilde{\Lambda}_R$  from Construction 3.44 (see (3.21)), we obtain an explicit presentation of  $\tilde{\Lambda}_{R,0}$  as follows:

$$\tilde{\Lambda}_{R,0} = (A_R \hat{\otimes}_{O_F} O_F[[\mu_0]]) \left\{ \frac{1 \otimes \tilde{p}}{\tilde{p} \otimes 1} \right\}_{(p, \tilde{p} \otimes 1)}^\wedge. \quad (3.27)$$

Using notations from Construction 3.44, we will denote the two projection maps as  $p_1 : A_R \rightarrow \tilde{\Lambda}_{R,0}$  and  $p_2 : O_F[[\mu_0]] \rightarrow \tilde{\Lambda}_{R,0}$ . Then, similar to Lemma 3.15, it is easy to see that both the preceding maps are faithfully flat, in particular,  $\tilde{\Lambda}_{R,0}$  is  $p_1(\mu)$ -torsion free.

Furthermore, note that from (3.23), we have a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$ . Now, using (A.1), the action of  $1 \times \mathbb{F}_p^\times$  induces a decomposition  $\Lambda_R = \bigoplus_{i=0}^{p-2} (\Lambda_R)_i$ . Note that  $p_1(\mu)$  in  $\tilde{\Lambda}_R$  is invariant under the action of  $1 \times \mathbb{F}_p^\times$  and  $p-1$  is invertible in  $\mathbb{Z}_p$ , in particular,  $H^1(1 \times \mathbb{F}_p^\times, \tilde{\Lambda}_R) = 0$ . Therefore, it follows that for each  $n \geq 1$ , we have a  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism of rings

$$\tilde{\Lambda}_{R,0}/p_1(\mu)^n \xrightarrow{\sim} (\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \mathbb{F}_p^\times}. \quad (3.28)$$

In particular, for  $n = 1$ , we have a  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism of rings  $\tilde{\Lambda}_{R,0}/p_1(\mu) \xrightarrow{\sim} (\tilde{\Lambda}_R/p_1(\mu))^{1 \times \mathbb{F}_p^\times}$ . Additionally, using (3.23) induces a  $(\varphi, \Gamma_0)$ -equivariant isomorphism of rings

$$\tilde{\Lambda}_{R,0}/p_1(\mu) \xrightarrow{\sim} (\tilde{\Lambda}_R/p_1(\mu))^{1 \times \mathbb{F}_p^\times} \xrightarrow{\sim} \Lambda_{R,0}^{\mathbb{F}_p^\times} =: \Lambda_{R,0}. \quad (3.29)$$

Let us describe the ring  $\Lambda_{R,0}$  more explicitly. From Proposition 3.21, recall that we have  $\Lambda_R = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ . Moreover, recall that  $t = \log(1 + \mu)$  is an element of  $\Lambda_F$  and since  $t/\mu$  is a unit in  $\Lambda_F$  (see Lemma 3.24), we may write  $\Lambda_R = R[t, (t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ . Now, since the action of  $\mathbb{F}_p^\times$  is trivial on  $t^{p-1}$ , it follows that  $\Lambda_{R,0} = \Lambda_{R,0}^{\mathbb{F}_p^\times} = R[(t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ . Furthermore, recall that  $\mu_0$  is the product of  $\mu^{p-1}$  with a unit in  $A_F$  (see Lemma 3.7) and since  $t/\mu$  is a unit in  $\Lambda_F$ , therefore, we can write  $\mu_0 = vt^{p-1}$ , where  $v$  is a unit in  $\Lambda_F$ . As the action of  $\mathbb{F}_p^\times$  is trivial on  $\mu_0$  and  $t^{p-1}$ , it follows that  $v$  belongs to  $\Lambda_{F,0} = \Lambda_{F,0}^{\mathbb{F}_p^\times}$ . In particular,  $\mu_0$  is a product of  $t^{p-1}$  with a unit in  $\Lambda_{F,0}$  and we have  $(\varphi, \Gamma_0)$ -equivariant identifications

$$\Lambda_{R,0} = \Lambda_{R,0}^{\mathbb{F}_p^\times} = R[(t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = R[(\mu_0/p)^{[k]}, k \in \mathbb{N}]_p^\wedge. \quad (3.30)$$

Note that  $\mu^{p-1}$  is not an element of  $\Lambda_{R,0}$  as the action of  $\mathbb{F}_p^\times$  is non-trivial on  $\mu^{p-1}$ . ■

**3.4.3. The action of  $1 + p\mathbb{Z}_p$ .** In this subsection, we will assume that  $p \geq 3$  and consider the invariants of the exact sequence (3.26), for the action of  $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + p\mathbb{Z}_p)$ , and show the following:

**Lemma 3.53.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times 1)$ -equivariant sequence is exact:*

$$0 \longrightarrow (\tilde{\Lambda}_R/p_1(\mu))^{1 \times \Gamma_F} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1 \times \Gamma_F} \longrightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \Gamma_F} \longrightarrow 0. \quad (3.31)$$

**Remark 3.54.** Via the natural  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism  $\iota_{\tilde{\Lambda}}$  in Lemma 3.47 (see (3.24)), we see that the exact sequence in (3.26) is the invariants, under the natural action of  $1 \times \mathbb{F}_p^\times$ , of the exact sequence (3.19) in Lemma 3.43. Then it follows that the exact sequence in (3.31) is the invariants, under the natural action of  $1 \times \Gamma_F$ , of the exact sequence (3.19) in Lemma 3.43.

Note that using the  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism in (3.28) and (3.29), the sequence in (3.31) can be rewritten as the following  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant sequence:

$$0 \longrightarrow \Lambda_{R,0}^{\Gamma_0} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1})^{1 \times \Gamma_0} \longrightarrow (\tilde{\Lambda}_{R,0}/p_1(\mu)^n)^{1 \times \Gamma_0} \longrightarrow 0. \quad (3.32)$$

In order to prove that (3.32) is exact, we will now look at the action of  $\Gamma_R \times \Gamma_0$  on the rings introduced in Construction 3.52. We start with the following observation:

**Lemma 3.55.** *The action of  $1 \times \Gamma_0$  is trivial on  $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$  and the action of  $\Gamma_R \times 1$  is trivial on  $\tilde{\Lambda}_{R,0}/p_1(\mu)$ .*

*Proof.* The proof of the claim is similar to that of Lemma 3.48. Let us first note that if  $g$  is any element of  $\Gamma_0$ , then  $(g-1)\mu_0$  is an element of  $\mu_0 O_F[[\mu_0]]$  (see Lemma 3.6). Now, for the first part, let us note that the action of  $1 \times \Gamma_0$  is trivial modulo  $p_2(\mu_0)$  on  $A_R \hat{\otimes}_{O_F} O_F[[\mu_0]]$ . Then, from the explicit description of  $\tilde{\Lambda}_{R,0}$  in (3.27), it is enough to show that for any  $m \in \mathbb{N}$  and  $g$  in  $1 \times \Gamma_0$ , we have that

$$(g-1)\delta^m \left( \frac{1 \otimes \tilde{p}}{\tilde{p} \otimes 1} \right) \in p_2(\mu_0) \tilde{\Lambda}_{R,0}.$$

We can reduce this claim further, as follows. Using Lemma 3.9 (2), we first note that  $p_2(\mu_0) \tilde{\Lambda}_{R,0}$  is a  $\delta$ -stable ideal of  $\tilde{\Lambda}_{R,0}$ , in the sense of [BS22, Example 2.10]. Then, using Lemma 3.19 for  $A = \tilde{\Lambda}_{R,0}$  and  $\alpha = p_2(\mu_0)$ , we see that to prove our claim, it is enough to show that  $(g-1) \left( \frac{1 \otimes \tilde{p}}{\tilde{p} \otimes 1} \right)$  belongs to  $p_2(\mu_0) \tilde{\Lambda}_{R,0}$ . The assertion now follows from the earlier observation that  $(g-1)\mu_0$  is an element of  $\mu_0 O_F[[\mu_0]]$ . In particular, we have shown that the action of  $1 \times \Gamma_0$  is trivial on  $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$ . The second claim easily follows from Lemma 3.48.  $\blacksquare$

**Remark 3.56.** From Lemma 3.55, note that the action of  $1 \times \Gamma_0$  is trivial on  $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$  and the element  $p_1(\mu)$  is invariant under this action. Therefore, it follows that for any  $g$  in  $1 \times \Gamma_0$  and any  $x$  in  $\tilde{\Lambda}_{R,0}/p_1(\mu)^n$  we have that  $(g-1)x$  is an element of  $p_2(\mu_0) \tilde{\Lambda}_{R,0}/p_1(\mu)^n$ . In particular, for  $n=1$ , using the isomorphism  $\tilde{\Lambda}_{R,0}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,0}$  from (3.29), we get that for any  $g$  in  $\Gamma_0$  and any  $x$  in  $\Lambda_{R,0}$  the element  $(g-1)x$  belongs to  $\mu_0 \Lambda_{R,0}$ . We can show this more explicitly. Indeed, let  $\gamma_0$  be the topological generator of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + pa$ , for a unit  $a$  in  $\mathbb{Z}_p$ , and we will write  $(1+pa)^{p-1} = 1 + pb$ , where  $b$  is also a unit in  $\mathbb{Z}_p$ . Now, from Construction 3.52, recall that  $\mu_0$  is the product of  $t^{p-1}$  with a unit in  $\Lambda_{F,0}$  (see (3.30) and the discussion preceding it). Therefore, it is enough to show that for any  $x$  in  $\Lambda_{R,0}$ , we have that  $(\gamma_0 - 1)x$  is an element of  $t^{p-1} \Lambda_{R,0}$ . Recall that  $\Lambda_{R,0} = R[(t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge$ , so we can write  $x = \sum_{k \in \mathbb{N}} x_k (t^{p-1}/p)^{[k]}$ , where  $x_k$  is an element of  $R$  for each  $k \in \mathbb{N}$ . Then we have,

$$\begin{aligned} (\gamma_0 - 1)x &= (\gamma_0 - 1) \left( \sum_{k \in \mathbb{N}} x_k \left( \frac{t^{p-1}}{p} \right)^{[k]} \right) \\ &= \sum_{k \in \mathbb{N}} (\chi(\gamma_0)^{(p-1)k} - 1) x_k \left( \frac{t^{p-1}}{p} \right)^{[k]} = t^{p-1} \sum_{k \in \mathbb{N}} \frac{((1+pb)^k - 1)}{pk} x_k \left( \frac{t^{p-1}}{p} \right)^{[k-1]} = t^{p-1} y, \end{aligned}$$

where  $y$  converges in  $\Lambda_{F,0}$  because for any  $k \in \mathbb{N}$ , an easy computation shows that we have  $(1+pb)^k - 1 = pku_k$ , for some unit  $u_k$  in  $\mathbb{Z}_p$ . Therefore, it follows that  $(\gamma_0 - 1)x$  is an element of  $t^{p-1} \Lambda_{R,0} = \mu_0 \Lambda_{R,0}$ .

Using the action of  $1 \times \Gamma_0$  on  $\tilde{\Lambda}_{R,0}$ , we will define a  $q$ -de Rham complex (see Definition 3.31). For such a definition we will treat the following element in  $\tilde{\Lambda}_{R,0}$  as a parameter:

$$\tilde{s} := \frac{1 \otimes \tilde{p} - \tilde{p} \otimes 1}{\tilde{p} \otimes 1} = \frac{p_2(\tilde{p}) - p_1(\tilde{p})}{p_1(\tilde{p})} \quad (3.33)$$

**Lemma 3.57.** *Let  $\gamma_0$  be any element of  $1 \times \Gamma_0$ . Then we have that  $(\gamma_0 - 1)\tilde{s} = up_2(\mu_0)$ , for some unit  $u$  in  $\tilde{\Lambda}_{R,0}$ .*

*Proof.* Note that it is enough to show the claim for a topological generator  $\gamma_0$  of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + pa$ , for a unit  $a$  in  $\mathbb{Z}_p$ . Now, recall that we have  $\tilde{p} = \mu_0 + p$ . Moreover, from Lemma 3.6, recall that we can write  $(\gamma_0 - 1)\mu_0 = \tilde{p}\mu_0 x$ , for some  $x$  in  $O_F[[\mu_0]]$ . So from (3.33) we can write

$$(\gamma_0 - 1)\tilde{s} = \frac{(\gamma_0 - 1)p_2(\tilde{p})}{p_1(\tilde{p})} = \frac{(\gamma_0 - 1)p_2(\mu_0)}{p_1(\tilde{p})} = \frac{p_2(\tilde{p}\mu_0 x)}{p_1(\tilde{p})} = \frac{p_2(\tilde{p})}{p_1(\tilde{p})} p_2(x) p_2(\mu_0). \quad (3.34)$$

From Lemma 3.3 and Lemma 3.15, it follows that  $\frac{1 \otimes \tilde{p}}{\tilde{p} \otimes 1}$  is a unit in  $A_R(1)$ . Moreover, from the description of  $\tilde{\Lambda}_{R,0}$  as the  $\mathbb{F}_p^\times$ -invariants of  $\tilde{\Lambda}_R$ , in Construction 3.52, it follows that  $\frac{1 \otimes \tilde{p}}{\tilde{p} \otimes 1}$  is also a unit in  $\tilde{\Lambda}_{R,0}$ . Therefore, to show the claim, it is enough to show that  $p_2(x)$  is a unit in  $\tilde{\Lambda}_{R,0}$ . Again, note that from Construction 3.52, the ring  $\tilde{\Lambda}_{R,0}$  is  $p_1(\mu)$ -adically complete and we have a  $(\varphi, \Gamma_0)$ -equivariant isomorphism  $\tilde{\Lambda}_{R,0}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,0}$  (see (3.29)), therefore, we are reduced to showing that  $p_2(\bar{x})$ , the image of  $p_2(x)$  under the preceding isomorphism, is a unit. Also note that, under the preceding



isomorphism,  $p_2(\mu_0)$  and  $p_2(\tilde{p})$  are the respective images of  $\mu_0$  and  $\tilde{p}$  in  $\Lambda_{R,0}$ . Now, reducing the equalities in (3.34) modulo  $p_1(\mu)$ , we obtain the following expression in  $\Lambda_{R,0}$ :

$$\frac{(\gamma_0-1)\mu_0}{p} = p_2(\bar{x})\mu_0\frac{\tilde{p}}{p}.$$

Note that  $\tilde{p}/p$  is a unit in  $\Lambda_{R,0}$ , because it is the image of the unit  $p_2(\tilde{p})/p_1(\tilde{p})$  in  $\tilde{\Lambda}_{R,0}$  (also see Lemma 3.24). Now, from Lemma 3.58, it follows that  $p_2(\bar{x})$  is a unit in  $\Lambda_{R,0}$ . Therefore,  $p_2(x)$  is a unit in  $\tilde{\Lambda}_{R,0}$ . Hence, we have shown that  $(\gamma_0 - 1)\tilde{s} = up_2(\mu_0)$ , for  $u = (p_2(\tilde{p})/p_1(\tilde{p}))x$  a unit in  $\tilde{\Lambda}_{R,0}$ . ■

The following observation was used above:

**Lemma 3.58.** *Let  $\gamma_0$  be a topological generator of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + pa$ , for a unit  $a$  in  $\mathbb{Z}_p$ . Then we have  $(\gamma_0 - 1)\frac{\mu_0}{p} = v\mu_0$ , for some unit  $v$  in  $\Lambda_{F,0} = \Lambda_{F^{\times}}^{\times}$ .*

*Proof.* From Construction 3.52, recall that  $\mu_0/p$  is the product of  $t^{p-1}/p$  with a unit in  $\Lambda_{F,0}$  (see (3.30) and the discussion preceding it). So let us write  $\mu_0/p = ut^{p-1}/p$ , for some unit  $u$  in  $\Lambda_{F,0}$ . Note that, from Remark 3.56, we have that  $(\gamma_0 - 1)u = \mu_0x$ , for some  $x$  in  $\Lambda_{F,0}$ . Moreover, we write  $(1 + pa)^{p-1} - 1 = pb$ , where  $b$  is a unit in  $\mathbb{Z}_p$ . Therefore, we can write

$$\begin{aligned} (\gamma_0 - 1)\frac{\mu_0}{p} &= (\gamma_0 - 1)\frac{ut^{p-1}}{p} = \frac{t^{p-1}}{p}(\gamma_0 - 1)u + \gamma_0(u)(\gamma_0 - 1)\frac{t^{p-1}}{p} \\ &= \frac{t^{p-1}}{p}\mu_0x + \gamma_0(u)(\chi(\gamma_0)^{p-1} - 1)\frac{t^{p-1}}{p} \\ &= \mu_0\left(\frac{t^{p-1}}{p}x + \gamma_0(u)u^{-1}b\right) = \mu_0v, \end{aligned}$$

where  $v = (\frac{t^{p-1}}{p}x + \gamma_0(u)u^{-1}b)$  is a unit in  $\Lambda_{F,0}$  because  $\gamma_0(u)u^{-1}b$  is a unit and  $t^{p-1}/p$  is  $p$ -adically nilpotent in  $\tilde{\Lambda}_{F,0}$ . Hence, the lemma is proved. ■

In the rest of this subsection, we will fix the choice of a topological generator  $\gamma_0$  of  $1 \times \Gamma_0$  such that  $\chi(\gamma_0) = 1 + pa$ , for a unit  $a$  in  $\mathbb{Z}_p$ . Let us now consider the following operator on  $\tilde{\Lambda}_{R,0}$ :

$$\begin{aligned} \nabla_{q,\tilde{s}} : \tilde{\Lambda}_{R,0} &\longrightarrow \tilde{\Lambda}_{R,0} \\ x &\longmapsto \frac{(\gamma_0-1)x}{(\gamma_0-1)\tilde{s}}. \end{aligned} \tag{3.35}$$

From the triviality of the action of  $1 \times \Gamma_0$  on  $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$  (see Lemma 3.55) and from Lemma 3.57, it follows that the operator  $\nabla_{q,\tilde{s}}$  is well-defined. For each  $n \in \mathbb{N}$ , using Remark 3.56, the operator in (3.35), induces well-defined operators  $\nabla_{q,\tilde{s}} : \tilde{\Lambda}_{R,0}/p_1(\mu)^n \longrightarrow \tilde{\Lambda}_{R,0}/p_1(\mu)^n$ . In particular, for  $n = 1$ , set  $s := \mu_0/p$  in  $\Lambda_{R,0}$ , then using Remark 3.56 and Lemma 3.58, we have a well-defined operator

$$\begin{aligned} \nabla_{q,s} : \Lambda_{R,0} &\longrightarrow \Lambda_{R,0} \\ x &\longmapsto \frac{(\gamma_0-1)x}{(\gamma_0-1)s}. \end{aligned} \tag{3.36}$$

**Remark 3.59.** Considering  $\tilde{s}$  as a parameter, the operator  $\nabla_{q,\tilde{s}}$  in (3.35), may be considered as a  $q$ -differential operator in non-logarithmic coordinates, in the sense of Definition 3.31 and Remark 3.32, where  $q\Omega_{A/D}^1$  identifies with the  $(p, p_1(\mu))$ -adic completion of the module of Kähler differentials of  $\tilde{\Lambda}_{R,0}$  with respect to  $p_1 : A_R \rightarrow \tilde{\Lambda}_{R,0}$ . Similarly, considering  $s$  as a parameter, the operator  $\nabla_{q,s}$  in (3.36), may be also considered as a non-logarithmic  $q$ -differential operator in the sense of Definition 3.31 and Remark 3.32, where the  $q\Omega_{A/D}^1$  identifies with the  $p$ -adic completion of the module of Kähler differentials of  $\Lambda_{R,0}$  with respect to  $R$ .

For each  $n \in \mathbb{N}$ , the operator  $\nabla_{q,\tilde{s}}$  is an endomorphism of  $\tilde{\Lambda}_{R,0}/p_1(\mu)^n$ , so we can define the following two term Koszul complex:

$$K_{\tilde{\Lambda}_{R,0}/p_1(\mu)^n}(\nabla_{q,\tilde{s}}) := [\tilde{\Lambda}_{R,0}/p_1(\mu)^n \xrightarrow{\nabla_{q,\tilde{s}}} \tilde{\Lambda}_{R,0}/p_1(\mu)^n]. \tag{3.37}$$

For  $n = 1$ , we have the following claim:

**Lemma 3.60.** *The cohomology of the Koszul complex  $K_{\Lambda_{R,0}}(\nabla_{q,s})$  vanishes in degree 1, i.e. we have  $H^1(K_{\Lambda_{R,0}}(\nabla_{q,s})) = 0$ .*

*Proof.* The proof of this claim will be carried out in Proposition 4.24, where the claim will be shown, more generally, for finite  $\Lambda_{R,0}$ -modules admitting a continuous action of  $\Gamma_0$  (trivial modulo  $\mu_0$ ), in particular,  $\Lambda_{R,0}$  itself. We give a brief sketch of the main steps below.

Recall that,  $s = \mu_0/p$  is the product of  $t^{p-1}/p$  with a unit in  $\Lambda_{F,0}$  (see Construction 3.52, in particular, (3.30) and the discussion preceding it). Moreover, from Lemma 3.58 we have that  $(\gamma_0 - 1)s$  is the product of  $\mu_0$  with a unit in  $\Lambda_{F,0}$ . Now let  $z := t^{p-1}/p$ , then in the proof of Proposition 4.16, it will be shown that  $(\gamma_0 - 1)z$  is the product of  $\mu_0$  with a unit in  $\Lambda_{F,0}$ . Therefore, it follows that the complex  $K_{\Lambda_{R,0}}(\nabla_{q,s})$  is quasi-isomorphic to the following complex

$$K_{\Lambda_{R,0}}(\nabla_{q,z}) := [\Lambda_{R,0} \xrightarrow{\nabla_{q,z}} \Lambda_{R,0}].$$

To show the vanishing of  $H^1(K_{\Lambda_{R,0}}(\nabla_{q,z}))$ , we switch from the  $q$ -differential operator to a differential operator. So, in Proposition 4.24, we show that  $\nabla_0^{\log} := \frac{\log(\gamma_0)}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}}{k+1}$ , converge as a series of operators on  $\Lambda_{R,0}$  and that  $\nabla_0(-) \otimes dz : \Lambda_{R,0} \rightarrow \Omega_{\Lambda_{R,0}/R}^1$ , coincides with the universal  $R$ -linear continuous de Rham differential operator  $d : \Lambda_{R,0} \rightarrow \Omega_{\Lambda_{R,0}/R}^1$ . In particular, we have the following identification of complexes:

$$\Omega_{\Lambda_{R,0}/R}^\bullet = K_{\Lambda_{R,0}}(\nabla_0) := [\Lambda_{R,0} \xrightarrow{\nabla_0} \Lambda_{R,0}].$$

Recall that  $\Lambda_{R,0} = R[z^{[k]}, k \in \mathbb{N}]_p^\wedge$  (see (3.30)), therefore it follows that the de Rham complex  $\Omega_{\Lambda_{R,0}/R}^\bullet = K_{\Lambda_{R,0}}(\nabla_0)$  is acyclic in positive degrees and we have that  $R \xrightarrow{\sim} \Lambda_{R,0}^{\nabla_0=0}$ . Furthermore, in Proposition 4.24, we show a natural quasi-isomorphism of complexes

$$K_{\Lambda_{R,0}}(\nabla_{q,z}) \xrightarrow{\sim} K_{\Lambda_{R,0}}(\nabla_0).$$

Hence, we conclude that  $H^1(K_{\Lambda_{R,0}}(\nabla_{q,s})) \xrightarrow{\sim} H^1(K_{\Lambda_{R,0}}(\nabla_0)) = 0$ .  $\blacksquare$

*Proof of Lemma 3.53.* Using the  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism of rings from (3.28), the exact sequence in (3.26) can be rewritten as follows:

$$0 \longrightarrow \Lambda_{R,0} \xrightarrow{p_1(\mu)^n} \tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1} \longrightarrow \tilde{\Lambda}_{R,0}/p_1(\mu)^n \longrightarrow 0.$$

Then, using the operator  $\nabla_{q,\bar{s}}$  in (3.35) and the Koszul complex defined in (3.37), we obtain an exact sequence of Koszul complexes:

$$0 \longrightarrow K_{\Lambda_{R,0}}(\nabla_{q,s}) \xrightarrow{p_1(\mu)^n} K_{\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1}}(\nabla_{q,\bar{s}}) \longrightarrow K_{\tilde{\Lambda}_{R,0}/p_1(\mu)^n}(\nabla_{q,\bar{s}}) \longrightarrow 0.$$

Considering the associated long exact sequence, and noting that  $H^1(K_{\Lambda_{R,0}}(\nabla_{q,s})) = 0$  from Lemma 3.60, we obtain the following exact sequence:

$$0 \longrightarrow \Lambda_{R,0}^{\nabla_{q,s}=0} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1})^{\nabla_{q,\bar{s}}=0} \longrightarrow (\tilde{\Lambda}_{R,0}/p_1(\mu)^n)^{\nabla_{q,\bar{s}}=0} \longrightarrow 0.$$

Since the action of  $1 \times \Gamma_0$  on  $\tilde{\Lambda}_{R,0}$  is continuous for the  $(p, p_1(\mu))$ -adic topology, therefore, we that  $(\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1})^{\nabla_{q,\bar{s}}=0} = (\tilde{\Lambda}_{R,0}/p_1(\mu)^{n+1})^{1 \times \Gamma_0}$ , for each  $n \in \mathbb{N}$ . Hence, from the preceding exact sequence we obtain that the sequence in (3.32) is exact, therefore, the sequence (3.31) is also exact.  $\blacksquare$

**3.4.4. The case  $p = 2$ .** In this subsection, our goal is to prove a statement similar to Lemma 3.53, for  $p = 2$ . From (1.6), recall that  $\Gamma_F$  fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\text{tor}} \longrightarrow 1,$$

where, we have  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$  and  $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$ , as groups.

We will first look at the action of  $\Gamma_{\text{tor}}$  on  $\tilde{\Lambda}_R$ . Let  $\sigma$  denote a generator of  $\Gamma_{\text{tor}}$ . Then, from (A.2), recall that by setting  $\tilde{\Lambda}_{R,+} := \{x \in \tilde{\Lambda}_R \text{ such that } \sigma(x) = x\}$  and  $\tilde{\Lambda}_{R,-} := \{x \in \tilde{\Lambda}_R \text{ such that } \sigma(x) = -x\}$ , we have a natural injective map of  $\tilde{\Lambda}_{R,+}$ -modules

$$\tilde{\Lambda}_{R,+} \oplus \tilde{\Lambda}_{R,-} \longrightarrow \tilde{\Lambda}_R, \quad (3.38)$$

given as  $(x, y) \mapsto x + y$ . Note that the action of  $1 \times \Gamma_F$  on  $\tilde{\Lambda}_R$ , is continuous for the  $(p, p_1(\mu))$ -adic topology, so it follows that  $\tilde{\Lambda}_{R,+}$  is a  $(p, p_1(\mu))$ -adically complete subring of  $\tilde{\Lambda}_R$  stable under the action of  $(\varphi, \Gamma_R \times \Gamma_F)$  on  $\tilde{\Lambda}_R$  and similarly,  $\tilde{\Lambda}_{R,-}$  is a complete  $\tilde{\Lambda}_{R,+}$ -submodule stable under the action of  $(\varphi, \Gamma_R \times \Gamma_F)$ . Equipping  $\tilde{\Lambda}_{R,+}$  and  $\tilde{\Lambda}_{R,-}$  with induced structures, we see that (3.38) is  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant.

Now, from (3.23), recall that  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$ . Similar to above, by setting  $\Lambda_{R,+} := \{x \in \Lambda_R \text{ such that } \sigma(x) = x\}$  and  $\Lambda_{R,-} := \{x \in \Lambda_R \text{ such that } \sigma(x) = -x\}$ , from (A.2), we have a natural  $(\varphi, \Gamma_F)$ -equivariant injective map of  $\Lambda_{R,+}$ -modules

$$\Lambda_{R,+} \oplus \Lambda_{R,-} \longrightarrow \Lambda_R. \quad (3.39)$$

**Lemma 3.61.** *The natural map in (3.39) is bijective.*

*Proof.* Recall that  $t = \log(1 + \mu)$  is an element of  $\Lambda_F$  and  $t/\mu$  is a unit in  $\Lambda_F$  (see Lemma 3.24), so we may write,  $\Lambda_R = R[t, (t^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^\wedge = R[(t/2)^{[k]}, k \in \mathbb{N}]_p^\wedge$ . Since  $\sigma(t) = -t$ , therefore it easily follows that  $\Lambda_{R,+} = R[(t/2)^{[k]}, k \in 2\mathbb{N}]_p^\wedge$  and  $\Lambda_{R,-} = R[(t/2)^{[k]}, k \in 2\mathbb{N} + 1]_p^\wedge$ . Then it is immediate that  $\Lambda_R = \Lambda_{R,+} \oplus \Lambda_{R,-}$ . Hence, (3.39) is bijective.  $\blacksquare$

Next, we will give another description of  $\Lambda_{R,+}$  and  $\Lambda_{R,-}$ . Let us consider the following element in  $A_F$  from [Fon94, Subsection 5.2.5]:

$$\nu := q - 1 + \sigma(q - 1) = q + q^{-1} - 2 = \frac{(q-1)^2}{q} = \frac{\mu^2}{1+\mu}. \quad (3.40)$$

Using Lemma 3.24, note that the element  $\nu$  is the product of  $t^2$  with a unit in  $\Lambda_F$ . Let  $\tau := \nu/8$  and we claim the following:

**Lemma 3.62.** *The element  $\tau$  is the product of  $t^2/8$  with a unit in  $\Lambda_{F,+}$ , i.e. we have  $\tau = ut^2/8$ , for some unit  $u$  in  $\Lambda_{F,+}$ . In particular, we have*

$$\begin{aligned} \Lambda_{R,+} &= R[(t^2/8)^{[k]}, k \in \mathbb{N}]_p^\wedge = R[\tau^{[k]}, k \in \mathbb{N}]_p^\wedge, \\ \Lambda_{R,-} &= R[(t/2)(t^2/8)^{[k]}, k \in \mathbb{N}]_p^\wedge = R[(t/2)\tau^{[k]}, k \in \mathbb{N}]_p^\wedge = (t/2)\Lambda_{R,+}, \end{aligned} \quad (3.41)$$

*Proof.* Recall that  $\sigma(t) = -t$ , so  $t^2/8$  is an element of  $\Lambda_{F,+}$ . Moreover,  $\nu = ut^2$  for some unit  $u$  in  $\Lambda_F$ , so  $\tau$  is an element of  $\Lambda_F$ . Since,  $\sigma(\nu) = \nu$ , it follows that  $\tau = \nu/8$  belongs to  $\Lambda_{F,+}$ , and therefore,  $u$  is a unit in  $\Lambda_{F,+}$ . Next, from the proof of Lemma 3.61, note that  $\Lambda_{R,+} = R[(t/2)^{[n]}, n \in 2\mathbb{N}]_p^\wedge$  and  $\Lambda_{R,-} = R[(t/2)(t/2)^{[n]}, n \in 2\mathbb{N}]_p^\wedge$ . Now, let  $n = 2k$  for  $k \in \mathbb{N}$ , and note that

$$\binom{t}{2}^{[n]} = \frac{t^{2k}}{(2k)!4^k} = \frac{k!2^k}{(2k)!} \binom{t^2}{8}^{[k]} = \frac{k!2^k}{(2k)!u^k} \tau^{[k]}.$$

The equalities in (3.41) now follow since an easy computation shows that we have  $v_2(2^k) + v_2(k!) = v_2((2k)!)$ .  $\blacksquare$

Let us now consider a lifting of the element  $\tau = \nu/8$  to  $\tilde{\Lambda}_R$ , under the surjective map  $\tilde{\Lambda}_R \rightarrow \tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$ , where the isomorphism is the  $(\varphi, \Gamma_F)$ -equivariant isomorphism in (3.23). We have the following element in  $\tilde{\Lambda}_R$ :

$$\tilde{\tau} := \frac{1}{p_2(q)} \delta\left(\frac{p_2([p]_q)}{p_1([p]_q)}\right) = \frac{1}{1 \otimes q} \delta\left(\frac{1 \otimes [p]_q}{[p]_q \otimes 1}\right) = \frac{1}{1 \otimes q} \delta\left(\frac{1 \otimes \tilde{p}}{\tilde{p} \otimes 1}\right), \quad (3.42)$$

where the last equality follows from Remark 3.2.

**Lemma 3.63.** *The element  $\tilde{\tau}$  belongs to  $\tilde{\Lambda}_{R,+}$  and we have  $\tilde{\tau} = \tau \bmod p_1(\mu)\tilde{\Lambda}_R$ .*

*Proof.* Let  $\sigma$  be a generator of  $1 \times \Gamma_{\text{tor}}$  and note that  $\sigma(1 \otimes q) = 1 \otimes q^{-1}$ . Moreover, since the action of  $1 \times \Gamma_R$  commutes with the  $\delta$ -structure on  $\tilde{\Lambda}_R$ , we have

$$\sigma(\tilde{\tau}) = \frac{1}{\sigma(1 \otimes q)} \delta\left(\frac{\sigma(1 \otimes [p]_q)}{[p]_q \otimes 1}\right) = (1 \otimes q) \delta\left(\frac{1 \otimes [p]_q}{1 \otimes q [p]_q \otimes 1}\right) = \frac{1}{1 \otimes q} \delta\left(\frac{1 \otimes [p]_q}{[p]_q \otimes 1}\right) = \tilde{\tau},$$

where the third equality follows from using the product formula for  $\delta$ -structure from (2.1) and the fact that  $\delta(1 \otimes q) = 0$ . Therefore,  $\tilde{\tau}$  is an element of  $\tilde{\Lambda}_{R,+}$ . Next, since the isomorphism  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$  in (3.23) is compatible with the respective Frobenii, in particular, with respective  $\delta$ -structures, we have the following:

$$\tilde{\tau} \bmod p_1(\mu) = \frac{1}{p_2(q)} \delta\left(\frac{p_2([p]_q)}{p_1([p]_q)}\right) \bmod p_1(q-1) = \frac{1}{q} \delta\left(\frac{[p]_q}{2}\right) = \frac{(q-1)^2}{8q} = \frac{\nu}{8} = \tau.$$

This proves the second claim. ■

**Lemma 3.64.** *For each  $n \geq 1$ , reduction modulo  $p_1(\mu)^n$  of (3.38), induces a natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism*

$$\tilde{\Lambda}_{R,+}/p_1(\mu)^n \xrightarrow{\sim} (\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}}. \quad (3.43)$$

Moreover, for  $n = 1$ , the  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$  from (3.23), induces a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism

$$\tilde{\Lambda}_{R,+}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,+}. \quad (3.44)$$

*Proof.* Let us consider the following natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \tilde{\Lambda}_{R,+}/p_1(\mu) & \xrightarrow{p_1(\mu)^n} & \tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1} & \longrightarrow & \tilde{\Lambda}_{R,+}/p_1(\mu)^n \longrightarrow 0 \\ & & \downarrow (3.38) & & \downarrow (3.38) & & \downarrow (3.38) \\ 0 & \longrightarrow & (\tilde{\Lambda}_R/p_1(\mu))^{1 \times \Gamma_{\text{tor}}} & \xrightarrow{p_1(\mu)^n} & (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1 \times \Gamma_{\text{tor}}} & \longrightarrow & (\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}}, \end{array} \quad (3.45)$$

where the vertical maps are injective because we have  $p_1(\mu)^n \tilde{\Lambda}_R \cap \tilde{\Lambda}_{R,+} = p_1(\mu)^n \tilde{\Lambda}_{R,+}$ , as  $p_1(\mu)$  is invariant under the action of  $1 \times \Gamma_F$ . Composing the left vertical arrow in (3.45) with the  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R$  from (3.23), we obtain a natural  $(\varphi, \Gamma_F)$ -equivariant injective map  $\tilde{\Lambda}_{R,+}/p_1(\mu) \rightarrow \Lambda_{R,+}$ , and we claim that it is surjective as well. Indeed, note that  $\tilde{\Lambda}_{R,+}/p_1(\mu)$  is a  $p$ -torsion free ring equipped with an induced Frobenius, in particular, a  $\delta$ -structure compatible and the left vertical map in (3.45) is compatible with the respective  $\delta$ -structures. Now, from (3.41) in Lemma 3.62, recall that we have  $\Lambda_{R,+} = R[\tau^{[k]}, k \in \mathbb{N}]_p^\wedge$ . If we again denote by  $\tau$ , its preimage under the isomorphism (3.23), then from Lemma 3.63 we have that  $\tau$  is an element of  $\tilde{\Lambda}_{R,+}/p_1(\mu)$  and we need to show that  $\tau^{[k]}$  belongs to  $\tilde{\Lambda}_{R,+}/p_1(\mu)$ , for each  $k \in \mathbb{N}$ . Moreover, using [BS22, Lemma 2.35], we see that it is enough to show that  $\frac{\tau^2}{2}$  is an element of  $\tilde{\Lambda}_{R,+}/p_1(\mu)$ . Since,  $\tilde{\Lambda}_{R,+}/p_1(\mu)$  is a  $\delta$ -ring, we have the following:

$$\delta(\tau) = \delta\left(\frac{(q-1)^2}{8q}\right) = \frac{1}{2q^2} \left(\frac{(q^2-1)^2}{8} - \frac{(q-1)^4}{64}\right) = \frac{1}{2q} (\tau(q+1))^2 - \frac{\tau^2}{q}.$$

As  $[2]_q = q + 1$  can be written as the product of 2 with a unit in  $\Lambda_F$  (see Lemma 3.24), therefore, it follows that  $\frac{\tau^2}{2} = \frac{q\tau(q+1)^2}{2} - q^2\delta(\tau)$  is an element of  $\tilde{\Lambda}_{R,+}/p_1(\mu)$ . Hence, we conclude that  $\tilde{\Lambda}_{R,+}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,+}$ , in particular, the composition in (3.44) and the left vertical arrow in (3.45) are bijective. Now, using the diagram (3.45), an easy induction on  $n \geq 1$ , gives that for each  $n \geq 1$ , the right vertical arrow is bijective and the bottom right horizontal arrow is surjective. Hence, it follows that the natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant map  $\tilde{\Lambda}_{R,+}/p_1(\mu)^n \rightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}}$ , induced by (3.38), is bijective for each  $n \geq 1$ . ■

**Remark 3.65.** Lemma 3.64 can be proved using an alternative method as in the proof of Lemma 5.20, where a crucial input is Lemma A.11. However, the proof given above is conceptually more satisfying.

From Lemma 3.64, we obtain the following:

**Lemma 3.66.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant sequence is exact:*

$$0 \longrightarrow (\tilde{\Lambda}_R/p_1(\mu))^{1 \times \Gamma_{\text{tor}}} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1 \times \Gamma_{\text{tor}}} \longrightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}} \longrightarrow 0. \quad (3.46)$$

*Proof.* The sequence (3.46) is the same as the second row of the diagram (3.43), which was shown to be exact in the proof of Lemma 3.64. ■

Next, we will look at the action of  $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + 4\mathbb{Z}_2)$  on  $\tilde{\Lambda}_{R,+}$  and prove a result similar to Lemma 3.53, for  $p = 2$ . In particular, we will show the following:

**Lemma 3.67.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times 1)$ -equivariant sequence is exact:*

$$0 \longrightarrow (\tilde{\Lambda}_R/p_1(\mu))^{1 \times \Gamma_F} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_R/p_1(\mu)^{n+1})^{1 \times \Gamma_F} \longrightarrow (\tilde{\Lambda}_R/p_1(\mu)^n)^{1 \times \Gamma_F} \longrightarrow 0. \quad (3.47)$$

**Remark 3.68.** Via the natural  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism  $\iota_{\tilde{\Lambda}}$  in Lemma 3.47 (see (3.24)), we see that the exact sequence in (3.47) is the invariants, under the natural action of  $1 \times \Gamma_F$ , of the exact sequence in (3.19) in Lemma 3.43, for  $p = 2$ .

Note that using the  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism in (3.43) and (3.44), the sequence in (3.47) can be rewritten as the following  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant sequence:

$$0 \longrightarrow \Lambda_{R,+}^{\Gamma_0} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1})^{1 \times \Gamma_0} \longrightarrow (\tilde{\Lambda}_{R,+}/p_1(\mu)^n)^{1 \times \Gamma_0} \longrightarrow 0. \quad (3.48)$$

In order to show that the sequence (3.48) is exact, we start with the following observation:

**Lemma 3.69.** *The action of  $1 \times \Gamma_0$  is trivial on  $\tilde{\Lambda}_{R,+}/p_2(\nu)$  and the action of  $\Gamma_R \times 1$  is trivial on  $\tilde{\Lambda}_{R,+}/p_1(\mu)$ .*

*Proof.* For the first claim, note that we have  $p_2(\nu)\tilde{\Lambda}_R \cap \tilde{\Lambda}_{R,+} = p_2(\nu)\tilde{\Lambda}_{R,+}$ . So, if  $x$  is an element of  $\tilde{\Lambda}_{R,+}$  and  $g$  any element of  $1 \times \Gamma_0$ , then it is enough to show that  $(g - 1)x$  is an element of  $p_2(\nu)\tilde{\Lambda}_R$ . Moreover, recall that  $\nu$  is the product of  $\mu^2$  with a unit in  $A_F$ . Therefore, we are reduced to showing that  $(g - 1)x$  is an element of  $p_2(\mu)^2\tilde{\Lambda}_R$ . Now, using Lemma 3.48, we can write  $(g - 1)x = p_2(\mu)y$ , for some  $y$  in  $\tilde{\Lambda}_R$ . Let  $\sigma$  be a generator of  $1 \times \Gamma_{\text{tor}}$  and note that  $\sigma(x) = x$ . Then, we have  $\sigma(p_2(\mu))\sigma(y) = p_2(\mu)y$ , in particular,  $(\sigma - 1)y = -(2 + p_2(\mu))y$ . Again, using Lemma 3.48, we can write  $-p_2([p]_q)y = (\sigma - 1)y = p_2(\mu)z$ , for some  $z$  in  $\tilde{\Lambda}_R$ . So we get that  $-py = 0 \pmod{p_2(\mu)\tilde{\Lambda}_R}$ . Note that  $(p_2(\mu), p)$  is a regular sequence on  $\tilde{\Lambda}_R$  since  $p_2 : A_F \rightarrow \tilde{\Lambda}_R$  is flat (see Construction 3.44). Therefore, we conclude that  $y = 0 \pmod{p_2(\mu)\tilde{\Lambda}_R}$ , i.e.  $y$  is an element of  $p_2(\mu)\tilde{\Lambda}_R$  and  $(g - 1)x = p_2(\mu)y$  is an element of  $p_2(\mu)^2\tilde{\Lambda}_R$ , as desired. The second claim easily follows from Lemma 3.48. ■

**Remark 3.70.** From Lemma 3.69, note that the action of  $1 \times \Gamma_0$  is trivial on  $\tilde{\Lambda}_{R,+}/p_2(\nu)$  and the element  $p_1(\mu)$  is invariant under this action. Therefore, it follows that for any  $g$  in  $1 \times \Gamma_0$  and any  $x$  in  $\tilde{\Lambda}_{R,+}/p_1(\mu)^n$  we have that  $(g - 1)x$  is an element of  $p_2(\nu)\tilde{\Lambda}_{R,+}/p_1(\mu)^n$ . In particular, for  $n = 1$ , using the isomorphism  $\tilde{\Lambda}_{R,+}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,+}$  from (3.44), we get that for any  $g$  in  $\Gamma_0$  and any  $x$  in  $\Lambda_{R,+}$  the element  $(g - 1)x$  belongs to  $\nu\Lambda_{R,+}$ .

Using the action of  $1 \times \Gamma_0$  on  $\tilde{\Lambda}_{R,+}$ , we will define a  $q$ -de Rham complex (see Definition 3.31). For such a definition we will treat the element  $\tilde{\tau}$  in  $\tilde{\Lambda}_{R,+}$  as a parameter. We start with the following observation:

**Lemma 3.71.** *Let  $\gamma_0$  be any element of  $1 \times \Gamma_0$ . Then we have  $(\gamma_0 - 1)\tilde{\tau} = up_2(\nu)$ , for some unit  $u$  in  $\tilde{\Lambda}_{R,+}$ .*

*Proof.* Note that it is enough to show the claim for a topological generator  $\gamma_0$  of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + 4a$ , for a unit  $a$  in  $\mathbb{Z}_2$ . Let us set  $v := \frac{p_2(\lfloor p/q \rfloor)}{p_1(\lfloor p/q \rfloor)}$ , then we have  $\tilde{\tau} = \frac{\delta(v)}{p_2(q)}$ . Now observe that

$$\begin{aligned} (\gamma_0 - 1)\tilde{\tau} &= (\gamma_0 - 1)\left(\frac{\delta(v)}{p_2(q)}\right) = (\gamma_0 - 1)\left(\frac{1}{p_2(q)}\right)\delta(v) + \frac{(\gamma_0 - 1)\delta(v)}{\gamma_0(p_2(q))} \\ &= \frac{1}{p_2(q^5)}((\gamma_0 - 1)\delta(v) - (p_2(q^4) - 1)\delta(v)) \end{aligned}$$

Since  $\delta$  and  $\gamma_0$  commute with each other, an easy computation shows that

$$\begin{aligned} (\gamma_0 - 1)\delta(v) &= \delta((\gamma_0 - 1)v) - v(\gamma_0 - 1)v \\ (p_2(q^4) - 1)\delta(v) &= \delta((p_2(q^4) - 1)v) - v^2(p_2(q^2) - 1). \end{aligned}$$

Then we note that

$$\begin{aligned} v(\gamma_0 - 1)v - v^2(p_2(q^2) - 1) &= v(p_2(q^2) - 1)\left(\frac{p_2(q^3) + p_2(q)}{p_1(q) + 1} - v\right) = -v^2(p_2(q) - 1)(p_2(q^3) - 1) \\ &= -v^2 p_2(q)(p_2(q^2) + p_2(q) + 1)p_2(\nu) = p_2(\nu)x_1, \end{aligned}$$

for some  $x_1$  in  $\tilde{\Lambda}_R$  and  $(\gamma_0 - 1)v = p_2(q)(p_2(q) - 1)(p_2(q^2) + 1)v$ . Now, let  $a = p_2(q)(p_2(q^2) + 1)$ ,  $b = p_2(q) + 1$  and  $c = (p_2(q) - 1)v$ , and note that  $a - b$ ,  $\delta(a - b)$  and  $\delta(c)$  are elements of  $(p_2(q) - 1)\tilde{\Lambda}_R$ , since the latter is a  $\delta$ -stable ideal of  $\tilde{\Lambda}_R$ , in the sense of [BS22, Example 2.10]. So we obtain that

$$\begin{aligned} \delta((\gamma_0 - 1)v) - \delta((p_2(q^4) - 1)v) &= \delta(ac) - \delta(bc) = \delta(ac - bc) + b^2c^2 + abc^2 \\ &= \delta(a - b)c^2 + (a - b)^2\delta(c) + 2\delta(a - b)\delta(c) + b^2c^2 + abc^2 = p_2(\nu)x_2, \end{aligned}$$

for some  $x_2$  in  $\tilde{\Lambda}_R$ , and in the third equality we have used that  $\nu = \frac{(q-1)^2}{q}$  from (3.40). Set  $u := \frac{x_1 + x_2}{p_2(q^5)}$  in  $\tilde{\Lambda}_R$ , and by putting everything together, we have that

$$(\gamma_0 - 1)\tilde{\tau} = up_2(\nu). \quad (3.49)$$

Since  $\sigma(\tilde{\tau}) = \tilde{\tau}$  (see Lemma 3.63),  $\sigma(\nu) = \nu$ , the group  $1 \times \Gamma_F$  is commutative and  $\tilde{\Lambda}_{R,+}$  is  $p_2(\nu)$ -torsion free, therefore, we get that  $u$  is an element of  $\tilde{\Lambda}_{R,+}$ . So to show the claim, it is enough to show that  $u$  is a unit in  $\tilde{\Lambda}_R$ . Now, note that from the discussion after (3.38), the ring  $\tilde{\Lambda}_{R,+}$  is  $p_1(\mu)$ -adically complete and we have a  $(\varphi, \Gamma_0)$ -equivariant isomorphism  $\tilde{\Lambda}_{R,+}/p_1(\mu) \xrightarrow{\sim} \Lambda_{R,+}$  (see (3.44) in Lemma 3.64), therefore, we are reduced to showing that  $\bar{u}$ , the image of  $u$  under the preceding isomorphism, is a unit. By reducing the equalities in (3.49), modulo  $p_1(\mu)$ , we obtain the following expression in  $\Lambda_{R,+}$ :

$$(\gamma_0 - 1)\tau = (\gamma_0 - 1)\frac{\nu}{8} = \bar{u}\nu.$$

But from Lemma 3.58, we see that  $\bar{u}$  must be a unit in  $\Lambda_{R,+}$ . Hence,  $u$  is a unit in  $\tilde{\Lambda}_{R,+}$ , proving the claim.  $\blacksquare$

The following observation was used above:

**Lemma 3.72.** *Let  $\gamma_0$  be a topological generator of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + 4a$ , for a unit  $a$  in  $\mathbb{Z}_2$ . Then we have  $(\gamma_0 - 1)\tau = (\gamma_0 - 1)\frac{\nu}{8} = u\nu$ , for some unit  $u$  in  $\Lambda_{F,+} = \Lambda_F^{\Gamma_{\text{tor}}}$ .*

*Proof.* From Lemma 3.62, recall that  $\nu/8$  is the product of  $t^2/8$  with a unit in  $\Lambda_{F,+}$ . So let us write  $\nu/8 = et^2/8$ , for some unit  $e$  in  $\Lambda_{F,+}$ . Note that, from Remark 3.70, we have that  $(\gamma_0 - 1)e = \nu x$ , for some  $x$  in  $\Lambda_{F,+}$ . Therefore, we can write

$$\begin{aligned} (\gamma_0 - 1)\frac{\nu}{8} &= (\gamma_0 - 1)\frac{et^2}{8} = \frac{t^2}{8}(\gamma_0 - 1)e + \gamma_0(e)(\gamma_0 - 1)\frac{t^2}{8} \\ &= \frac{t^2}{8}\nu x + \gamma_0(e)(\chi(\gamma_0)^2 - 1)\frac{t^2}{8} = \nu\left(\frac{t^2}{8}x + \gamma_0(e)e^{-1}a(2a + 1)\right) = \nu u, \end{aligned}$$

where  $u = \left(\frac{t^2}{8}x + \gamma_0(e)e^{-1}a(2a + 1)\right)$  is a unit in  $\Lambda_{F,+}$  because  $\gamma_0(e)e^{-1}a(2a + 1)$  is a unit and  $t^2/8$  is  $p$ -adically nilpotent in  $\Lambda_{F,+}$ . Hence, the lemma is proved.  $\blacksquare$

In the rest of this subsection, we will fix a topological generator  $\gamma_0$  of  $1 \times \Gamma_0$  such that  $\chi(\gamma_0) = 1 + 4a$ , for a unit  $a$  in  $\mathbb{Z}_2$ . Similar to (3.35), let us now consider the following operator on  $\tilde{\Lambda}_{R,+}$ :

$$\begin{aligned} \nabla_{q,\tilde{\tau}} : \tilde{\Lambda}_{R,+} &\longrightarrow \tilde{\Lambda}_{R,+} \\ x &\mapsto \frac{(\gamma_0-1)x}{(\gamma_0-1)\tilde{\tau}}. \end{aligned} \tag{3.50}$$

From the triviality of the action of  $1 \times \Gamma_0$  on  $\tilde{\Lambda}_{R,+}/p_2(\nu)$  (see Lemma 3.69) and from Lemma 3.71, it follows that the operator  $\nabla_{q,\tilde{\tau}}$  is well-defined. For each  $n \in \mathbb{N}$ , using Remark 3.70, the operator in (3.50), induces well-defined operators  $\nabla_{q,\tilde{\tau}} : \tilde{\Lambda}_{R,+}/p_1(\mu)^n \longrightarrow \tilde{\Lambda}_{R,+}/p_1(\mu)^n$ . In particular, for  $n = 1$ , we have  $\tau = \nu/8$  in  $\Lambda_{R,0}$ , and using Remark 3.70 and Lemma 3.72, we have a well-defined operator

$$\begin{aligned} \nabla_{q,\tau} : \Lambda_{R,+} &\longrightarrow \Lambda_{R,+} \\ x &\mapsto \frac{(\gamma_0-1)x}{(\gamma_0-1)\tau}. \end{aligned} \tag{3.51}$$

**Remark 3.73.** Considering  $\tilde{\tau}$  as a variable, the operator  $\nabla_{q,\tilde{\tau}}$  in (3.35), may be considered as a  $q$ -differential operator in non-logarithmic coordinates, in the sense of Definition 3.31 and Remark 3.32, where  $q\Omega_{A/D}^1$  identifies with the  $(p, p_1(\mu))$ -adic completion of the module of Kähler differentials of  $\tilde{\Lambda}_{R,+}$  with respect to  $p_1 : A_R \rightarrow \tilde{\Lambda}_{R,+}$ . Similarly, considering  $\tau$  as a variable, the operator  $\nabla_{q,\tau}$  in (3.51), may be also considered as a non-logarithmic  $q$ -differential operator in the sense of Definition 3.31 and Remark 3.32, where the  $q\Omega_{A/D}^1$  identifies with the  $p$ -adic completion of the module of Kähler differentials of  $\Lambda_{R,+}$  with respect to  $R$ .

For each  $n \in \mathbb{N}$ , the operator  $\nabla_{q,\tilde{\tau}}$  is an endomorphism of  $\tilde{\Lambda}_{R,+}/p_1(\mu)^n$ , so we can define the following two term Koszul complex:

$$K_{\tilde{\Lambda}_{R,+}/p_1(\mu)^n}(\nabla_{q,\tilde{\tau}}) : [\tilde{\Lambda}_{R,+}/p_1(\mu)^n \xrightarrow{\nabla_{q,\tilde{\tau}}} \tilde{\Lambda}_{R,+}/p_1(\mu)^n]. \tag{3.52}$$

For  $n = 1$ , we have the following claim:

**Lemma 3.74.** *The cohomology of the Koszul complex  $K_{\Lambda_{R,+}}(\nabla_{q,\tau})$  vanishes in degree 1, i.e. we have  $H^1(K_{\Lambda_{R,+}}(\nabla_{q,\tau})) = 0$ .*

*Proof.* The idea of the proof is similar to the proof of Lemma 3.60, with slightly different computations. Main arguments for the proof of the claim will be given in Proposition 4.24, where the claim will be shown, more generally, for finite  $\Lambda_{R,0}$ -modules admitting a continuous action of  $\Gamma_0$  (trivial modulo  $\mu_0$ ), in particular,  $\Lambda_{R,0}$  itself. We give a brief sketch of the main steps below.

Recall that  $\tau = \nu/8$  is the product of  $t^2/8$  with a unit in  $\Lambda_{F,+}$  (see Lemma 3.62). Moreover, from Lemma 3.72 we have that  $(\gamma_0 - 1)\tau$  is the product of  $\nu$  with a unit in  $\Lambda_{F,+}$ . Now let  $w := t^2/8$ , then from the proof of Proposition 4.30, we note that  $(\gamma_0 - 1)w$  is the product of  $\nu$  with a unit in  $\Lambda_{F,+}$ . Therefore, it follows that the complex  $K_{\Lambda_{R,+}}(\nabla_{q,\tau})$  is quasi-isomorphic to the following complex

$$K_{\Lambda_{R,+}}(\nabla_{q,w}) : [\Lambda_{R,+} \xrightarrow{\nabla_{q,w}} \Lambda_{R,+}].$$

To show the vanishing of  $H^1(K_{\Lambda_{R,+}}(\nabla_{q,w}))$ , we switch from the  $q$ -differential operator to a differential operator. So, in Proposition 4.30 we show that  $\nabla_0^{\log} := \frac{\log(\gamma_0)}{\log(\chi(\gamma_0))}$ , converge as a series of operators on  $\Lambda_{R,+}$  and  $\nabla_0(-) \otimes dz : \Lambda_{R,+} \rightarrow \Omega_{\Lambda_{R,+}/R}^1$ , coincides with the universal  $R$ -linear continuous de Rham differential operator  $d : \Lambda_{R,+} \rightarrow \Omega_{\Lambda_{R,+}/R}^1$ . In particular, we obtain the following identification of complexes:

$$\Omega_{\Lambda_{R,+}/R}^\bullet = K_{\Lambda_{R,+}}(\nabla_0) : [\Lambda_{R,+} \xrightarrow{\nabla_0} \Lambda_{R,+}].$$

Recall that  $\Lambda_{R,+} = R[w^{[k]}, k \in \mathbb{N}]_p^\wedge$  (see Lemma 3.62), therefore it follows that the de Rham complex  $\Omega_{\Lambda_{R,+}/R}^\bullet = K_{\Lambda_{R,+}}(\nabla_0)$  is acyclic in positive degrees and we have that  $R \xrightarrow{\sim} \Lambda_{R,+}^{\nabla_0=0}$ . Furthermore, in Proposition 4.30, we show a natural quasi-isomorphism of complexes

$$K_{\Lambda_{R,+}}(\nabla_{q,w}) \xrightarrow{\sim} K_{\Lambda_{R,+}}(\nabla_0).$$

Hence, we conclude that  $H^1(K_{\Lambda_{R,+}}(\nabla_{q,\tau})) \xrightarrow{\sim} H^1(K_{\Lambda_{R,+}}(\nabla_0)) = 0$ .  $\blacksquare$

*Proof of Lemma 3.67.* The proof is similar to the proof of Lemma 3.53. Using the  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism in (3.43), the exact sequence in (3.46) can be rewritten as follows:

$$0 \longrightarrow \Lambda_{R,+} \xrightarrow{p_1(\mu)^n} \tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1} \longrightarrow \tilde{\Lambda}_{R,+}/p_1(\mu)^n \longrightarrow 0.$$

Then, using the operator  $\nabla_{q,\tilde{\tau}}$  in (3.50) and the Koszul complex defined in (3.52), we obtain an exact sequence of Koszul complexes:

$$0 \longrightarrow K_{\Lambda_{R,+}}(\nabla_{q,\tau}) \xrightarrow{p_1(\mu)^n} K_{\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1}}(\nabla_{q,\tilde{\tau}}) \longrightarrow K_{\tilde{\Lambda}_{R,+}/p_1(\mu)^n}(\nabla_{q,\tilde{\tau}}) \longrightarrow 0.$$

Considering the associated long exact sequence, and noting that  $H^1(K_{\Lambda_{R,+}}(\nabla_{q,\tau})) = 0$  from Lemma 3.74, we obtain the following exact sequence:

$$0 \longrightarrow \Lambda_{R,+}^{\nabla_{q,\tau}=0} \xrightarrow{p_1(\mu)^n} (\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{\tau}}=0} \longrightarrow (\tilde{\Lambda}_{R,+}/p_1(\mu)^n)^{\nabla_{q,\tilde{\tau}}=0} \longrightarrow 0.$$

Since the action of  $1 \times \Gamma_0$  is continuous on  $\tilde{\Lambda}_{R,+}$  for the  $(p, p_1(\mu))$ -adic topology, therefore, we have  $(\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{\tau}}=0} = (\tilde{\Lambda}_{R,+}/p_1(\mu)^{n+1})^{1 \times \Gamma_0}$  for each  $n \in \mathbb{N}$ . Hence, from the preceding exact sequence we obtain that the sequence in (3.48) is exact, therefore, the sequence (3.47) is also exact.  $\blacksquare$

**3.4.5. Proof of Proposition 3.41.** Note that from the explicit description of  $A_R(1)/p_1(\mu)$  in Proposition 3.25 it is easy to see that reduction modulo  $\mu$  of  $p_1 : A_R \rightarrow A_R(1)$  induces an isomorphism  $p_1 : R \xrightarrow{\sim} (A_R(1)/p_1(\mu))^{1 \times \Gamma_R}$ . More generally, we have the following:

**Lemma 3.75.** *For  $n \in \mathbb{N}_{\geq 1}$ , reduction modulo  $\mu^n$  of the  $(\varphi, \Gamma_R^2)$ -equivariant map  $p_1 : A_R \rightarrow A_R(1)$  induces a  $(\varphi, \Gamma_R \times 1)$ -equivariant isomorphism  $p_1 : A_R/\mu^n \xrightarrow{\sim} (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R}$ .*

*Proof.* Note that  $p_1$  is  $(\varphi, \Gamma_R^2)$ -equivariant, so it is enough to show the bijectivity of the map modulo  $\mu^n$ . By considering the following diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & A_R/\mu & \xrightarrow{\mu^n} & A_R/\mu^{n+1} & \longrightarrow & A_R/\mu^n \longrightarrow 0 \\ & & \downarrow p_1 & & \downarrow p_1 & & \downarrow p_1 \\ 0 & \longrightarrow & (A_R(1)/p_1(\mu))^{1 \times \Gamma_R} & \xrightarrow{p_1(\mu)^n} & (A_R(1)/p_1(\mu)^{n+1})^{1 \times \Gamma_R} & \longrightarrow & (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R}, \end{array} \quad (3.53)$$

an easy induction on  $n \geq 1$  gives the claimed  $(\varphi, \Gamma_R \times 1)$ -equivariant isomorphism  $p_1 : A_R/\mu^{n+1} \xrightarrow{\sim} (A_R(1)/p_1(\mu)^{n+1})^{1 \times \Gamma_R}$ .  $\blacksquare$



**Remark 3.76.** From the description of the action of  $\Gamma_R^3$  on  $A_R(2)$  in Remark 3.20, we note that there is an induced action of  $\Gamma_R^3$  on  $A_R(2)/p_1(\mu)$ , where the action of the first component is identity. Moreover, we have a  $(\varphi, \Gamma_R^3)$ -equivariant map  $r_1 : A_R \rightarrow A_R(2)$ , where  $A_R$  is equipped with an action of  $\Gamma_R^3$  via projection on to the first coordinate. Then similar to above it can easily be shown that reduction modulo  $\mu$  of  $r_1 : A_R \rightarrow A_R(2)$  induces an isomorphism  $r_1 : R \xrightarrow{\sim} (A_R(2)/p_1(\mu))^{1 \times \Gamma_R \times \Gamma_R}$ .

Now, recall that we have the  $\varphi$ -equivariant multiplication map  $\Delta : A_R(1) \rightarrow A_R$ . The map  $\Delta$  induces an  $A_R/\mu^n$ -linear (via  $p_1$ ) and  $\varphi$ -equivariant maps  $\Delta : A_R(1)/p_1(\mu)^n \rightarrow A_R/\mu^n$  for  $n \in \mathbb{N}_{\geq 1}$ . For  $n = 1$ , using Lemma 3.75, the map  $\Delta$  restricts to a  $\varphi$ -equivariant isomorphism  $(A_R(1)/p_1(\mu))^{1 \times \Gamma_R} \xrightarrow{\sim} R$ . More generally, we have the following:

**Lemma 3.77.** *For  $n \in \mathbb{N}_{\geq 1}$ , reduction modulo  $p_1(\mu)^n$  of the  $\varphi$ -equivariant homomorphism  $\Delta : A_R(1) \rightarrow A_R$  restricts to a  $\varphi$ -equivariant isomorphism  $(A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R} \xrightarrow{\sim} A_R/\mu^n$ .*

*Proof.* Let us first note that using Lemma 3.43, together with Lemma 3.51, Lemma 3.53 and Remark 3.54 for  $p \geq 3$  and Lemma 3.66, Lemma 3.67 and Remark 3.68 for  $p = 2$ , we obtain that for each  $n \geq 1$ , the following  $(\varphi, \Gamma_R \times 1)$ -equivariant sequence is exact:

$$0 \longrightarrow (A_R(1)/p_1(\mu))^{1 \times \Gamma_R} \xrightarrow{p_1(\mu)^n} (A_R(1)/p_1(\mu)^{n+1})^{1 \times \Gamma_R} \longrightarrow (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R} \longrightarrow 0. \quad (3.54)$$

Next, we note that  $\Delta$  is  $\varphi$ -equivariant, so it is enough to show that the map modulo  $p_1(\mu)^n$  is bijective. Now, consider the following natural diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (A_R(1)/p_1(\mu))^{1 \times \Gamma_R} & \xrightarrow{p_1(\mu)^n} & (A_R(1)/p_1(\mu)^{n+1})^{1 \times \Gamma_R} & \longrightarrow & (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R} \longrightarrow 0 \\ & & \downarrow \Delta & & \downarrow \Delta & & \downarrow \Delta \\ 0 & \longrightarrow & A_R/\mu & \xrightarrow{\mu^n} & A_R/\mu^{n+1} & \longrightarrow & A_R/\mu^n \longrightarrow 0, \end{array}$$

where the top row is the exact sequence in (3.54). Using the diagram, an easy induction on  $n \geq 1$ , gives the  $\varphi$ -equivariant isomorphism  $\Delta : (A_R(1)/p_1(\mu)^{n+1})^{1 \times \Gamma_R} \xrightarrow{\sim} A_R/\mu^{n+1}$ . ■

Finally, recall that the ring  $A_R(1)$  is equipped with an action of  $(\varphi, \Gamma_R^2)$ , and the rings  $A_R(1)^{1 \times \Gamma_R}$  and  $(A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R}$  are equipped with a residual action of  $\Gamma_R = \Gamma_R \times 1 \subset \Gamma_R^2$ . Then the following observation proves Proposition 3.41:

**Lemma 3.78.** *For  $n \in \mathbb{N}_{\geq 1}$  the isomorphism  $(A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R} \xrightarrow{\sim} A_R/\mu^n$  of Proposition 3.41, is compatible with  $(\varphi, \Gamma_R)$ -action. Passing to the limit over  $n$  gives a  $(\varphi, \Gamma_R)$ -equivariant isomorphism  $A_R(1)^{1 \times \Gamma_R} \xrightarrow{\sim} A_R$ .*

*Proof.* The isomorphism in Proposition 3.41 is  $\varphi$ -equivariant. To check  $\Gamma_R$ -equivariance note that if  $g$  is in  $\Gamma_R$  and  $a$  in  $(A_R(1)/p_1(\mu)^n)$ , then we have that  $\Delta((g, g)a) = g(a)$ . So if  $a$  is  $(1 \times \Gamma_R)$ -invariant, then for  $g_1, g_2$  in  $\Gamma_R$ , we have that  $\Delta((g_1, g_2)f) = \Delta((g_1, g_1)f) = g_1(\Delta(f))$ . This proves the first claim. Next, as inverse limit commutes with right adjoint functors, in particular, with taking  $(1 \times \Gamma_R)$ -invariants, therefore, it follows that we have a  $(\varphi, \Gamma_R)$ -equivariant isomorphism  $A_R(1)^{1 \times \Gamma_R} = (\lim_n A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R} = \lim_n (A_R(1)/p_1(\mu)^n)^{1 \times \Gamma_R} \xrightarrow{\sim} \lim_n A_R/\mu^n = A_R$ . This proves the second claim. ■

## 4. AN INTEGRAL COMPARISON ISOMORPHISM

In this section, we will prove an integral comparison isomorphism for Wach modules, which will be the most important input in building a stratification on Wach modules in Subsection 5.2.5. We will use the setup and notations of Subsection 1.6.

**Definition 4.1** ([Abh23b, Definition 1.3], Lemma 3.10). A *Wach module* over  $A_R$  is a finitely generated  $A_R$ -module  $N$  equipped with a semilinear action of  $\Gamma_R$  satisfying the following:

- (1) The sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular on  $N$ .
- (2) The action of  $\Gamma_R$  is trivial on  $N/\mu N$ .
- (3)  $N$  is equipped with a Frobenius of finite  $[p]_q$ -height, i.e. an  $A_R$ -linear and  $\Gamma_R$ -equivariant isomorphism  $\varphi_N : (\varphi^* N)[1/[p]_q] = (A_R \otimes_{\varphi, A_R} N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ .

Say that  $N$  is *effective* if  $\varphi_N$  carries  $\varphi^* N$  into  $N$ . Denote the category of Wach modules over  $A_R$  as  $(\varphi, \Gamma)\text{-Mod}_{A_R}^{[p]_q}$  with morphisms between objects being  $A_R$ -linear,  $\Gamma_R$ -equivariant and  $\varphi_N$ -equivariant (after inverting  $[p]_q$ ) morphisms.

**Remark 4.2.** Note that the action of  $\Gamma_R$  is automatically continuous on  $N$  for the  $(p, \mu)$ -adic topology (see [Abh23b, Lemma 3.7]). Moreover, from [Abh23b, Proposition 3.9], note that for a Wach module  $N$  over  $A_R$ , the  $A_R[1/p]$ -module  $N[1/p]$  is finite projective, the  $A_R[1/\mu]$ -module  $N[1/\mu]$  is finite projective and by [Abh23b, Remark 3.10] the  $A_R[1/[p]_q]$ -module  $N[1/[p]_q]$  is finite projective. Furthermore, from loc. cit., the sequences  $\{p, [p]_q\}$  and  $\{[p]_q, p\}$  are regular on  $N$  and equivalent to condition (1) in Definition 4.1.

**Remark 4.3.** For  $R = O_F$ , a Wach module over  $A_F$  is necessarily finite free (see [Abh23b, Remark 1.6]).

From Remark 3.20 recall that we have a  $(\varphi, \Gamma_R^2)$ -equivariant maps  $p_i : A_R \rightarrow A_R(1)$  for  $i = 1, 2$ , where  $A_R$  is equipped with a  $\Gamma_R^2$ -action via projection onto the  $i^{\text{th}}$  coordinate. Moreover, we note that there is an induced action of  $\Gamma_R^2$  on  $A_R(1)/p_1(\mu)$ , where the action of the first component is identity. In this section, we will identify  $\Gamma_R$  with  $1 \times \Gamma_R$  and say that  $A_R(1)/p_1(\mu)$  is equipped with a natural continuous action of  $\Gamma_R$ .

**Remark 4.4.** Let  $N$  be a Wach module over  $A_R$ , then extending the isomorphism in Definition 4.1 (3) along the  $(\varphi, 1 \times \Gamma_R)$ -equivariant map  $p_2 : A_R \rightarrow A_R(1)/p_1(\mu)$ , we obtain an isomorphism  $(A_R(1)/p_1(\mu) \otimes_{\varphi, A_R} N)[1/[p]_q] \xrightarrow{\sim} (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)[1/[p]_q]$ . Now note that  $p_1([p]_q) = p \bmod p_1(\mu)A_R(1)$  and  $p_1([p]_q)/p_2([p]_q)$  is a unit in  $A_R(1)$  (see Lemma 3.15). So, by setting  $M := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\Gamma_R}$ , and taking  $\Gamma_R$ -invariants in the preceding isomorphism gives an  $R[1/p]$ -linear isomorphism  $(\varphi^* M)[1/p] = (R \otimes_{\varphi, R} M)[1/p] \xrightarrow{\sim} M[1/p]$ .

*Notation.* In this section, by the  $\varphi$ -equivariance of a morphism we always mean  $\varphi$ -equivariance after inverting  $p$ . However, we will not mention this explicitly everytime.

The goal of this section is to prove Theorem 4.5 below, which is an important ingredient for the proof of Theorem 5.9.

**Theorem 4.5.** *Let  $N$  be a Wach module over  $A_R$  and set  $M := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\Gamma_R}$  as an  $R$ -module equipped with tensor product Frobenius. Then we have a natural  $(\varphi, \Gamma_R)$ -equivariant isomorphism*

$$\begin{aligned} A_R(1)/p_1(\mu) \otimes_{p_1, R} M &\xrightarrow{\sim} A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N \\ a \otimes b \otimes x &\longmapsto ab \otimes x. \end{aligned} \tag{4.1}$$

Moreover,  $M$  is a finitely generated  $p$ -torsion free  $R$ -module and we have a  $\varphi$ -equivariant isomorphism of  $R$ -modules  $M \xrightarrow{\sim} N/\mu N$ .

**Remark 4.6.** For  $R = O_F$ , the  $R$ -module  $M$  in Theorem 4.5 is finite free of rank  $= \text{rk}_{A_F} N$ .

In Subsections 4.2 and 4.3, we will prove the comparison isomorphism (4.1) claimed in Theorem 4.5. Our proof is broadly divided into three main steps: a geometric descent for the action of  $\Gamma'_R$  (see Proposition 4.17) and a “two-step” arithmetic descent for the action of  $\Gamma_F$  (see Propositions 4.22 and 4.25, for  $p \geq 3$ , and Propositions 4.26 and 4.31, for  $p = 2$ ). Then in Subsection 4.4, we will put everything together to complete the proof of Theorem 4.5. We begin by interpreting the action of  $\Gamma_R$  on a Wach module as a  $q$ -connection.

**4.1. Wach modules and  $q$ -connections.** In this subsection, we will interpret Wach modules and its scalar extension to  $A_R(1)/p_1(\mu)$  as modules with  $q$ -connections similar to [Abh23b, Section 5]. We will work with the notation described in Subsection 3.3.3.

**Definition 4.7** ([MT20, Definition 2.2]). A module with  $q$ -connection over  $D$  is a right  $D$ -module  $N$  equipped with an  $A$ -linear map  $\nabla_q : N \rightarrow N \otimes q\Omega_{D/A}^1$  satisfying the Leibniz rule  $\nabla_q(xf) = \nabla_q(x)f + x \otimes d_q(f)$  for all  $f$  in  $D$  and  $x$  in  $N$ . The  $q$ -connection  $\nabla_q$  extends uniquely to a map of graded  $A$ -modules  $\nabla_q : N \otimes q\Omega_{D/A}^{\bullet} \rightarrow N \otimes q\Omega_{D/A}^{\bullet+1}$  satisfying  $\nabla_q((n \otimes \omega) \cdot \omega') = \nabla_q(n \otimes \omega) \cdot \omega' + (-1)^{\deg \omega} (n \otimes \omega) \cdot d_q(\omega')$ . The  $q$ -connection  $\nabla_q$  is said to be *flat* or *integrable* if  $\nabla_q \circ \nabla_q = 0$ .

**Example 4.8.** From Example 3.34, take  $D = A_F = O_F[[\mu]]$ ,  $A = A_R$  equipped with the action of  $\Gamma_R$  and  $\{\gamma_1, \dots, \gamma_d\}$  as topological generators of  $\Gamma'_R$  (see Subsection 3.1). Taking  $q = 1 + \mu$  and  $U_i = [X_i^b]$  for  $1 \leq i \leq d$ , we know that  $A_R$  satisfies the hypotheses of Definition 3.31. In particular,  $A_R$  is equipped with an  $A_F$ -linear  $q$ -connection  $\nabla_q : A_R \rightarrow q\Omega_{A_R/A_F}^1$ , given as  $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)} d \log(p_2([X_i^b]))$ . Now let  $N$  be a Wach module  $A_R$ . Then from [Abh23b, Proposition 5.3], the geometric  $q$ -connection  $\nabla_q : N \rightarrow N \otimes_{A_R} \Omega_{A_R/A_F}^1$  given as  $x \mapsto \sum_{i=1}^d \frac{\gamma_i(x)-x}{\mu} d \log([X_i^b])$  describes  $(N, \nabla_q)$  as a  $\varphi$ -module equipped with  $(p, [p]_q)$ -adically quasi-nilpotent flat  $q$ -connection over  $A_R$  (see [Abh23b, Subsection 5.2] for details).

**Example 4.9.** From Example 3.35, take  $D = \Lambda_F$  and  $A = A^{\text{PD}}$  as a  $\Lambda_F$ -algebra. Note that  $A^{\text{PD}}$  is equipped with a  $\Lambda_F$ -linear action of  $\Gamma'_R$  and we have  $\{\gamma_1, \dots, \gamma_d\}$  as topological generators of  $\Gamma'_R$  (see Subsection 3.1). Then setting  $q = 1 + \mu$  and  $U_i = [X_i^b]$  for  $1 \leq i \leq d$ , we know that  $A^{\text{PD}}$  satisfies the hypotheses of Definition 3.31. In particular,  $A^{\text{PD}}$  is equipped with a  $\Lambda_F$ -linear  $q$ -connection  $\nabla_q : A^{\text{PD}}(1) \rightarrow q\Omega_{A^{\text{PD}}(1)/\Lambda_F}^1$ , given as  $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)} d \log(p_2([X_i^b]))$ . Now let  $N$  be a Wach module over  $A_R$  and set  $N^{\text{PD}} := A^{\text{PD}} \otimes_{A_R} N$ , equipped with tensor product Frobenius and tensor product action of  $\Gamma_R$ . Then note that for any  $f \otimes y$  in  $N^{\text{PD}}$  and  $g$  in  $\Gamma_R$ , we have  $(g-1)(f \otimes y) = (g-1)f \otimes y + g(f) \otimes (g-1)y$  in  $\mu N^{\text{PD}}$ . Therefore, the operator  $\nabla_q : N^{\text{PD}} \rightarrow N^{\text{PD}} \otimes_{A^{\text{PD}}} \Omega_{A^{\text{PD}}/\Lambda_F}^1$  given as  $x \mapsto \sum_{i=1}^d \frac{\gamma_i(x)-x}{\mu} d \log([X_i^b])$  satisfies the assumptions of Definition 4.7. Moreover, the  $q$ -connection  $\nabla_q$  on  $N^{\text{PD}}$  is  $p$ -adically quasi-nilpotent using Example 3.35 and Example 4.8, and it is flat because  $\gamma_i$  commute with each other.

**Example 4.10.** From Example 3.36, take  $D$  to be  $\Lambda_R$  and  $A$  to be  $\overline{A}(1) = A_R(1)/p_1(\mu)$  as a  $\Lambda_R$ -algebra via the morphism of rings  $\iota_\Lambda : \Lambda_R \rightarrow \overline{A}(1)$ . Then, the  $\Lambda_R$ -algebra  $\overline{A}(1)$  is equipped with a  $\Lambda_R$ -linear (via  $\iota_\Lambda$ ) action of  $\Gamma'_R$  and we take  $\{\gamma_1, \dots, \gamma_d\}$  as topological generators of  $\Gamma'_R$  (see Subsection 3.1). Then setting  $q = 1 + \mu$  and  $U_i = [X_i^b]$  for  $1 \leq i \leq d$ , we know that  $\overline{A}(1)$  satisfies the hypotheses of Definition 3.31. In particular,  $\overline{A}(1)$  is equipped with a  $\Lambda_R$ -linear  $q$ -connection  $\nabla_q : \overline{A}(1) \rightarrow q\Omega_{\overline{A}(1)/\Lambda_R}^1$ , given as  $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f)-f}{p_2(\mu)} d \log(p_2([X_i^b]))$ . Now let  $N$  be a Wach module over  $A_R$  and set  $N(1) := A_R(1) \otimes_{p_2, A_R} N$ , equipped with tensor product Frobenius and tensor product action of  $\Gamma_R^2$ , where  $\Gamma_R^2$  acts on  $N$  via projection onto the second coordinate. Let  $\overline{N}(1) := N(1)/p_1(\mu) = \overline{A}(1) \otimes_{p_2, A_R} N$  equipped with induced action of Frobenius and  $\Gamma_R = 1 \times \Gamma_R$ . Note that for any  $f \otimes y$  in  $\overline{N}(1)$  and  $g$  in  $\Gamma_R$ , we have that  $(g-1)(f \otimes y) = (g-1)y \otimes y + g(f) \otimes (g-1)y$  is in  $p_2(\mu)\overline{N}(1)$ . Therefore, the operator  $\nabla_q : \overline{N}(1) \rightarrow \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^1$ , given as  $x \mapsto \sum_{i=1}^d \frac{\gamma_i(x)-x}{\mu} d \log([X_i^b])$  satisfies the assumptions of Definition 4.7. Moreover, the  $q$ -connection  $\nabla_q$  on  $\overline{N}(1)$  is  $p$ -adically quasi-nilpotent using Example 3.36 and Example 4.8, and it is flat because  $\gamma_i$  commute with each other.

**Lemma 4.11.** *Let  $N$  be a Wach module over  $A_R$  and  $\overline{A}(1) = A_R(1)/p_1(\mu)$ . Then the  $\overline{A}(1)$ -module  $\overline{A}(1) \otimes_{p_2, A_R} N$  is  $p$ -adically complete and  $p$ -torsion free.*

*Proof.* Note that the morphism  $p_2 : A_R \rightarrow A_R(1)$  is faithfully flat from Lemma 3.15. Moreover, from Definition 4.1 recall that  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular sequences on  $N$ . Then from the flatness of  $p_2$ , it follows that  $\{p, p_2(\mu)\}$  and  $\{p_2(\mu), p\}$  are regular sequences on  $A_R(1) \otimes_{p_2, A_R} N$ . Using the fact that  $p_1([p]_q)/p_2([p]_q)$  is a unit in  $A_R(1)$  (see Lemma 3.15), we get that  $p_1(\mu)^{p-1}$  is an element of  $(p, p_2(\mu))_{A_R(1)}$ . Now, since both  $\{p, p_2(\mu)\}$  and  $\{p_2(\mu), p\}$  are regular sequences on  $A_R(1) \otimes_{p_2, A_R} N$ ,

therefore, it follows that both  $\{p, p_1(\mu)^{p-1}\}$  and  $\{p_1(\mu)^{p-1}, p\}$  are regular sequences on  $A_R(1) \otimes_{p_2, A_R} N$ . In particular, using [Sta23, Tag 07DV] we get that both  $\{p, p_1(\mu)\}$  and  $\{p_1(\mu), p\}$  are regular sequences on  $A_R(1) \otimes_{p_2, A_R} N$ , and we conclude that  $\overline{A}(1) \otimes_{p_1, A_R} N$  is  $p$ -torsion free. Next, note that  $N$  is a finitely generated and  $(p, \mu)$ -adically complete  $A_R$ -module. So, it follows that  $A_R(1) \otimes_{p_2, A_R} N$  is a finitely generated and  $(p, p_2(\mu))$ -adically =  $(p, p_1(\mu))$ -adically complete  $A_R(1)$ -module, in particular, it is  $(p, p_1(\mu))$ -adically separated. Consequently, we get that  $\overline{A}(1) \otimes_{p_2, A_R} N$  is  $p$ -adically separated and it is clearly finitely generated over  $\overline{A}(1)$ -module, therefore,  $p$ -adically complete. ■

**Remark 4.12.** Let  $N$  be a Wach module over  $A_R$  and let  $A^{\text{PD}}$  be the ring defined in Example 3.35. Moreover, recall that we defined a ring  $E$  in Remark 3.46, which admits faithfully flat maps  $p_1 : A_F \rightarrow E$  and  $p_2 : A_R \rightarrow E$ , and we have that  $E/p_1(\mu) \xrightarrow{\sim} A^{\text{PD}}$ . Then, by an argument similar to the proof of Lemma 4.11, it follows that the  $A^{\text{PD}}$ -module  $A^{\text{PD}} \otimes_{A_R} N$  is  $p$ -adically complete and  $p$ -torsion free.

From Lemma 3.24, we have that  $t = \log(1 + \mu)$  converges in  $\mu\Lambda_F \subset \mu\Lambda_R$  and  $t/\mu$  is a unit.

**Proposition 4.13.** *Let  $N$  be a Wach module over  $A_R$  and  $\overline{A}(1) = A_R(1)/p_1(\mu)$ . Then, for  $1 \leq i \leq d$ , the series of operators  $\nabla_i^{\log} = \frac{\log \gamma_i}{t} = \frac{1}{t} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1}$  converge  $p$ -adically on  $\overline{N}(1) = \overline{A}(1) \otimes_{p_2, A_R} N$ . This defines a  $\Lambda_R$ -linear  $p$ -adically quasi-nilpotent flat connection on  $\overline{N}(1)$ , denoted as  $\nabla : \overline{N}(1) \rightarrow \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^1$  and given as  $x \mapsto \sum_{i=1}^d \nabla_i^{\log}(x) d\log([X_i^{\flat}])$ . The data of the connection  $\nabla$  on  $\overline{N}(1)$  is equivalent to the data of the  $q$ -connection  $\nabla_q$  described in Example 4.10, i.e. either may be recovered from the other. Moreover, the  $q$ -de Rham complex  $\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega_{\overline{A}(1)/\Lambda_R}^{\bullet}$  is naturally quasi-isomorphic to the de Rham complex  $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^{\bullet}$ . In particular, we have  $\overline{N}(1)^{\nabla_q=0} = \overline{N}(1)^{\nabla=0}$ .*

*Proof.* Our proof employs ideas similar to that of [Abh21, Lemmas 4.36 & 4.38], [MT20, Theorem 4.2] and [BMS18, Corollary 12.5]. Note that the  $\overline{A}(1)$ -module  $\overline{N}(1)$  is  $p$ -adically complete and  $p$ -torsion free by Lemma 4.11. Let us first show that for  $1 \leq i \leq d$  the series of operators  $\nabla_i^{\log} = \frac{1}{t} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1}$  converge on  $\overline{N}(1)$ . Indeed, note that we have

$$\frac{(\gamma_i - 1)^{k+1}}{k+1} = \mu k! \frac{\mu^k}{(k+1)!} \frac{(\gamma_i - 1)^{k+1}}{\mu^{k+1}},$$

where  $\mu^k/(k+1)!$  converges  $p$ -adically to 0 in  $\Lambda_R$  as  $k$  goes to  $+\infty$ , and  $t/\mu$  is a unit in  $\Lambda_R$ .

Next, let us check that the operator  $\nabla_i^{\log}$  satisfies the Leibniz rule. To show this, we claim that for  $x$  in  $\overline{N}(1)$ , we have

$$\lim_{m \rightarrow +\infty} \frac{\gamma_i^{p^m} - 1}{p^m}(x) = t \nabla_i^{\log}(x). \quad (4.2)$$

Indeed, note that since  $t^k/k!$  converges  $p$ -adically to 0 in  $\overline{A}(1)$  as  $k$  goes to  $+\infty$ , so we can write  $\gamma_i^n = \exp(nt \nabla_i^{\log})$ , for  $n \in \mathbb{Z}$ . Expanding the preceding exponential, we see that

$$\frac{\gamma_i^n - 1}{n} = t \nabla_i^{\log} + n \sum_{k \geq 2} n^{k-2} \frac{t^k}{k!} (\nabla_i^{\log})^k : \overline{N}(1) \rightarrow \overline{N}(1),$$

is well-defined. Taking  $n = p^m$  and letting  $m \rightarrow +\infty$ , we get the formula in (4.2). Now, for any  $f$  in  $\overline{A}(1)$  and  $x$  in  $\overline{N}(1)$ , we have that

$$(\gamma_i^{p^m} - 1)(fx) = (\gamma_i^{p^m} - 1)(f) \cdot x + \gamma_i^{p^m}(f)(\gamma_i^{p^m} - 1)(x).$$

Dividing out the preceding equality by  $tp^m$ , letting  $m \rightarrow +\infty$  and using (4.2) we get that  $\nabla_i^{\log}(fx) = \nabla_i^{\log}(f)x + f \nabla_i^{\log}(x)$ , where the first operator on the right is  $\nabla_i^{\log} := \log(\gamma_i)/t : \overline{A}(1) \rightarrow \overline{A}(1)$ , whose well-definedness and the equality (4.2) can be checked similar to above. In particular, we have shown that the operators  $\nabla_i^{\log}$  are well defined and satisfy a Leibniz rule.

To show that  $\nabla : \overline{N}(1) \rightarrow \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^1$ , given as  $x \mapsto \sum_{i=1}^d \nabla_i^{\log}(x) d\log([X_i^{\flat}])$ , is a well defined connection, we need to show that  $\nabla : \overline{A}(1) \rightarrow \Omega_{\overline{A}(1)/\Lambda_R}^1$  is the usual de Rham differential

$d : \overline{A}(1) \rightarrow \Omega_{\overline{A}(1)/\Lambda_R}^1$ . As each  $\nabla_i^{\log}$  is a continuous  $\Lambda_R$ -linear derivation, we can write  $\nabla_i^{\log} = h_i \circ d$  for some unique continuous  $\Lambda_R$ -linear map  $h_i : \Omega_{\overline{A}(1)/\Lambda_R}^1 \rightarrow \Lambda_R$ . Then it is easy to see that  $h_i(d[X_i^b]) = \nabla_i^{\log}([X_i^b]) = [X_i^b]$  for  $i = j$  and 0 otherwise. Therefore, we have  $d = \sum_i [X_i^b]^{-1} \nabla_i^{\log}(-) \otimes d[X_i^b] = \sum_i \nabla_i^{\log}(-) \otimes d \log([X_i^b])$ , as desired.

Next, let us show that the operators  $\nabla_i = \nabla_i^{\log}/[X_i^b] = (\log \gamma_i)/(t[X_i^b])$  are  $p$ -adically quasi-nilpotent. Indeed, we first note that from the commutativity of  $\varphi$  and  $\gamma_i$ , it follows that  $\log \gamma_i \circ \varphi = \varphi \circ \log \gamma_i$ . Therefore, it is easy to see that  $\nabla_i \circ \varphi = p[X_i^b]^{p-1} \varphi \circ \nabla_i$ . Recall that  $N$  is equipped with an  $A_R$ -linear isomorphism  $\varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ , and  $[p]_q/p$  is a unit in  $\Lambda_F$  (see Lemma 3.24), therefore,  $\overline{N}(1)$  is equipped with an  $\overline{A}(1)$ -linear isomorphism  $\varphi^*(\overline{N}(1))[1/p] \xrightarrow{\sim} \overline{N}(1)[1/p]$ . In particular, for any  $x$  in  $\overline{N}(1)$ , there exists  $r \in \mathbb{N}$  large enough, such that  $p^r x$  belongs to  $\varphi^*(\overline{N}(1))$ . Then, from the relation  $\nabla_i \circ \varphi = p\varphi \circ \nabla_i$ , we see that  $\nabla_i^k(p^r x)$  converges  $p$ -adically to 0 as  $k \rightarrow +\infty$ . Hence, it follows that  $\nabla_i^k(x) = p^{-r} \nabla_i^k(p^r x)$  converges  $p$ -adically to 0 as  $k \rightarrow +\infty$ , in particular,  $\nabla_i$  are  $p$ -adically quasi-nilpotent.

So far, we have shown that  $\nabla : \overline{N}(1) \rightarrow \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^1$  is a  $p$ -adically quasi-nilpotent connection and it is flat since  $\gamma_i$ , and therefore  $\nabla_i^{\log}$ , commute with each other. Moreover, we defined the connection  $\nabla$  using the action of  $\Gamma'_R$  and conversely, we have shown that the action of  $\Gamma'_R$  can be recovered by the formula  $\gamma_i := \exp(t\nabla_i^{\log})$  and it remains to check that, the action of  $\gamma_i$  thus obtained, is semi-linear. Note that the Leibniz rule implies that  $\frac{1}{k!}(t\nabla_i^{\log})^k(xf) = \sum_{a+b=k} \frac{1}{a!}(t\nabla_i^{\log})^a(x) \frac{1}{b!}(t\nabla_i^{\log})^b(f)$  for  $x$  in  $\overline{N}(1)$  and  $f$  in  $\overline{A}(1)$ . Now taking the sum over all  $k > 0$ , we get that  $\exp(t\nabla_i^{\log}(xf)) = \exp(t\nabla_i^{\log})(x) \exp(t\nabla_i^{\log})(f) = \exp(t\nabla_i^{\log})(x) \gamma_i(f)$ , as required.

From the discussion above, it is clear that we have a  $q$ -de Rham complex  $\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega_{\overline{A}(1)/\Lambda_R}^\bullet$  and a de Rham complex  $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^\bullet$ . We claim that these complexes are naturally quasi-isomorphic. Indeed, let us first note that the endomorphisms  $\nabla_{q,i}^{\log} = \frac{\gamma_i - 1}{\mu} : \overline{N}(1) \rightarrow \overline{N}(1)$  are commuting. So, let  $K_{\overline{N}(1)}(\nabla_{q,1}^{\log}, \dots, \nabla_{q,d}^{\log})$  denote the corresponding Koszul complex (see Definition A.8). Then we have a natural identification of complexes

$$K_{\overline{N}(1)}(\nabla_{q,1}^{\log}, \dots, \nabla_{q,d}^{\log}) = \overline{N}(1) \otimes_{\overline{A}(1)} q\Omega_{\overline{A}(1)/\Lambda_R}^\bullet.$$

Next, let  $\nabla : \overline{A}(1) \rightarrow \Omega_{\overline{A}(1)/\Lambda_R}^1$  denote the  $\Lambda_R$ -linear  $p$ -adically quasi-nilpotent flat connection on  $\overline{A}(1)$  (arising from the  $q$ -connection, see Proposition 3.37), given as  $f \mapsto \sum_{i=1}^d \nabla_i^{\log}(f) d \log([X_i^b])$ . Next, let  $t = \log(1 + \mu)$ , and again note that the endomorphisms  $\nabla_i^{\log} = \frac{\log \gamma_i}{t} : \overline{N}(1) \rightarrow \overline{N}(1)$  are commuting. So, let  $K_{\overline{N}(1)}(\nabla_1^{\log}, \dots, \nabla_d^{\log})$  denote the corresponding Koszul complex, and again we have a natural identification of complexes

$$K_{\overline{N}(1)}(\nabla_1^{\log}, \dots, \nabla_d^{\log}) = \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^\bullet.$$

Now, from the discussion above, recall that we have the relation  $\gamma_i = \exp(t\nabla_i^{\log})$  on  $\overline{N}(1)$ . Therefore, we can write

$$\nabla_{q,i} = \nabla_i^{\log} (1 + \sum_{k \geq 1} \frac{t^k}{k! \mu} (\nabla_i^{\log})^{k-1}), \quad (4.3)$$

where the term inside the parentheses is invertible, because  $t/\mu$  is a unit in  $\overline{A}(1)$  (see Lemma 3.24) and  $\mu^{k-1}/(k!)$  is topologically nilpotent because  $(\mu^{p-1}/p)^k$  is topologically nilpotent in  $\overline{A}(1)$  (see Proposition 3.25). Now, in the notation of Lemma A.9, let us set  $M = \overline{N}(1)$ ,  $f_i = \nabla_i^{\log}$  and take  $h_i$  to be the formula in the parentheses in (4.3), in particular,  $f_i h_i = \nabla_{q,i}^{\log}$ . Then, from Lemma A.9, we obtain a natural quasi-isomorphism of complexes

$$\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega_{\overline{A}(1)/\Lambda_R}^\bullet = K_{\overline{N}(1)}(\nabla_{q,1}^{\log}, \dots, \nabla_{q,d}^{\log}) \xrightarrow{\sim} K_{\overline{N}(1)}(\nabla_1^{\log}, \dots, \nabla_d^{\log}) = \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^\bullet.$$

Finally, from the quasi-isomorphism above, it follows that we have  $\overline{N}(1)^{\nabla_q=0} = \overline{N}(1)^{\Gamma'_R} \xrightarrow{\sim} \overline{N}(1)^{\nabla=0}$ . This concludes our proof.  $\blacksquare$

**Remark 4.14.** From remark 4.12, recall that the  $A^{\text{PD}}$ -module  $N^{\text{PD}} = A^{\text{PD}} \otimes_{A_R} N$  is  $p$ -adically complete and  $p$ -torsion free. Then, in Proposition 4.13, replacing  $\Lambda_R$  by  $\Lambda_F$ ,  $\overline{A}(1)$  by  $A^{\text{PD}}$  and  $\overline{N}(1)$  by  $N^{\text{PD}} = A^{\text{PD}} \otimes_{A_R} N$ , and using essentially the same arguments, we see that for  $1 \leq i \leq d$ , the series of operators  $\nabla_i^{\log} = \frac{\log \gamma_i}{t} = \frac{1}{t} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^k}{k+1}$  converge  $p$ -adically on  $N^{\text{PD}}$ . This defines a  $\Lambda_F$ -linear  $p$ -adically quasi-nilpotent flat connection on  $N^{\text{PD}}$ , denoted as  $\nabla : N^{\text{PD}} \rightarrow N^{\text{PD}} \otimes_{A^{\text{PD}}} \Omega_{A^{\text{PD}}/\Lambda_F}^1$  and given as  $x \mapsto \sum_{i=1}^d \nabla_i^{\log}(x) d \log([X_i^b])$ . The data of the connection  $\nabla$  on  $N^{\text{PD}}$  is equivalent to the data of the  $q$ -connection  $\nabla_q$  described in Example 4.9, i.e. either may be recovered from the other. Moreover, the  $q$ -de Rham complex  $N^{\text{PD}} \otimes_{A^{\text{PD}}} q\Omega_{A^{\text{PD}}/\Lambda_F}^\bullet$  is naturally quasi-isomorphic to the de Rham complex  $N^{\text{PD}} \otimes_{A^{\text{PD}}} \Omega_{A^{\text{PD}}/\Lambda_F}^\bullet$ . In particular, we have  $(N^{\text{PD}})^{\nabla_q=0} \xrightarrow{\sim} (N^{\text{PD}})^{\nabla=0}$ .

**Remark 4.15.** Let  $\overline{A}(1) = A_R(1)/p_1(\mu)$  and  $A^{\text{PD}}$  as in Example 3.35. Recall that we have a natural injective  $(\varphi, \Gamma_R)$ -equivariant homomorphism of rings  $p_2 : A^{\text{PD}} \rightarrow \overline{A}(1)$  (see Remark 3.39). Now let  $N$  be a Wach module over  $A_R$ , then we have that  $\overline{A}(1) \otimes_{p_2, A^{\text{PD}}} N^{\text{PD}} = \overline{A}(1) \otimes_{p_2, A_R} N = \overline{N}(1)$ , compatible with the action of  $(\varphi, \Gamma_R)$ . Then, using the compatibility of the corresponding connections on  $\overline{A}(1)$  and  $A^{\text{PD}}$  (see Remark 3.40) and the Leibniz rule for the connection on  $\overline{N}(1)$ , proven in Proposition 4.13, it follows that the respective connections on  $\overline{N}(1)$  and  $N^{\text{PD}}$  are compatible, in particular, the connection on  $\overline{N}(1)$  is given as the tensor product of respective connections on  $\overline{A}(1)$  and  $N^{\text{PD}}$ .

**4.2. Geometric descent.** From Example 4.8, recall that a Wach module  $N$  can be seen as a  $\varphi$ -module over  $A_R$  equipped with a  $(p, [p]_q)$ -adically quasi-nilpotent flat  $q$ -connection. Then from Proposition 4.13 we have that  $\overline{N}(1) = A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$  is a finite module over  $\overline{A}(1) := A_R(1)/p_1(\mu)$ , equipped with a  $p$ -adically quasi-nilpotent flat connection and a Frobenius-semilinear endomorphism  $\varphi$  (after inverting  $p$ ). Similarly, from Remark 4.14 we have that  $N^{\text{PD}} = A^{\text{PD}} \otimes_{A_R} N$  is a finite module over  $A^{\text{PD}}$ , equipped with a  $p$ -adically quasi-nilpotent flat connection and a Frobenius-semilinear endomorphism  $\varphi$  (after inverting  $p$ ). In order to prove Theorem 4.5, we need to first interpret the preceding  $\varphi$ -modules with connection, as  $F$ -crystals over the crystalline site  $\text{CRIS}(A^{\text{PD}}/\Lambda_F)$ . So we begin this subsection by recalling some standard facts about relevant crystalline sites and crystals. Using the formalism of crystalline sites, we will also be able to obtain the geometric descent step in the proof of Theorem 4.5 mentioned above.

**4.2.1. Crystalline site and  $F$ -crystals.** Let  $A$  be a  $p$ -adically complete PD-ring and let  $D$  be the  $p$ -adic completion of a divided power  $A$ -algebra. For  $m \geq 1$ , we set  $\Sigma_m = \text{Spec}(A/p^m)$  and  $X_m = \text{Spec}(D/p^m)$ . Let  $\text{CRIS}(X_m/\Sigma_m)$  denote the big crystalline site of  $X_m$  over  $\Sigma_m$  with the PD-structure given by  $p(A/p^m) + JA/p^m$ , where  $J$  denotes the PD-ideal of  $A$ , and let  $\mathcal{O}_{X_m/\Sigma_m}$  denote the structure sheaf of rings. Let  $\text{CR}(X_m/\Sigma_m)$  denote the category of finitely generated crystals of  $\mathcal{O}_{X_m/\Sigma_m}$ -modules over  $\text{CRIS}(X_m/\Sigma_m)$ . Note that the homomorphisms  $\Sigma_m \rightarrow \Sigma_{m+1}$  and  $X_m \rightarrow X_{m+1}$  induce the pullback functor  $i_{m,m+1}^* : \text{CR}(X_{m+1}/\Sigma_{m+1}) \rightarrow \text{CR}(X_m/\Sigma_m)$ . One can define  $\text{CRIS}(X_1/\Sigma_m)$  and  $\text{CR}(X_1/\Sigma_m)$  similarly and the pullback functor  $i_m^* : \text{CR}(X_m/\Sigma_m) \xrightarrow{\sim} \text{CR}(X_1/\Sigma_m)$  is an equivalence of categories by [Ber74, Chapitre IV, Théorème 1.4.1]. We define a finitely generated crystal of  $\mathcal{O}_{X/\Sigma}$ -modules  $\mathcal{E}$  on  $X/\Sigma$  to be a system  $(\mathcal{E}_m)_{m \geq 1}$ , where  $\mathcal{E}_m$  is an object of  $\text{CR}(X_m/\Sigma_m)_{\text{cris}}$  for each  $m \geq 1$  and we have isomorphisms  $i_{m,m+1}^* \mathcal{E}_{m+1} \xrightarrow{\sim} \mathcal{E}_m$ . A locally free crystal on  $X_1/\Sigma$  can be defined similarly, using  $\text{CRIS}(X_1/\Sigma_m)$ . Write  $\text{CR}(X/\Sigma)$  and  $\text{CR}(X_1/\Sigma)$  for the category of finitely generated crystals on  $X/\Sigma$  and  $X_1/\Sigma$  respectively, then the obvious pullback functor  $i^* : \text{CR}(X/\Sigma) \rightarrow \text{CR}(X_1/\Sigma)$  is an equivalence of categories. Let  $\text{MIC}_{\text{conv}}(D)$  denote the category of finitely generated  $D$ -modules equipped with an  $A$ -linear  $p$ -adically quasi-nilpotent flat connection. Then by [Ber74, Chapitre IV, Théorème 1.6.5], we have an equivalence of categories

$$\text{CR}(X/\Sigma) \xrightarrow{\sim} \text{MIC}_{\text{conv}}(D), \quad (4.4)$$

obtained by sending  $(\mathcal{E}_m)_{m \geq 1}$  to the inverse limit of the evaluation of  $\mathcal{E}_m$  on the object  $\text{Spec}(X_m) \xrightarrow{id} \text{Spec}(X_m)$  of the site  $\text{CRIS}(X_m/\Sigma_m)$ , equipped with the natural  $A$ -linear  $p$ -adically quasi-nilpotent flat connection.

Let us now describe  $F$ -crystals on  $\text{CRIS}(X/\Sigma)$ . Note that the absolute Frobenius on  $X_1$  and the natural Frobenius on  $\Sigma$  induce Frobenius pullbacks  $\varphi^* : \text{CR}(X_1/\Sigma_m) \rightarrow \text{CR}(X_1/\Sigma_m)$  and  $\varphi^* : \text{CR}(X_1/\Sigma) \rightarrow \text{CR}(X_1/\Sigma)$ . A finitely generated  $F$ -crystal  $\mathcal{E}$  on  $\text{CRIS}(X/\Sigma)$  is an object of  $\text{CR}(X_1/\Sigma)$  equipped with an isomorphism  $\varphi_{\mathcal{E}} : (\varphi^* i^* \mathcal{F})_{\mathbb{Q}} \xrightarrow{\sim} (i^* \mathcal{F})_{\mathbb{Q}}$  in the isogeny category  $\text{CR}(X/\Sigma) \otimes \mathbb{Q}$ . We will denote the category of finitely generated  $F$ -crystals on  $X/\Sigma$  as  $\text{CR}^{\varphi}(X/\Sigma)$ . Let  $\text{MIC}^{\varphi}(D)$  denote the following category : an object  $M$  is a finitely generated  $D$ -module, equipped with an  $A$ -linear  $p$ -adically quasi-nilpotent flat connection and an isomorphism  $\varphi_M : \varphi^* M[1/p] \xrightarrow{\sim} M[1/p]$ ; morphisms between two objects are  $D$ -linear maps compatible with respective Frobenii and connections. In fact, in the presence of Frobenius structures, similar to [MT20, Lemma 2.24], it can be shown that the  $p$ -adic quasi-nilpotence of the connection is automatic. Therefore, the equivalence in (4.4) refines to an equivalence,

$$\text{CR}^{\varphi}(X/\Sigma) \xrightarrow{\sim} \text{MIC}^{\varphi}(D), \quad (4.5)$$

**Remark 4.16.** In the discussion above, we can take  $A$  to be  $\Lambda_F$  with the PD-structure on it given by  $p(\Lambda_F/p^m) + J\Lambda_F/p^m$ , where  $J = ((\mu^{p-1}/p)^{[k]}, k \geq 1) \subset \Lambda_F$ , and  $D$  to be  $A^{\text{PD}}$ , i.e. we set  $\Sigma_m = \text{Spec}(\Lambda_F/p^m)$  and  $X_m = \text{Spec}(A^{\text{PD}}/p^m)$ . Now, recall that we have an isomorphism of rings  $\iota : \Lambda_R \xrightarrow{\sim} A^{\text{PD}}$  from (3.13). Moreover, we have an inclusion  $\text{Spec}(\Lambda_R/p^m) \hookrightarrow \text{Spec}(\overline{A}(1)/p^m)$  induced by the surjection  $\overline{A}(1) \twoheadrightarrow \Lambda_R$ , which is defined by sending  $[X_i^b] \mapsto X_i$  and  $T_i^{[k_i]} \mapsto 0$ , for all  $1 \leq i \leq d$ , and note that the composition  $\Lambda_R \xrightarrow{\iota_{\Lambda}} \overline{A}(1) \twoheadrightarrow \Lambda_R$  is the identity. In particular,  $\text{Spec}(\Lambda_R/p^m) \hookrightarrow \text{Spec}(\overline{A}(1)/p^m)$  is an object of  $\text{CRIS}(X_m/\Sigma_m)$ . Furthermore, using the isomorphism  $\iota : \Lambda_R \xrightarrow{\sim} A^{\text{PD}}$ , it is easy to see that  $\overline{A}(1)$  is the self product of  $A^{\text{PD}}$  in  $\text{CRIS}(X/\Sigma)$  with the two maps  $\iota_{\Lambda} : \Lambda_R \rightarrow \overline{A}(1)$  and  $p_2 : A^{\text{PD}} \rightarrow \overline{A}(1)$  describing the two projection maps.

**4.2.2. The action of  $\Gamma'_R$ .** As mentioned at the beginning of this subsection,  $N$  is a Wach module over  $A_R$ . From Example 4.10, recall that we have an isomorphism of  $\Lambda_F$ -algebras  $\Lambda_R = R[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]_p^{\wedge} \xrightarrow{\sim} A^{\text{PD}}$  (see the isomorphism  $\iota$  in (3.13)) and  $\overline{A}(1) = A_R(1)/p_1(\mu)$ . Then, the  $\overline{A}(1)$ -module  $\overline{N}(1) = \overline{A}(1) \otimes_{p_2, A_R} N$  is equipped with a Frobenius-semilinear endomorphism  $\varphi$  (after inverting  $p$ ) and an action of  $1 \times \Gamma_R$ , which induces a  $\Lambda_R$ -linear flat  $q$ -connection  $\nabla_q : \overline{N}(1) \rightarrow \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^1$ . Moreover, from Proposition 4.13 recall that the data of the  $q$ -connection on  $\overline{N}(1)$  is equivalent to the data of a connection  $\nabla : \overline{N}(1) \rightarrow \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^1$ . Now, we make the following claim:

**Proposition 4.17.** *Let  $N$  be a Wach module over  $A_R$  and set  $M_{\Lambda} := (\overline{A}(1) \otimes_{p_2, A_R} N)^{\nabla=0}$  as a  $\Lambda_R$ -module via  $\iota_{\Lambda} : \Lambda_R \xrightarrow{\sim} \overline{A}(1)^{\nabla=0}$  and equipped with an induced  $(\varphi, \Gamma_F)$ -action. Then we have a natural  $(\varphi, \nabla, \Gamma_F)$ -equivariant isomorphism*

$$\begin{aligned} \overline{A}(1) \otimes_{\Lambda_R} M_{\Lambda} &\xrightarrow{\sim} \overline{A}(1) \otimes_{p_2, A_R} N \\ a \otimes b \otimes x &\mapsto ab \otimes x. \end{aligned} \quad (4.6)$$

Moreover, we have  $(\varphi, \Gamma_F)$ -equivariant isomorphisms of finitely generated  $p$ -adically complete modules over  $\Lambda_R \xrightarrow[\sim]{(3.13)} A^{\text{PD}}$ ,

$$M_{\Lambda} \xrightarrow{\sim} A^{\text{PD}} \otimes_{A_R} N \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N. \quad (4.7)$$

Furthermore, the de Rham complex  $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^{\bullet}$  is acyclic in positive degrees.

*Proof.* To use the results from Subsection 4.2.1, we will work with the notations in Remark 4.16, i.e. we set  $\Sigma_m = \text{Spec}(\Lambda_F/p^m)$  and  $X_m = \text{Spec}(A^{\text{PD}}/p^m)$ . Now, note that the  $A^{\text{PD}}$ -module  $N^{\text{PD}}$  is equipped with a  $\Lambda_F$ -linear  $p$ -adically quasi-nilpotent flat connection  $\nabla : N^{\text{PD}} \rightarrow N^{\text{PD}} \otimes_{A^{\text{PD}}} \Omega_{A^{\text{PD}}/\Lambda_F}^1$ , in particular,  $N^{\text{PD}}$  is an object of  $\text{MIC}^{\varphi}(A^{\text{PD}})$  (see Remark 4.14). Then from the equivalence in (4.5), there exists a finitely generated  $F$ -crystal  $\mathcal{E}$  over  $\text{CRIS}(X/\Sigma)$  and we have  $\mathcal{E}(A^{\text{PD}}) = N^{\text{PD}}$ . Moreover, recall that we have a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings  $\iota : \Lambda_R \xrightarrow{\sim} A^{\text{PD}}$  from (3.13). Next, from Proposition 3.25 and Remark 3.27 recall that  $\overline{A}(1)$  is the  $p$ -adic completion of a PD-polynomial algebra over  $A^{\text{PD}}$  in variables  $T_1, \dots, T_d$ . Moreover,  $\overline{A}(1)$  is equipped with a  $\Lambda_R$ -linear flat

connection  $\nabla : \overline{A}(1) \rightarrow \Omega_{\overline{A}(1)/\Lambda_R}^1 = \overline{A}(1) \otimes_{A^{\text{PD}}} \Omega_{A^{\text{PD}}/\Lambda_F}^1$  (see Proposition 3.37 and Remark 3.40). Now consider the injection  $\text{Spec}(\Lambda_R/p^m) \hookrightarrow \text{Spec}(\overline{A}(1)/p^m)$  induced by the  $(\varphi, \Gamma_F)$ -equivariant surjection  $\overline{A}(1) \twoheadrightarrow \Lambda_R$  from Remark 4.16 and note that the composition  $\Lambda_R \xrightarrow{\iota_\Lambda} \overline{A}(1) \twoheadrightarrow \Lambda_R$  is the identity and  $(\varphi, \Gamma_F)$ -equivariant. So we have the following  $(\varphi, \Gamma_F)$ -equivariant morphisms in  $\text{CRIS}(X_m/\Sigma_m)$ , for  $m \geq 1$ ,

$$\begin{array}{ccc} \text{Spec}(\Lambda_R/p^m) & \hookrightarrow & \text{Spec}(\overline{A}(1)/p^m) & \text{Spec}(\Lambda_R/p^m) & \hookrightarrow & \text{Spec}(\overline{A}(1)/p^m) \\ \downarrow \iota & & \downarrow p_2 & \downarrow \text{id} & & \downarrow T_i^{[k_i] \mapsto 0} \\ \text{Spec}(A^{\text{PD}}/p^m) & \xrightarrow{\text{id}} & \text{Spec}(A^{\text{PD}}/p^m) & \text{Spec}(\Lambda_R/p^m) & \hookrightarrow & \text{Spec}(\overline{A}(1)/p^m). \end{array} \quad (4.8)$$

Evaluating the  $F$ -crystal  $\mathcal{E}$  on  $\text{Spec}(\Lambda_R/p^m) \hookrightarrow \text{Spec}(\overline{A}(1)/p^m)$  for each  $m \geq 1$ , and taking the limit over  $m$ , gives a finitely generated  $\overline{A}(1)$ -module  $\mathcal{E}(\overline{A}(1))$  equipped with a  $(\varphi, \Gamma_F)$ -action and a  $\Lambda_R$ -linear  $p$ -adically quasi-nilpotent flat connection. Now, using the diagram on the left in (4.8) and the fact that  $\mathcal{E}$  is an  $F$ -crystal, we get that  $\mathcal{E}(\overline{A}(1)) \xleftarrow{\sim} \overline{A}(1) \otimes_{p_2, A^{\text{PD}}} N^{\text{PD}}$  compatible with the respective  $(\varphi, \Gamma_F)$ -actions and connections, where the right hand term is equipped with the tensor product  $(\varphi, \Gamma_F)$ -action and the tensor product of respective connections on  $\overline{A}(1)$  and  $N^{\text{PD}}$ , described above. Next, note that the right vertical arrow of the right hand diagram of (4.8), factors through  $\text{Spec}(\overline{A}(1)/p^m) \xrightarrow{p_1} \text{Spec}(\Lambda_R/p^m) \hookrightarrow \text{Spec}(\overline{A}(1)/p^m)$ . Therefore, from the fact that  $\mathcal{E}$  is an  $F$ -crystal, we obtain a  $(\varphi, \Gamma_F)$ -equivariant isomorphism

$$\overline{\varepsilon} : \overline{A}(1) \otimes_{\Lambda_R} \mathcal{E}(\Lambda_R) \xrightarrow{\sim} \mathcal{E}(\overline{A}(1)) \xleftarrow{\sim} \overline{A}(1) \otimes_{p_2, A^{\text{PD}}} N^{\text{PD}} = \overline{N}(1),$$

compatible with the respective  $\Lambda_R$ -linear connections, where the  $\Lambda_R$ -linear connection on the source is given as  $\nabla \otimes 1$  with  $\nabla$  being the  $\Lambda_R$ -linear connection on  $\overline{A}(1)$  described above. Explicitly, for  $x$  in  $\overline{A}(1) \otimes_{\Lambda_R} \mathcal{E}(\Lambda_R)$ , the map  $\overline{\varepsilon}$  is given by the formula  $\overline{\varepsilon}(x) = \sum_{j_1, \dots, j_d \geq 0} (-1)^{j_1 + \dots + j_d} \prod_{i=1}^d \nabla_i^{j_i}(x) \otimes T_i^{[j_i]}$ , where  $\nabla_i = [X_i^\flat]^{-1} \nabla_i^{\log}$  (see Proposition 4.13 for the definition of  $\nabla_i^{\log}$ ). Now, using the  $(\varphi, \Gamma_R)$ -equivariant isomorphism  $\iota : \Lambda_R \xrightarrow{\sim} A^{\text{PD}}$ , let us note that we have a  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $\mathcal{E}(\Lambda_R) \xrightarrow{\sim} N^{\text{PD}}$ . Then from the interpretation of  $\overline{A}(1)$  as the self product of  $A^{\text{PD}}$  in  $\text{CRIS}(X/\Sigma)$  (see Remark 4.16), we see that the isomorphism  $\overline{\varepsilon}$  is just the stratification of  $N^{\text{PD}}$  over  $\overline{A}(1)$ .

Next, consider the de Rham complex  $\Omega_{\overline{A}(1)/\Lambda_R}^\bullet \xrightarrow{\sim} \overline{A}(1) \otimes_{P_R} \Omega_{P_R/R}^\bullet$  (see Remark 3.38 for the isomorphism and Remark 3.27 for the definition of  $P_R$ ), regarded as a complex of  $P_R$ -modules via the map  $P_R \rightarrow \overline{A}(1)$  induced by the inverse of the map (3.17). Then from the proof of [Ber74, Chapitre V, Lemme 2.1.2], it follows that the de Rham complex  $\Omega_{P_R/R}^\bullet$  is acyclic in positive degrees. Hence, from the isomorphism of the de Rham complexes

$$(\overline{A}(1) \otimes_{\Lambda_R} \mathcal{E}(\Lambda_R)) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^\bullet \xrightarrow{\sim} \overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^\bullet,$$

it follows that the de Rham complex  $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^\bullet$  is acyclic in positive degrees.

Now, by taking the horizontal sections for the respective connections in the isomorphism  $\overline{\varepsilon}$ , we obtain  $(\varphi, \Gamma_F)$ -equivariant isomorphisms of finite  $\Lambda_R$ -modules  $\mathcal{E}(\Lambda_R) \xrightarrow{\sim} \mathcal{E}(\overline{A}(1))^{\nabla=0} \xleftarrow{\sim} \overline{N}(1)^{\nabla=0}$ . In particular, from the isomorphism  $\overline{\varepsilon}$ , we deduce that  $\overline{A}(1)$ -linearly extending the natural inclusion  $\overline{N}(1)^{\nabla=0} \subset \overline{N}(1)$ , we obtain the following  $(\varphi, \nabla, \Gamma_F)$ -equivariant diagram

$$\begin{array}{ccc} \overline{A}(1) \otimes_{\Lambda_R} \mathcal{E}(\Lambda_R) & \xrightarrow{\sim} & \overline{N}(1) \\ \downarrow \iota_{\overline{\varepsilon}} & & \parallel \\ \overline{A}(1) \otimes_{\Lambda_R} \overline{N}(1)^{\nabla=0} & \xrightarrow{(4.6)} & \overline{A}(1) \otimes_{p_2, A_R} N. \end{array} \quad (4.9)$$

From the diagram, it follows that (4.6) is an isomorphism. Moreover, from the preceding discussions we also obtain  $(\varphi, \Gamma_F)$ -equivariant isomorphisms  $N^{\text{PD}} \xleftarrow{\sim} \mathcal{E}(\Lambda_R) \xrightarrow{\sim} \overline{N}(1)^{\nabla=0}$ , i.e. the claimed isomorphisms in (4.7). This allows us to conclude.  $\blacksquare$



**Remark 4.18.** In Subsection 3.4.1, we defined a  $(\varphi, 1 \times \Gamma_F)$ -equivariant map  $\Delta' : A_R(1) \rightarrow \tilde{\Lambda}_R$  and from the proof of Lemma 3.50, we have a  $(\varphi, \Gamma_F)$ -equivariant diagram

$$\Lambda_R \xrightarrow[\sim]{\iota_\Lambda} A_R(1)/p_1(\mu) \xrightarrow{\Delta'} \tilde{\Lambda}_R/p_1(\mu) \xleftarrow[\sim]{(3.23)} \Lambda_R.$$

Now, base changing the  $(\varphi, \nabla, \Gamma_F)$ -equivariant diagram in (4.9) along  $\Delta' \bmod p_1(\mu)$ , we obtain a  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$ , which is precisely the isomorphism (4.7), proven in Proposition 4.17. In particular, we see that the  $(\varphi, \Gamma_F)$ -equivariant isomorphism in (4.7) is the base change of the  $(\varphi, \nabla, \Gamma_F)$ -equivariant isomorphism in (4.6) along the map  $\Delta'$  from (3.25).

**4.3. Arithmetic descent.** In this subsection, we will carry out the descent for the arithmetic part of  $\Gamma_R$ , i.e.  $\Gamma_F$ . From (1.6), recall that  $\Gamma_F$  fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\text{tor}} \longrightarrow 1,$$

where, for  $p \geq 3$ , we have that  $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$  and for  $p = 2$ , we have that  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$ . Moreover, for  $p \geq 3$ , we have that  $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$  and the projection map  $\Gamma_F \rightarrow \Gamma_{\text{tor}}$ , admits a section  $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$ , where the second map is given as  $a \mapsto [a]$ , the Teichmüller lift of  $a$ , and the final isomorphism is induced by the  $p$ -adic cyclotomic character. Finally, for  $p = 2$ , we have that  $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$ , as groups.

**4.3.1. The action of  $\mathbb{F}_p^\times$ .** Our first goal is to carry out the descent for the action of  $\mathbb{F}_p^\times$  on Wach modules, in the case of  $p \geq 3$ . Let  $N$  be a Wach module over  $A_R$  and let us consider it as a module over  $R[[\mu]]$  via the  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings  $\iota : R[[\mu]] \xrightarrow{\sim} A_R$  (see Subsection 3.1.1). In particular, by abusing notations, we will consider  $N$  as an  $R[[\mu]]$ -module equipped with an  $R[[\mu]]$ -linear Frobenius isomorphism  $\varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$  and a semilinear and continuous action of  $\Gamma_F$  commuting with the Frobenius and such that the action of  $\Gamma_F$  is trivial on  $N/\mu N$ . Next, from Subsections 3.1.1 and 3.1.2, recall that we have  $\mu_0 = -p + \sum_{a \in \mathbb{F}_p} (1 + \mu)^{[a]}$  and  $\tilde{p} = \mu_0 + p$ , as elements of  $R[[\mu]]^{\mathbb{F}_p^\times}$ . Moreover, from Lemma 3.4, we have a  $(\varphi, \Gamma_0)$ -equivariant isomorphism of rings  $R[[\mu_0]] \xrightarrow{\sim} R[[\mu]]^{\mathbb{F}_p^\times}$ . Then, we claim the following:

**Proposition 4.19.** *Let  $N$  be a Wach module over  $A_R$ . Then  $N_0 := N^{\mathbb{F}_p^\times}$  is a finitely generated  $R[[\mu_0]]$ -module, equipped with a continuous and semilinear action of  $\Gamma_0$  such that the action of  $\Gamma_0$  is trivial on  $N_0/\mu_0 N_0 \xrightarrow{\sim} N/\mu N$ , and we have a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $R[[\mu]]$ -modules  $R[[\mu]] \otimes_{R[[\mu_0]]} N_0 \xrightarrow{\sim} N$ . Moreover,  $N_0$  is equipped with an  $R[[\mu_0]]$ -linear isomorphism  $\varphi^*(N_0)[1/\tilde{p}] \xrightarrow{\sim} N_0[1/\tilde{p}]$ , compatible with the respective natural actions of  $\Gamma_0$ .*

*Proof.* From (A.1), note that we have an  $\mathbb{F}_p^\times$ -decomposition  $N = \bigoplus_{i=0}^{p-2} N_i$ , where each  $N_i$  is a  $(p, \mu_0)$ -adically complete  $R[[\mu_0]]$ -module equipped with a continuous and semilinear action of  $\Gamma_0$ . Moreover, recall that  $R[[\mu]]$  is flat and finite of degree  $p - 1$  over  $R[[\mu_0]]$  (see Remark 3.8), so it follows that  $N$  is finitely generated as an  $R[[\mu_0]]$ -module. Since  $R[[\mu_0]]$  is noetherian, therefore, we get that the  $R[[\mu_0]]$ -submodule  $N_0 \subset N$  is finitely generated.

Next, by extending the natural  $R[[\mu_0]]$ -linear and  $(\varphi, \Gamma_F)$ -equivariant inclusion  $N_0 \subset N$ , along the map  $R[[\mu_0]] \rightarrow R[[\mu]]$ , we obtain a natural  $(\varphi, \Gamma_F)$ -equivariant map

$$N' := R[[\mu]] \otimes_{R[[\mu_0]]} N_0 \longrightarrow N, \tag{4.10}$$

and we will show that it is bijective. Recall that  $\mu_0$  is the product of  $\mu^{p-1}$  with a unit in  $R[[\mu]]$  (see Lemma 3.7), so we get that  $N$  is  $\mu_0$ -torsion free, and it follows that  $N_0$  is  $\mu_0$ -torsion free as well. As the natural map  $R[[\mu_0]] \rightarrow R[[\mu]]$  is flat (see Remark 3.8), therefore, we get that  $N'$  is  $\mu_0$ -torsion free =  $\mu^{p-1}$ -torsion free, hence,  $\mu$ -torsion free. Moreover, as  $N_0$  is  $(p, \mu_0)$ -adically complete, it also follows that  $N'$  is  $(p, \mu_0)$ -adically =  $(p, \mu)$ -adically complete. Furthermore, as  $N$  is  $(p, \mu)$ -adically complete and  $\mu$ -torsion free, therefore, showing that (4.10) is bijective, is equivalent to showing that (4.10) is

bijjective modulo  $\mu$ . We will first show that (4.10) is surjective. Indeed, note that we have an  $R$ -linear and  $\Gamma_F$ -equivariant surjection  $N \rightarrow N/\mu N$ . Then, from the decomposition in (A.1), the preceding surjection can be rewritten as  $\bigoplus_{i=0}^{p-2} N_i \rightarrow \bigoplus_{i=0}^{p-2} (N/\mu N)_i$ , and since the map is  $\Gamma_F$ -equivariant, it is termwise surjective, i.e. the induced  $R$ -linear map  $N_i \rightarrow (N/\mu N)_i$  is surjective for each  $0 \leq i \leq p-2$ . However, since the action of  $\Gamma_F$  is trivial on  $N/\mu N$ , we see that  $N/\mu N = \bigoplus_{i=0}^{p-2} (N/\mu N)_i = (N/\mu N)_0$ . Therefore, it follows that the natural  $R$ -linear map  $N_0 \rightarrow N/\mu N$  is surjective. As  $\mu$  is in the Jacobson radical of  $R[[\mu]]$ , therefore, by Nakayama Lemma, we get that the natural  $(\varphi, \Gamma_F)$ -equivariant map  $R[[\mu]] \otimes_{R[[\mu_0]]} N_0 \rightarrow N$  is surjective as well. Next, we consider the following diagram:

$$\begin{array}{ccc} N'/\mu N' & \longrightarrow & N/\mu N \\ \parallel & \nearrow & \\ N_0/\mu_0 N_0 & & \end{array} \quad (4.11)$$

where the top arrow is surjective by the discussion above, the slanted arrow is the natural  $R$ -linear and  $\Gamma_F$ -equivariant map induced by the inclusion  $N_0 \subset N$  and the left vertical equality follows from the following observation,

$$N'/\mu N' = (N'/\mu_0 N')/\mu N' = (R[[\mu]]/\mu^{p-1} \otimes_R N_0/\mu_0 N_0)/\mu = N_0/\mu_0.$$

To show that (4.10) is injective modulo  $\mu$ , it is enough to show that the slanted arrow in (4.11) is injective. So set  $N'' := \mu N \cap N_0 \subset N$  as an  $R[[\mu_0]]$ -module and note that we have a natural  $(\varphi, \Gamma_F)$ -equivariant inclusion  $N'' \rightarrow \mu N$ . The preceding inclusion induces a  $\Gamma_F$ -equivariant map  $N \rightarrow \mu N''/\mu^2 N''$ , where the source admits a trivial action of  $\mathbb{F}_p^\times$  and the target admits a non-trivial action of  $\mathbb{F}_p^\times$  (see Remark 4.20). So it follows that  $N'' = \mu^2 N \cap N_0 \subset N$ . Iterating the preceding argument  $p-2$  times, we obtain that  $N'' = \mu^{p-1} N \cap N_0 = \mu_0 N \cap N_0 \subset N = \mu_0 N_0$ , where the last equality follows because for any  $x$  in  $N$  and  $g$  in  $\mathbb{F}_p^\times$ , we have that  $g(\mu_0 x) = \mu_0 x$  if and only if  $g(x) = x$ , i.e.  $x$  is in  $N_0$ . Thus, from the preceding observation, it follows that the natural map  $N_0/\mu_0 N_0 \rightarrow N/\mu N$  is injective. Hence, from (4.11), we conclude that  $N_0/\mu_0 N_0 \xrightarrow{\sim} N/\mu N$  as  $R$ -modules and (4.10) is bijective. In particular, we also get that the action of  $\Gamma_0$  is trivial on  $N_0/\mu_0 N_0$ .

Finally, let us show the Frobenius finite height condition on  $N_0$ . Note that since  $\tilde{p}$  the product of  $[p]_q$  with a unit in  $R[[\mu]]$  (see Lemma 3.3), therefore, the Frobenius finite height condition on  $N$  can also be stated as an  $R[[\mu]]$ -linear isomorphism  $\varphi^*(N)[1/\tilde{p}] \xrightarrow{\sim} N[1/\tilde{p}]$ . Now, recall that the Frobenius on  $N$  commutes with the action of  $\Gamma_F$ , so taking the invariants of the preceding isomorphism under the action of  $\mathbb{F}_p^\times$  and using the  $(\varphi, \Gamma_F)$ -equivariant isomorphism in (4.10), we obtain that  $N_0$  is equipped with an  $R[[\mu_0]]$ -linear isomorphism  $\varphi^*(N_0)[1/\tilde{p}] \xrightarrow{\sim} N_0[1/\tilde{p}]$ , compatible with the natural action of  $\Gamma_0$  on each side. This allows us to conclude.  $\blacksquare$

**Remark 4.20.** Let  $N$  be a Wach module over  $A_R$ . Then, as the action of  $\Gamma_R$  is trivial on  $N/\mu N$ , therefore, we see that for each  $k \in \mathbb{N}$ , over the quotient  $\mu^k N/\mu^{k+1} N$ , the action of  $\Gamma_R$  is given via the  $p$ -adic cyclotomic character. In particular, it follows that  $\mathbb{F}_p^\times$  has a non-trivial action on  $\mu^k N/\mu^{k+1} N$ , for  $1 \leq k \leq p-2$ .

**Remark 4.21.** In Proposition 4.19, for  $R = O_F$ , note that the  $O_F[[\mu_0]]$ -module  $N_0$  is  $p$ -torsion free and  $\mu_0$ -torsion free. Moreover,  $N_0/\mu_0 N_0 \xrightarrow{\sim} N/\mu N$  is  $p$ -torsion free. Therefore, from [Abh23b, Lemma 3.5] and [Fon90, Proposition B.1.2.4], it follows that  $N_0$  is finite free over  $O_F[[\mu_0]]$ .

**Proposition 4.22.** *Let  $N$  be a Wach module over  $A_R$  and let  $M_\Lambda = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R}$  be the  $\Lambda_R$ -module from Proposition 4.17. Then  $M_{\Lambda,0} := M_\Lambda^{1 \times \mathbb{F}_p^\times}$  is a finitely generated module over  $\Lambda_{R,0} = \Lambda_R^{\mathbb{F}_p^\times}$ , equipped with an induced semilinear and continuous action of  $\Gamma_0$  such that the action of  $\Gamma_0$  is trivial on  $M_{\Lambda,0}/\mu_0 M_{\Lambda,0}$ , and we have a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $\Lambda_R$ -modules  $\Lambda_R \otimes_{\Lambda_{R,0}} M_{\Lambda,0} \xrightarrow{\sim} M_\Lambda$ . Moreover,  $M_{\Lambda,0}$  is equipped with an  $\Lambda_{R,0}$ -linear isomorphism  $\varphi^*(M_{\Lambda,0})[1/p] \xrightarrow{\sim} M_{\Lambda,0}[1/p]$ , compatible with the natural action of  $\Gamma_0$  on each side.*

*Proof.* From Proposition 4.17, let us recall that  $M_\Lambda$  is a finitely generated  $\Lambda_R$ -module and we have a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $\Lambda_R$ -modules  $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$  via the composition  $A_R \rightarrow A^{\text{PD}} \xrightarrow{\sim} \Lambda_R$ , where the last isomorphism is the inverse of (3.13). Now, consider  $N$  as an  $R[[\mu]]$ -module via the  $(\varphi, \Gamma_F)$ -equivariant isomorphism of rings  $\iota : R[[\mu]] \xrightarrow{\sim} A_R$  (see Subsection 3.1.1), equipped with a  $(\varphi, \Gamma_F)$ -action (see Subsection 4.3.1). Then, from Proposition 4.19, we have that  $N_0 := N^{\mathbb{F}_p^\times}$  is a finitely generated  $R[[\mu_0]]$ -module, equipped with an induced action of  $(\varphi, \Gamma_0)$ . Moreover, from Proposition 4.19, we note that  $R[[\mu]]$ -linearly extending the natural inclusion  $N_0 \subset N$ , induces a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $R[[\mu]]$ -modules  $R[[\mu]] \otimes_{R[[\mu_0]]} N_0 \xrightarrow{\sim} N$ . Combining this with the preceding discussion, it follows that we have a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $\Lambda_R$ -modules  $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{R[[\mu_0]]} N_0$ . Using that  $\Lambda_{R,0} = \Lambda_R^{\mathbb{F}_p^\times}$  (see Construction 3.52), together with the preceding isomorphism, we obtain a  $(\varphi, \Gamma_0)$ -equivariant isomorphism of  $\Lambda_{R,0}$ -modules

$$M_{\Lambda,0} = M_\Lambda^{\mathbb{F}_p^\times} \xrightarrow{\sim} \Lambda_{R,0} \otimes_{R[[\mu_0]]} N_0. \quad (4.12)$$

In particular, the  $\Lambda_{R,0}$ -module  $M_{\Lambda,0}$  is finitely generated. Now, let  $g$  be any element of  $\Gamma_0$ . Then note that for any  $f$  in  $\Lambda_{R,0}$  and  $y$  in  $N_0$ , we have that

$$(\gamma_0 - 1)fy = (\gamma_0 - 1)f \cdot y + \gamma_0(f)(\gamma_0 - 1)y \in \mu_0(\Lambda_{R,0} \otimes_{A_R} N).$$

From the  $(\varphi, \Gamma_0)$ -equivariant isomorphism in (4.12), it follows that for any  $x$  in  $M_{\Lambda,0}$ , we have that  $(g - 1)x$  is an element of  $\mu_0 M_{\Lambda,0}$ . Furthermore, from (4.12) it also follows that  $\Lambda_R$ -linearly extending the natural inclusion  $M_{\Lambda,0} \subset M_\Lambda$ , gives a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $\Lambda_R$ -modules

$$\begin{aligned} \Lambda_R \otimes_{\Lambda_{R,0}} M_{\Lambda,0} &\xrightarrow{\sim} M_\Lambda \\ a \otimes x &\longmapsto ax. \end{aligned} \quad (4.13)$$

Finally, note that from Proposition 4.19, we have an  $R[[\mu_0]]$ -linear isomorphism  $\varphi^*(N_0)[1/\tilde{p}] \xrightarrow{\sim} N_0[1/\tilde{p}]$  compatible with the action of  $\Gamma_0$  on each side. Then, by extending this isomorphism  $\Lambda_{R,0}$ -linearly, using the  $(\varphi, \Gamma_F)$ -equivariant isomorphism in (4.12) and noting that  $\tilde{p}/p$  is a unit in  $\Lambda_{F,0}$  from Lemma 3.24, we obtain a  $\Lambda_{R,0}$ -linear isomorphism  $\varphi^*(M_{\Lambda,0})[1/p] \xrightarrow{\sim} M_{\Lambda,0}[1/p]$  compatible with the action of  $\Gamma_0$  on each side. Hence, the proposition is proved.  $\blacksquare$

**4.3.2. The action of  $1 + p\mathbb{Z}_p$ .** In this subsection, we will assume  $p \geq 3$  and show the descent step, for the action of  $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + p\mathbb{Z}_p)$ . Let  $N$  be a Wach module over  $A_R$ , let  $M_\Lambda = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma'_R}$  be the  $\Lambda_R$ -module from Proposition 4.17 and let  $M_{\Lambda,0} = M_\Lambda^{1 \times \mathbb{F}_p^\times}$  be the  $\Lambda_{R,0}$ -module from Proposition 4.22. Let  $\gamma_0$  be a topological generator of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + pa$ , for a unit  $a$  in  $\mathbb{Z}_p$ . Then, from Proposition 4.22 note that for any  $x$  in  $M_{\Lambda,0}$ , we have that  $(\gamma_0 - 1)x$  is an element of  $\mu_0 M_{\Lambda,0}$ . Set  $s := \mu_0/p$  in  $\Lambda_{R,0}$ , and from Lemma 3.58 recall that  $(\gamma_0 - 1)s = u\mu_0$ , for some unit  $u$  in  $\Lambda_{F,0}$ . Therefore, we see that the following operator is well-defined

$$\begin{aligned} \nabla_{q,s} : M_{\Lambda,0} &\longrightarrow M_{\Lambda,0} \\ x &\longmapsto \frac{(\gamma_0 - 1)x}{(\gamma_0 - 1)s}. \end{aligned} \quad (4.14)$$

As the operator  $\nabla_{q,s}$  is an endomorphism of  $M_{\Lambda,0}$ , we can define the following two term Koszul complex:

$$K_{M_{\Lambda,0}}(\nabla_{q,s}) : [M_{\Lambda,0} \xrightarrow{\nabla_{q,s}} M_{\Lambda,0}]. \quad (4.15)$$

**Remark 4.23.** Considering  $s$  as a variable, similar to Remark 3.59, the operator  $\nabla_{q,s}$  in (4.14), may also be considered as a  $q$ -connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32. Then, (4.15) is the  $q$ -de Rham complex arising from such a  $q$ -connection.

**Proposition 4.24.** *The series of operators  $\nabla_0^{\log} = \frac{\log \gamma_0}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}}{k+1}$  converge  $p$ -adically on  $M_{\Lambda,0}$ . Let  $z := t^{p-1}/p$  in  $\Lambda_{R,0}$ , then the operator  $\nabla_0 := \frac{1}{(p-1)z} \nabla_0^{\log}$  defines an*

$R$ -linear  $p$ -adically quasi-nilpotent flat connection on  $M_{\Lambda,0}$ , denoted  $\nabla : M_{\Lambda,0} \rightarrow M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega_{\Lambda_{R,0}/R}^1$  and given as  $x \mapsto \nabla_0(x)dz$ . The data of the connection  $\nabla$  on  $M_{\Lambda,0}$  is equivalent to the data of the  $q$ -connection  $\nabla_{q,s}$  from (4.14), i.e. either may be recovered from the other. Moreover, the  $q$ -de Rham complex  $K_{M_{\Lambda,0}}(\nabla_{q,s})$  in (4.15) is naturally quasi-isomorphic to the de Rham complex  $M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega_{\Lambda_{R,0}/R}^\bullet$ . In particular, we have that  $M_{\Lambda,0}^{\nabla_{q,s}=0} \xrightarrow{\sim} M_{\Lambda,0}^{\nabla_0=0}$ .

*Proof.* Recall that  $s = \mu_0/p = ut^{p-1}/p$ , for a unit  $u$  in  $\Lambda_{F,0}$  (see Construction 3.52, in particular, (3.30) and the discussion preceding it). Moreover, from Lemma 3.57 we have that  $(\gamma_0 - 1)s = (\gamma_0 - 1)\frac{\mu_0}{p} = v\mu_0$ , for a unit  $v$  in  $\Lambda_{F,0}$ . Now let  $z = t^{p-1}/p$  and we write  $(1 + pa)^{p-1} = 1 + pb$ , where  $b$  is a unit in  $\mathbb{Z}_p$ . Then note that we have

$$(\gamma_0 - 1)z = (\gamma_0 - 1)\frac{t^{p-1}}{p} = (\chi(\gamma_0)^{p-1} - 1)\frac{t^{p-1}}{p} = ((1 + pa)^{p-1} - 1)\frac{t^{p-1}}{p} = bt^{p-1} = u^{-1}b\mu_0. \quad (4.16)$$

Therefore, it follows that the complex  $K_{M_{\Lambda,0}}(\nabla_{q,s})$  is quasi-isomorphic to the following complex

$$K_{M_{\Lambda,0}}(\nabla_{q,z}) : [M_{\Lambda,0} \xrightarrow{\nabla_{q,z}} M_{\Lambda,0}]. \quad (4.17)$$

Rest of the proof is similar to the proof of Proposition 4.13, with some changes. To avoid confusion, we provide a sketch.

Let us first show that  $\nabla_0^{\log} := \frac{\log(\gamma_0)}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}}{k+1}$ , converge as a series of operators on  $M_{\Lambda,0}$ . Indeed, let  $x$  be any element of  $M_{\Lambda,0}$ , then using Proposition 4.22, we can write  $(\gamma_0 - 1)x = t^{p-1}x_1$ , for some  $x_1$  in  $M_{\Lambda,0}$ . Let us note that  $\log(\chi(\gamma_0)) = \log(1 + pa) = pc$ , where  $c$  is a unit in  $\mathbb{Z}_p$ , and we also have  $(\gamma_0 - 1)t^{p-1} = ((1 + pa)^{p-1} - 1)t^{p-1} = pbt^{p-1}$ , where  $b$  is a unit in  $\mathbb{Z}_p$ . Therefore, an easy induction on  $k \in \mathbb{N}$ , shows that  $(\gamma_0 - 1)^{k+1}x = p^k t^{p-1} x_{k+1}$ , for some  $x_{k+1}$  in  $M_{\Lambda,0}$ . In particular, we get that

$$\begin{aligned} \nabla_0^{\log}(x) &= \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}(x)}{k+1} \\ &= \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{p^k t^{p-1} x_{k+1}}{k+1} \\ &= \frac{t^{p-1}}{pc} \sum_{k \in \mathbb{N}} (-1)^k \frac{p^k x_{k+1}}{k+1}, \end{aligned}$$

which converges in  $\frac{t^{p-1}}{p} M_{\Lambda,0} = z M_{\Lambda,0}$ , since  $M_{\Lambda,0}$  is  $p$ -adically complete. Moreover, it also follows that the operator  $\nabla_0 := \frac{1}{(p-1)z} \nabla_0^{\log}$  is well-defined on  $M_{\Lambda,0}$ .

Next, similar to the case of  $\nabla_i^{\log}$  in the proof of Proposition 4.13, it can be shown that  $\nabla_0^{\log}$ , and therefore  $\nabla_0 = \frac{1}{(p-1)z} \nabla_0^{\log}$  satisfies a Leibniz rule, i.e.  $\nabla_0(fx) = \nabla_0(f)x + f\nabla_0(x)$ , where the first operator on the right is  $\nabla_0 := \frac{\log(\gamma_0)}{(p-1)z \log(\chi(\gamma_0))} : \Lambda_{R,0} \rightarrow \Lambda_{R,0}$  whose well-definedness can be checked similar to above. Moreover, note that the operator  $\nabla_0$  is flat by definition. Furthermore, similar to the case of  $\nabla_i$  in the proof of Proposition 4.13, it can be shown that the operator  $\nabla_0 : \Lambda_{R,0} \rightarrow \Omega_{\Lambda_{R,0}/R}^1$  is the continuous de Rham differential operator  $d : \Lambda_{R,0} \rightarrow \Omega_{\Lambda_{R,0}/R}^1$ . So, in particular, the operator  $\nabla : M_{\Lambda,0} \rightarrow M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega_{\Lambda_{R,0}/R}^1$ , given as  $x \mapsto \nabla_0(x)dz$ , is a well defined flat connection. Now, let us show that the operator  $\nabla_0$  is  $p$ -adically quasi-nilpotent. Indeed, we first note that from the commutativity of  $\varphi$  and  $\gamma_0$ , it follows that  $\nabla_0^{\log} \circ \varphi = \varphi \circ \nabla_0^{\log}$ . Therefore, it is easy to see that  $\nabla_0 \circ \varphi = p^{p-1} \varphi \circ \nabla_0$ . Recall that  $M_{\Lambda,0}$  is equipped with a  $\Lambda_{R,0}$ -linear isomorphism  $\varphi^*(M_{\Lambda,0})[1/p] \xrightarrow{\sim} M_{\Lambda,0}[1/p]$ , compatible with action of  $\Gamma_0$  on each side. In particular, for any  $x$  in  $M_{\Lambda,0}$ , there exists  $r \in \mathbb{N}$  large enough such that  $p^r x$  belongs to  $\varphi^*(M_{\Lambda,0})$ . Then from the relation,  $\nabla_0 \circ \varphi = p^{p-1} \varphi \circ \nabla_0$ , we see that  $\nabla_0^k(p^r x)$  converges  $p$ -adically to 0 as  $k \rightarrow +\infty$ . Hence, it follows that  $\nabla_0^k(x) = p^{-r} \nabla_0^k(p^r x)$  converges  $p$ -adically to 0 as  $k \rightarrow +\infty$ , in particular,  $\nabla_0$  is  $p$ -adically quasi-nilpotent.

So far, we have defined the  $p$ -adically quasi-nilpotent flat connection  $\nabla$  using the action of  $\Gamma_0$  and conversely, we have shown that the action of  $\Gamma_0$  can be recovered by the formula  $\gamma_0 := \exp(\log(\chi(\gamma_0))\nabla_0^{\log})$ . Again, similar to case of  $\gamma_i$  in the proof of Proposition 4.13, using the Leibniz rule for  $\nabla_0$ , it can be checked that the action of  $\gamma_0$  thus obtained, is semilinear.

Finally, it remains to compare the  $q$ -de Rham complex in (4.15) with the de Rham complex  $M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega_{\Lambda_{R,0}/R}^\bullet = M_{\Lambda,0} \otimes_{\Lambda_{F,0}} \Omega_{\Lambda_{F,0}/O_F}^\bullet$ . As  $\nabla_0$  is an endomorphism of  $M_{\Lambda,0}$ , let  $K_{M_{\Lambda,0}}(\nabla_0)$  denote the corresponding Koszul complex in the sense of Definition A.8. Then we have an identification of complexes:

$$M_{\Lambda,0} \otimes_{\Lambda_{F,0}} \Omega_{\Lambda_{F,0}/O_F}^\bullet = K_{M_{\Lambda,0}}(\nabla_0) : [\Lambda_{R,0} \xrightarrow{\nabla_0} \Lambda_{R,0}].$$

Now, recall that we have  $\gamma_0 = \exp(\log(\chi(\gamma_0))\nabla_0^{\log})$ . Therefore, we can write

$$\nabla_{q,z} = \frac{\gamma_0 - 1}{(\gamma_0 - 1)z} = \frac{\log(\chi(\gamma_0))(p-1)z}{(\gamma_0 - 1)z} \nabla_0 (1 + \sum_{k \geq 1} \frac{\log(\chi(\gamma_0))^k}{(k+1)!} (\nabla_0^{\log})^k). \quad (4.18)$$

Recall that  $(\gamma_0 - 1)z = pbz$  and  $\log(\chi(\gamma_0)) = pc$ , for units  $b$  and  $c$  in  $\mathbb{Z}_p$ . Therefore, in (4.18), we have that  $\frac{\log(\chi(\gamma_0))(p-1)z}{(\gamma_0 - 1)z} = \frac{c(p-1)}{b}$  is a unit and it is clear that the term inside the parentheses converges  $p$ -adically to a unit. Now, in the notation of Lemma A.9, let us set  $i = 1$ ,  $M = M_{\Lambda,0}$ ,  $f_1 = \nabla_0$  and take  $h_1$  to be the product of  $\frac{c(p-1)}{b}$  with the formula in the parentheses of (4.18), in particular,  $f_1 h_1 = \nabla_{q,z}$ . Then, from Lemma A.9, we obtain a natural quasi-isomorphism of complexes

$$K_{M_{\Lambda,0}}(\nabla_{q,s}) \xrightarrow{\sim} K_{M_{\Lambda,0}}(\nabla_{q,z}) \xrightarrow{\sim} K_{M_{\Lambda,0}}(\nabla_0).$$

In particular, we get that  $M_{\Lambda,0}^{\nabla_{q,s}=0} \xrightarrow{\sim} M_{\Lambda,0}^{\nabla_0=0}$ . This allows us to conclude.  $\blacksquare$

**Proposition 4.25.** *Set  $M := M_{\Lambda,0}^{\nabla_0=0}$  as an  $R$ -module via the isomorphism  $R \xrightarrow{\sim} \Lambda_{R,0}^{\nabla_0=0}$ . Then  $M$  is finitely generated over  $R$  and we have a natural  $(\varphi, \nabla_0)$ -equivariant isomorphism*

$$\begin{aligned} \Lambda_{R,0} \otimes_R M &\xrightarrow{\sim} M_{\Lambda,0} \\ a \otimes x &\longmapsto ax. \end{aligned} \quad (4.19)$$

Moreover, the de Rham complex  $M_{\Lambda,0} \otimes_{\Lambda_{F,0}} \Omega_{\Lambda_{F,0}/O_F}^\bullet$  is acyclic in positive degrees. In particular, from Proposition 4.24, we have  $H^1(K_{M_{\Lambda,0}}(\nabla_{q,s})) = 0$ .

*Proof.* The proof is similar to the proof of Proposition 4.17. For the sake of completeness and to avoid confusion, we give the details.

We will use the results from Subsection 4.2.1, by setting  $\Sigma_m = \text{Spec}(R/p^m)$  and  $X_m = \text{Spec}(\Lambda_{R,0}/p^m)$ . Let us first note that from Proposition 4.24, the  $\Lambda_{R,0}$ -module  $M_{\Lambda,0}$  is equipped with a Frobenius endomorphism (after inverting  $p$ ) and an  $R$ -linear  $p$ -adically quasi-nilpotent flat connection  $\nabla : M_{\Lambda,0} \rightarrow M_{\Lambda,0} \otimes_{\Lambda_{F,0}} \Omega_{\Lambda_{F,0}/O_F}^1$ , in particular,  $M_{\Lambda,0}$  is an object of  $\text{MIC}^\varphi(\Lambda_{R,0})$ . Then, from the equivalence in (4.5), there exists a finitely generated  $F$ -crystal  $\mathcal{E}$  over  $\text{CRIS}(X/\Sigma)$  and we have  $\mathcal{E}(\Lambda_{R,0}) = M_{\Lambda,0}$ . Next, from Construction 3.52 recall that  $\Lambda_{R,0}$  is the  $p$ -adic completion of a PD-polynomial algebra over  $R$  in the variable  $s$  (see (3.30)). Moreover,  $\Lambda_{R,0}$  is equipped with an  $R$ -linear flat connection  $\nabla : \Lambda_{R,0} \rightarrow \Omega_{\Lambda_{R,0}/R}^1 = \Lambda_{R,0} \otimes_{\Lambda_{F,0}} \Omega_{\Lambda_{F,0}/O_F}^1$  (see the proofs of Proposition 4.24 and Lemma 3.60). Now, consider the injective map  $\text{Spec}(R/p^m) \hookrightarrow \text{Spec}(\Lambda_{R,0}/p^m)$  induced by the  $\varphi$ -equivariant surjection  $\Lambda_{R,0} \twoheadrightarrow R$ , sending  $s^{[k]} \mapsto 0$  for each  $k \geq 1$ , and note that the composition  $R \rightarrow \Lambda_{R,0} \twoheadrightarrow R$  is the identity and  $\varphi$ -equivariant. So we have the following  $\varphi$ -equivariant diagram in  $\text{CRIS}(X_m/\Sigma_m)$ , for  $m \geq 1$ ,

$$\begin{array}{ccc} \text{Spec}(R/p^m) & \hookrightarrow & \text{Spec}(\Lambda_{R,0}/p^m) \\ \downarrow \text{id} & & \downarrow \\ \text{Spec}(R/p^m) & \xrightarrow{\text{id}} & \text{Spec}(R/p^m). \end{array} \quad (4.20)$$

Evaluating the  $F$ -crystal  $\mathcal{E}$  on  $\text{Spec}(R/p^m) \hookrightarrow \text{Spec}(\Lambda_{R,0}/p^m)$  for each  $m \geq 1$ , and taking the limit over  $m$ , gives a finitely generated  $\Lambda_{R,0}$ -module  $\mathcal{E}(\Lambda_{R,0}) = M_{\Lambda,0}$ , equipped with a Frobenius and an  $R$ -linear  $p$ -adically quasi-nilpotent flat connection, described in Proposition 4.24 and the discussion preceding it. Now, using the fact that  $\mathcal{E}$  is an  $F$ -crystal and the diagram (4.20), we obtain a  $\varphi$ -equivariant isomorphism

$$\bar{\varepsilon} : \Lambda_{R,0} \otimes_R \mathcal{E}(R) \xrightarrow{\sim} \mathcal{E}(\Lambda_{R,0}) = M_{\Lambda,0},$$

compatible with respective  $R$ -linear connections, where the  $R$ -linear connection on the source is given as  $\nabla \otimes 1$  with  $\nabla$  being the  $R$ -linear connection on  $\Lambda_{R,0}$  described in the proof of Proposition 4.24. Explicitly, for  $x$  in  $\Lambda_{R,0} \otimes_R \mathcal{E}(R)$ , the map  $\bar{\varepsilon}$  is given by the formula  $\bar{\varepsilon}(x) = \sum_j (-1)^j \nabla_0^j(x) \otimes z^{[j]}$ , where  $\nabla_0 = \frac{1}{(p-1)z} \nabla_0^{\log}$  (see Proposition 4.24 for the definition of  $\nabla_0^{\log}$ ).

Next, consider the de Rham complex  $\Omega_{\Lambda_{R,0}/R}^\bullet = \Lambda_{R,0} \otimes_{\Lambda_{F,0}} \Omega_{\Lambda_{F,0}/O_F}^\bullet$ , regarded as a complex of  $\Lambda_{F,0}$ -modules via the natural map  $\Lambda_{F,0} \rightarrow \Lambda_{R,0}$ . Recall that  $\Lambda_{F,0} = O_F[s^{[k]}, k \in \mathbb{N}]_p^\wedge$ , therefore it follows that the de Rham complex  $\Omega_{\Lambda_{R,0}/R}^\bullet = \Lambda_{R,0} \otimes_{\Lambda_{F,0}} \Omega_{\Lambda_{F,0}/O_F}^\bullet$  is acyclic in positive degrees (also see [Ber74, Chapitre V, Lemme 2.1.2]). Hence, from the isomorphism of the de Rham complexes

$$(\Lambda_{R,0} \otimes_R \mathcal{E}(R)) \otimes_{\Lambda_{R,0}} \Omega_{\Lambda_{R,0}/R}^\bullet \xrightarrow{\sim} M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega_{\Lambda_{R,0}/R}^\bullet,$$

it follows that the de Rham complex  $M_{\Lambda,0} \otimes_{\Lambda_{R,0}} \Omega_{\Lambda_{R,0}/R}^\bullet$  is acyclic in positive degrees.

Now, taking horizontal sections for respective connections in the isomorphism  $\bar{\varepsilon}$ , we obtain a  $\varphi$ -equivariant isomorphism of finite  $R$ -modules  $\mathcal{E}(R) \xrightarrow{\sim} \mathcal{E}(\Lambda_{R,0})^{\nabla=0} = M_{\Lambda,0}^{\nabla_0=0}$ . In particular, from the isomorphism  $\bar{\varepsilon}$ , we deduce that  $\Lambda_{R,0}$ -linearly extending the natural inclusion  $M_{\Lambda,0}^{\nabla_0=0} \subset M_{\Lambda,0}$ , we obtain the following  $(\varphi, \nabla_0)$ -equivariant diagram

$$\begin{array}{ccc} \Lambda_{R,0} \otimes_R \mathcal{E}(R) & \xrightarrow{\sim \bar{\varepsilon}} & \mathcal{E}(\Lambda_{R,0}) \\ \downarrow \bar{\varepsilon} & & \parallel \\ \Lambda_{R,0} \otimes_{\Lambda_R} M_{\Lambda,0}^{\nabla_0=0} & \xrightarrow{(4.19)} & M_{\Lambda,0}. \end{array}$$

Hence, it follows that (4.19) is an isomorphism. This concludes our proof.  $\blacksquare$

**4.3.3. The case  $p = 2$ .** In this subsection, we will prove statements analogous to Proposition 4.22, Proposition 4.24 and Proposition 4.25, for  $p = 2$ . Let  $N$  be a Wach module over  $A_R$  and let  $M_\Lambda = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R}$  be the  $\Lambda_R$ -module from Proposition 4.17 equipped with an induced action of  $(\varphi, \Gamma_F)$ . First, we will look at the action of  $\Gamma_{\text{tor}}$  on  $M_\Lambda$ . Let  $\sigma$  denote a generator of  $\Gamma_{\text{tor}}$ . Then from (A.2), recall that by setting  $M_{\Lambda,+} := \{x \in M_\Lambda \text{ such that } \sigma(x) = x\}$  and  $M_{\Lambda,-} := \{x \in M_\Lambda \text{ such that } \sigma(x) = -x\}$ , we have a natural injective map of modules over  $\Lambda_{R,+}$  (see Subsection 3.4.4),

$$M_{\Lambda,+} \oplus M_{\Lambda,-} \longrightarrow M_\Lambda, \quad (4.21)$$

given as  $(x, y) \mapsto x + y$ . Note that the action of  $1 \times \Gamma_F$  is continuous for the  $(p, p_1(\mu))$ -adic topology on  $M_\Lambda$ , so it follows that  $M_{\Lambda,+}$  is a  $(p, p_1(\mu))$ -adically complete  $\Lambda_{R,+}$ -submodule of  $M_\Lambda$ , stable under the action of  $(\varphi, \Gamma_F)$  on  $M_\Lambda$ , and similarly,  $M_{\Lambda,-}$  is a complete  $\Lambda_{R,+}$ -submodule, stable under the action of  $(\varphi, \Gamma_F)$ . Equipping  $M_{\Lambda,+}$  and  $M_{\Lambda,-}$  with induced structures, we see that (4.21) is  $(\varphi, \Gamma_F)$ -equivariant.

**Proposition 4.26.** *The natural map in (4.21) is bijective. Moreover, we have that  $M_{\Lambda,-} = (t/2)M_{\Lambda,+}$ . In particular, we have a natural  $\Lambda_R$ -linear and  $(\varphi, \Gamma_F)$ -equivariant isomorphism*

$$\Lambda_R \otimes_{\Lambda_{R,+}} M_{\Lambda,+} \xrightarrow{\sim} M_\Lambda. \quad (4.22)$$

*Proof.* Let us first note that by  $\Lambda_R$ -linearly extending the  $\Lambda_{R,+}$ -linear and  $(\varphi, \Gamma_F)$ -equivariant injective map  $M_{\Lambda,+} \rightarrow M_\Lambda$  from (4.21), we obtain the  $\Lambda_R$ -linear and  $(\varphi, \Gamma_F)$ -equivariant map in (4.22). Recall that  $\Lambda_R = \Lambda_{R,+} \oplus (t/2)\Lambda_{R,+}$  from Lemma 3.61 and Lemma 3.62, and  $(t/2)M_{\Lambda,+} \subset M_{\Lambda,-}$ , so from the injectivity of (4.21), it follows that (4.22) is injective. To prove the claims, it is enough to show that (4.22) is surjective as well.

Let  $I$  denote the kernel of the surjective map  $\Lambda_R \rightarrow R$ . Then from the explicit description of  $\Lambda_R$  in Proposition 3.21, it follows that  $I$  is the ideal of  $\Lambda_R$  generated by the divided powers of  $t/2$ . As the divided powers of  $t/2$  are topologically nilpotent, it follows that  $I$  is contained in the Jacobson radical of  $\Lambda_R$ . Moreover, since the ideal  $I$  is stable under the action of  $(\varphi, \Gamma_F)$  on  $\Lambda_R$ , it follows that the map  $\Lambda_R \rightarrow R$  is  $(\varphi, \Gamma_F)$ -equivariant. Next, from (4.7) in Proposition 4.17, recall that we have a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $\Lambda_R$ -modules  $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$ . Base changing this

isomorphism along the surjective map  $\Lambda_R \twoheadrightarrow R$ , we obtain a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $R$ -modules  $M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N$ . Precomposing it with (4.22) gives a  $\Lambda_R$ -linear map

$$\Lambda_R \otimes_{\Lambda_{R,+}} M_{\Lambda,+} \longrightarrow M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N. \quad (4.23)$$

Since  $I$  is in the Jacobson radical of  $\Lambda_R$ , therefore, by Nakayama Lemma, to show that (4.22) is surjective, it is enough to show that the first map in (4.23) is surjective. So let  $\bar{x}$  be any element of  $M_\Lambda/IM_\Lambda$  and let  $x$  be a lift of  $\bar{x}$  in  $M_\Lambda$ . We first claim that  $(\sigma + 1)x$  is an element of  $2M_\Lambda$ . Indeed, let us first write  $x = \sum_i a_i \otimes x_i$ , for some  $a_i$  in  $\Lambda_R$  and  $x_i$  in  $N$ . Then, from the description of  $\Lambda_R$  in Lemma 3.61 and Lemma 3.62 we have that  $(\sigma - 1)a_i = tb_i$ , for some  $b_i$  in  $\Lambda_{R,+}$  and from the triviality of the action of  $\Gamma_R$  on  $N/\mu N$ , we can write  $(\sigma - 1)x_i = \mu y_i$ , for some  $y_i$  in  $N$ . So we have the following:

$$\begin{aligned} (\sigma + 1)x &= (\sigma - 1)x + 2x = (\sigma - 1)(\sum_i a_i \otimes x_i) + 2x \\ &= \sum_i (\sigma - 1)a_i \otimes x_i + \sum_i \sigma(a_i) \otimes (\sigma - 1)x_i + 2x \\ &= \sum_i tb_i \otimes x_i + \sum_i \sigma(a_i) \otimes \mu y_i + 2x, \end{aligned}$$

which is clearly in  $2M_\Lambda$  (since  $t/\mu$  is a unit in  $\Lambda_R$ , see Lemma 3.24). As  $M_\Lambda$  is  $p$ -torsion free, therefore, we set  $x' := \frac{(\sigma+1)}{2}(x)$  in  $M_\Lambda$  and note that  $\sigma(x') = x'$ , i.e.  $x'$  is in  $M_{\Lambda,+}$ . From the computation of  $(\sigma + 1)x$  above and the fact that  $M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N$  (see (4.23)) is  $p$ -torsion free, we see that  $x' = \bar{x} \bmod IM_\Lambda$ . In particular, we conclude that the first map of (4.23) is surjective, hence, (4.22) is bijective. Finally, from the decomposition  $\Lambda_R = \Lambda_{R,+} \oplus (t/2)\Lambda_{R,+}$ , the inclusion  $(t/2)M_{\Lambda,+} \subset M_{\Lambda,-}$  and from the bijectivity of (4.22), it follows that (4.21) is bijective and  $M_{\Lambda,-} = (t/2)M_{\Lambda,+}$ . This completes our proof.  $\blacksquare$

Next, we will look at the action of  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$  on  $M_{\Lambda,+}$ . From (3.40), recall that  $\nu = \frac{\mu^2}{1+\mu}$  is an element of  $\Lambda_{F,+}$  and we claim the following:

**Lemma 4.27.** *The action of  $\Gamma_0$  is trivial on  $M_{\Lambda,+}/\nu M_{\Lambda,+}$ .*

In the following, we provide an explicit proof of the claim. Note that the claim in Lemma 4.27 also follows from Remark 5.24. Moreover, the arguments in Remark 5.24 do not depend on the claim proven in Lemma 4.27 and the subsequent results of the current Subsection 4.3.3. In particular, Remark 5.24 gives a more conceptual proof of the claim.

*Proof.* Note that the action of  $\Gamma_0$  is continuous on  $M_{\Lambda,+}$ , so it is enough to show the claim for a topological generator of  $\Gamma_0$ . So let us fix  $\gamma_0$  to be a topological generator of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + 4a$ , for a unit  $a$  in  $\mathbb{Z}_2$ . Moreover, using Lemma 3.24, it is easy to see that  $\nu$  is the product of  $t^2$  with a unit in  $\Lambda_{F,+}$ . So for any  $m$  in  $M_{\Lambda,+}$ , it is enough to show that  $(\gamma_0 - 1)m$  is an element of  $t^2 M_{\Lambda,+}$ . Now, using Lemma 4.28, let us write  $(\gamma_0 - 1)m = 2tm'$ , for some  $m'$  in  $M_\Lambda$ . Since,  $\sigma(m) = m$ , we get that  $\sigma(m') = -m'$ , i.e.  $m'$  belongs to  $M_{\Lambda,-}$ . Using Proposition 4.26, we can write  $m' = (t/2)n$ , for some  $n$  in  $M_{\Lambda,+}$ . Hence, we get that  $(\gamma_0 - 1)m = t^2 n$ , as claimed.  $\blacksquare$

The following observation was used above:

**Lemma 4.28.** *Let  $\gamma_0$  be a topological generator of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + 4a$ , for a unit  $a$  in  $\mathbb{Z}_2$ . Then for any  $m$  in  $M_\Lambda$ , the element  $(\gamma_0 - 1)m$  belongs to  $2tM_\Lambda$ .*

*Proof.* Let us start with some observations. Let  $f$  be an element of  $\Lambda_R$ . Then using that  $\Lambda_R = \Lambda_{R,+} \oplus (t/2)\Lambda_{R,+}$  from Lemma 3.61 and Lemma 3.62 and the fact that the action of  $\gamma_0$  is trivial on  $\Lambda_{R,+}/\nu\Lambda_{R,+}$ , where  $\nu$  is the product of  $t^2$  with a unit in  $\Lambda_{F,+}$ , it follows that  $(\gamma_0 - 1)f$  is an element of  $2t\Lambda_R$ .

Next, let  $x$  be any element of  $M_{\Lambda,+} \subset M_\Lambda$ , then from the  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$  (see (4.7) in Proposition 4.17), the triviality of the action of  $\Gamma_F$  on  $\Lambda_R/\mu$  (see Lemma 3.48) and  $N/\mu N$  (see Definition 4.1) and the fact that  $t/\mu$  is a unit in  $\Lambda_F$  (see Lemma 3.24), it follows that we can write  $(\gamma_0 - 1)x = tx_1$ , for some  $x_1$  in  $M_\Lambda$ . As  $x$  is in  $M_{\Lambda,+}$ , we have  $\sigma(x) = x$  and since  $\Gamma_F$

is commutative and  $M_\Lambda$  is  $t$ -torsion free, it follows that  $\sigma(x_1) = -x_1$ , i.e.  $x_1$  is an element of  $M_{\Lambda,-}$ . Then using Proposition 4.26, we can write  $x_1 = (t/2)x_2$ , for some  $x_2$  in  $M_{\Lambda,+}$ , and get that  $(\gamma_0 - 1)x$  is an element of  $(t^2/2)M_{\Lambda,+}$ .

Next, let  $y$  be any element of  $M_{\Lambda,-}$  and using Proposition 4.26, write  $y = (t/2)y_1$ , for some  $y_1$  in  $M_{\Lambda,+}$ . From the preceding discussion, note that we have  $(\gamma_0 - 1)y_1 = (t^2/2)y_2$ , for some  $y_2$  in  $M_{\Lambda,+}$ . So, we get that

$$\begin{aligned} (\gamma_0 - 1)y &= (\gamma_0 - 1)\left(\frac{t}{2}y_1\right) = \frac{1}{2}(y_1(\gamma_0 - 1)t + \gamma_0(t)(\gamma_0 - 1)y_1) \\ &= \frac{1}{2}(4aty_2 + (1 + 4a)ty_2\frac{t^2}{2}) = 2t(ay_1 + (1 + 4a)y_2\frac{t^2}{8}), \end{aligned}$$

is an element of  $2tM_{\Lambda,+}$ .

Next, note that since  $N[1/p]$  is finite projective over  $A_R[1/p]$  (see Remark 4.2) and the map  $A_R \rightarrow \Lambda_R$  is injective, therefore, it follows that the  $A_R$ -linear and  $(\varphi, \Gamma_F)$ -equivariant map  $N \rightarrow M_\Lambda$  is injective. Now, let  $z$  be an element of  $N$  and let us denote its image in  $M_\Lambda$ , again by  $z$ . Then by Definition 4.1, we have that  $(\gamma_0 - 1)z = \mu z_0$ , for some  $z_0$  in  $N \subset M_\Lambda$ . Using Proposition 4.26, let us write  $z = x + y$ , for some  $x$  in  $M_{\Lambda,+}$  and  $y$  in  $M_{\Lambda,-} = (t/2)M_{\Lambda,+}$ . Then from the preceding discussions and the fact that  $t/\mu$  is a unit in  $\Lambda_F$ , we can write  $(\gamma_0 - 1)x = (\mu^2/2)x'$  and  $(\gamma_0 - 1)y = 2\mu y'$ , for some  $x'$  and  $y'$  in  $M_\Lambda$ . In particular, since  $M_\Lambda$  is  $\mu$ -torsion free, we get that  $z_0 = (\mu/2)x' + 2y'$ . Reducing the preceding equality modulo  $IM_\Lambda$ , where  $I$  is the kernel of the surjective map  $\Lambda_R \rightarrow R$  (see the proof of Proposition 4.26), we see that  $z_0 = 2y' \bmod IM_\Lambda$ , in  $M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N$  (see (4.23)). Since  $y' \bmod IM_\Lambda$  is an element of  $M_\Lambda/IM_\Lambda \xrightarrow{\sim} N/\mu N$  and  $z_0$  is in  $N$ , we get that  $z_0 \bmod \mu N$  is an element of  $2(N/\mu N)$ . In particular, we can write  $z_0 = 2z_1 + \mu z_2$  for some  $z_1, z_2$  in  $N$ . So, we have that,

$$(\gamma_0 - 1)z = \mu z_0 = 2\mu z_1 + \mu^2 z_2 = 2\mu(z_1 + \frac{\mu}{2}z_2),$$

is an element of  $2tM_\Lambda$ , since  $t/\mu$  is a unit in  $\Lambda_F$ .

Now, let  $f \otimes z$  be an element of  $\Lambda_R \otimes_{A_R} N$ , and using the discussion at the beginning of the proof, let us write  $(\gamma_0 - 1)f = 2te$ , for some  $e$  in  $\Lambda_R$ . Moreover, from the discussion above we can write  $(\gamma_0 - 1)z = 2tz'$ , for some  $z'$  in  $M_\Lambda$ . In particular, we see that

$$(\gamma_0 - 1)(f \otimes z) = ((\gamma_0 - 1)f) \otimes z + \gamma_0(f) \otimes (\gamma_0 - 1)z = 2te + \gamma_0(f)2tz' = 2t(e + \gamma_0(f)z'),$$

is an element of  $2tM_\Lambda$ . Using the  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $M_\Lambda \xrightarrow{\sim} \Lambda_R \otimes_{A_R} N$  (see (4.7) in Proposition 4.17) and the preceding observation, we conclude that for any  $m$  in  $M_\Lambda$ , the element  $(\gamma_0 - 1)m$  belongs to  $2tM_\Lambda$ . Hence, the lemma is proved.  $\blacksquare$

In the rest of this subsection, let us fix  $\gamma_0$  to be a topological generator of  $\Gamma_0$  such that  $\chi(\gamma_0) = 1 + 4a$ , for a unit  $a$  in  $\mathbb{Z}_2$ . Then from Lemma 4.27, note that for any  $x$  in  $M_{\Lambda,+}$ , we have that  $(\gamma_0 - 1)x$  is an element of  $\nu M_{\Lambda,+}$ . Set  $\tau := \nu/8$  in  $\Lambda_{R,+}$ , then from Lemma 3.72 we know that  $(\gamma_0 - 1)\tau = u\nu$ , for some unit  $u$  in  $\Lambda_{F,+}$ . Therefore, we see that the following operator is well-defined:

$$\begin{aligned} \nabla_{q,\tau} : M_{\Lambda,+} &\longrightarrow M_{\Lambda,+} \\ x &\longmapsto \frac{(\gamma_0 - 1)x}{(\gamma_0 - 1)\tau}. \end{aligned} \tag{4.24}$$

As the operator  $\nabla_{q,\tau}$  is an endomorphism of  $M_{\Lambda,+}$ , we can define the following two term Koszul complex:

$$K_{M_{\Lambda,+}}(\nabla_{q,\tau}) : [M_{\Lambda,+} \xrightarrow{\nabla_{q,\tau}} M_{\Lambda,+}]. \tag{4.25}$$

**Remark 4.29.** Considering  $\tau$  as a variable, similar to Remark 3.73, the operator  $\nabla_{q,\tau}$  in (4.24), may be also considered as a  $q$ -connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32. Then, (4.25) is the  $q$ -de Rham complex arising from such a  $q$ -connection.

**Proposition 4.30.** *The series of operators  $\nabla_0^{\log} = \frac{\log \gamma_0}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}}{k+1}$  converge  $p$ -adically on  $M_{\Lambda,+}$ . Let  $w := t^2/8$  in  $\Lambda_{R,+}$ , then the operator  $\nabla_0 := \frac{1}{2w} \nabla_0^{\log}$  defines an*



$R$ -linear  $p$ -adically quasi-nilpotent flat connection on  $M_{\Lambda,+}$ , denoted  $\nabla : M_{\Lambda,+} \rightarrow M_{\Lambda,+} \otimes_{\Lambda_{R,+}} \Omega_{\Lambda_{R,+}/R}^1$  and given as  $x \mapsto \nabla_0(x)dw$ . The data of the connection  $\nabla$  on  $M_{\Lambda,+}$  is equivalent to the data of the  $q$ -connection  $\nabla_{q,\tau}$  from (4.24), i.e. either may be recovered from the other. Moreover, the  $q$ -de Rham complex  $K_{M_{\Lambda,+}}(\nabla_{q,\tau})$  in (4.25) is naturally quasi-isomorphic to the de Rham complex  $M_{\Lambda,+} \otimes_{\Lambda_{R,+}} \Omega_{\Lambda_{R,+}/R}^\bullet$ . In particular, we have that  $M_{\Lambda,+}^{\nabla_{q,\tau}=0} \xrightarrow{\sim} M_{\Lambda,+}^{\nabla_0=0}$ .

*Proof.* The idea of the proof is similar to the proof of Proposition 4.24, with slightly different computations. We sketch it below. Recall that  $\tau = et^2/8$ , for a unit  $e$  in  $\Lambda_{F,+}$  (see Lemma 3.62). Moreover, from Lemma 3.72 we have that  $(\gamma_0 - 1)\tau = (\gamma_0 - 1)\frac{\nu}{8} = u\nu$ , for a unit  $u$  in  $\Lambda_{F,+}$ . Now let  $w = t^2/8$  and we write  $(1 + 4a)^2 = 1 + 8b$ , noting that  $b = a(2a + 1)$  is a unit in  $\mathbb{Z}_2$ . So we have

$$(\gamma_0 - 1)w = (\gamma_0 - 1)\frac{t^2}{8} = (\chi(\gamma_0)^2 - 1)\frac{t^2}{8} = ((1 + 4a)^2 - 1)\frac{t^2}{8} = bt^2 = e^{-1}b(2a + 1)\nu. \quad (4.26)$$

Therefore, it follows that the complex  $K_{M_{\Lambda,+}}(\nabla_{q,\tau})$  is quasi-isomorphic to the following complex

$$K_{M_{\Lambda,+}}(\nabla_{q,w}) : [M_{\Lambda,+} \xrightarrow{\nabla_{q,w}} M_{\Lambda,+}]. \quad (4.27)$$

Rest of the proof is similar to the proof of Proposition 4.13, with some changes. To avoid confusion, we provide a sketch.

Let us first show that  $\nabla_0^{\log} := \frac{\log(\gamma_0)}{\log(\chi(\gamma_0))} = \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}}{k+1}$ , converge as a series of operators on  $M_{\Lambda,+}$ . Indeed, let  $x$  be any element of  $M_{\Lambda,+}$ , then using Lemma 4.27, we can write  $(\gamma_0 - 1)x = t^2x_1$ , for some  $x_1$  in  $M_{\Lambda,+}$ . Let us note that  $\log(\chi(\gamma_0)) = \log(1 + 4a) = 4c$ , where  $c$  is a unit in  $\mathbb{Z}_2$ , and we also have  $(\gamma_0 - 1)t^2 = ((1 + 4a)^2 - 1)t^{p-1} = 8bt^2$ , where  $b$  is a unit in  $\mathbb{Z}_2$ . Therefore, an easy induction on  $k \in \mathbb{N}$ , shows that  $(\gamma_0 - 1)^{k+1}x = 8^k t^2 x_{k+1}$ , for some  $x_{k+1}$  in  $M_{\Lambda,+}$ . In particular, we get that

$$\begin{aligned} \nabla_0^{\log}(x) &= \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_0 - 1)^{k+1}(x)}{k+1} \\ &= \frac{1}{\log(\chi(\gamma_0))} \sum_{k \in \mathbb{N}} (-1)^k \frac{8^k t^2 x_{k+1}}{k+1} \\ &= \frac{t^2}{4c} \sum_{k \in \mathbb{N}} (-1)^k \frac{8^k x_{k+1}}{k+1}, \end{aligned}$$

which converges in  $\frac{t^2}{4}M_{\Lambda,+} = 2wM_{\Lambda,+}$ , since  $M_{\Lambda,+}$  is 2-adically complete. Then, it also follows that the operator  $\nabla_0 := \frac{1}{2w}\nabla_0^{\log}$  is well-defined on  $M_{\Lambda,+}$ .

Next, similar to the case of  $\nabla_i^{\log}$  in the proof of Proposition 4.13, it can be shown that  $\nabla_0^{\log}$ , and therefore  $\nabla_0 = \frac{1}{2w}\nabla_0^{\log}$  satisfies a Leibniz rule, i.e.  $\nabla_0(fx) = \nabla_0(f)x + f\nabla_0(x)$ , where the first operator on the right is  $\nabla_0 := \frac{\log(\gamma_0)}{2w\log(\chi(\gamma_0))} : \Lambda_{R,+} \rightarrow \Lambda_{R,+}$  whose well-definedness can be checked similar to above. Moreover, note that the operator  $\nabla_0$  is flat by definition. Furthermore, similar to the case of  $\nabla_i$  in the proof of Proposition 4.13, it can be shown that the operator  $\nabla_0 : \Lambda_{R,+} \rightarrow \Omega_{\Lambda_{R,+}/R}^1$  is the usual de Rham differential  $d : \Lambda_{R,+} \rightarrow \Omega_{\Lambda_{R,+}/R}^1$ . So, in particular, the map  $\nabla : M_{\Lambda,+} \rightarrow M_{\Lambda,+} \otimes_{\Lambda_{R,+}} \Omega_{\Lambda_{R,+}/R}^1$ , given as  $x \mapsto \nabla_0(x)dw$ , is a well defined connection. Now, let us show that the operator  $\nabla_0$  is  $p$ -adically quasi-nilpotent. Indeed, we first note that from the commutativity of  $\varphi$  and  $\gamma_0$ , it follows that  $\nabla_0^{\log} \circ \varphi = \varphi \circ \nabla_0^{\log}$ . Therefore, it is easy to see that  $\nabla_0 \circ \varphi = 4\varphi \circ \nabla_0$ . Next, recall that  $M_{\Lambda}$  is equipped with a  $\Lambda_R$ -linear isomorphism  $\varphi^*(M_{\Lambda})[1/p] \xrightarrow{\sim} M_{\Lambda}[1/p]$ , compatible with the action of  $\Gamma_F$  on each side. Then using the isomorphism (4.22) in Proposition 4.26, we easily obtain that  $M_{\Lambda,+}$  is equipped with a  $\Lambda_{R,+}$ -linear isomorphism  $\varphi^*(M_{\Lambda,+})[1/p] \xrightarrow{\sim} M_{\Lambda,+}[1/p]$ , compatible with action of  $\Gamma_0$  on each side. Using this observation and the relation  $\nabla_0 \circ \varphi = 4\varphi \circ \nabla_0$ , similar to the proof of Proposition 4.24, it can be shown that for any  $x$  in  $M_{\Lambda,+}$ , the sequence  $\nabla_0^k(x)$  converges  $p$ -adically to 0 as  $k \rightarrow +\infty$ , in particular,  $\nabla_0$  is  $p$ -adically quasi-nilpotent. Furthermore, note that so far we have defined the  $p$ -adically quasi-nilpotent flat connection  $\nabla$  using the action of  $\Gamma_0$  and conversely, we have shown that the action of  $\Gamma_0$  can be recovered by the formula  $\gamma_0 := \exp(\log(\chi(\gamma_0))\nabla_0^{\log})$ . Again, similar to case of  $\gamma_i$  in the proof of Proposition 4.13, using the Leibniz rule for  $\nabla_0$ , it can be checked that the action of  $\gamma_0$  thus obtained, is semilinear.

Finally, it remains to compare the  $q$ -de Rham complex in (4.15) with the de Rham complex  $M_{\Lambda,+} \otimes_{\Lambda_{R,+}} \Omega_{\Lambda_{R,+}/R}^{\bullet} = M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega_{\Lambda_{F,+}/O_F}^{\bullet}$ . As  $\nabla_0$  is an endomorphism of  $M_{\Lambda,+}$ , let  $K_{M_{\Lambda,+}}(\nabla_0)$  denote the corresponding Koszul complex in the sense of Definition A.8. Then we have an identification of complexes:

$$M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega_{\Lambda_{F,+}/O_F}^{\bullet} = K_{M_{\Lambda,+}}(\nabla_0) : [\Lambda_{R,+} \xrightarrow{\nabla_0} \Lambda_{R,+}].$$

Now, recall that we have  $\gamma_0 = \exp(\log(\chi(\gamma_0))\nabla_0^{\log})$ . Therefore, we can write

$$\nabla_{q,w} = \frac{\gamma_0 - 1}{(\gamma_0 - 1)w} = \frac{\log(\chi(\gamma_0))2w}{(\gamma_0 - 1)w} \nabla_0 (1 + \sum_{k \geq 1} \frac{\log(\chi(\gamma_0))^k}{(k+1)!} (\nabla_0^{\log})^k). \quad (4.28)$$

Recall that  $(\gamma_0 - 1)w = 8bw$  and  $\log(\chi(\gamma_0)) = 4c$ , for units  $b$  and  $c$  in  $\mathbb{Z}_2$ . Therefore, in (4.28), we have that  $\frac{\log(\chi(\gamma_0))2w}{(\gamma_0 - 1)w} = \frac{c}{b}$  is a unit and it is clear that the term inside the parentheses converges  $p$ -adically to a unit. Now, in the notation of Lemma A.9, let us set  $i = 1$ ,  $M = M_{\Lambda,+}$ ,  $f_1 = \nabla_0$  and take  $h_1$  to be the product of  $\frac{c}{b}$  with the formula in the parentheses in (4.28), in particular,  $f_1 h_1 = \nabla_{q,w}$ . Then, from Lemma A.9, we obtain a natural quasi-isomorphism of complexes

$$K_{M_{\Lambda,+}}(\nabla_{q,\tau}) \xrightarrow{\sim} K_{M_{\Lambda,+}}(\nabla_{q,w}) \xrightarrow{\sim} K_{M_{\Lambda,+}}(\nabla_0).$$

In particular, we get that  $M_{\Lambda,+}^{\nabla_{q,\tau}=0} \xrightarrow{\sim} M_{\Lambda,+}^{\nabla_0=0}$ . This allows us to conclude.  $\blacksquare$

**Proposition 4.31.** *Set  $M := M_{\Lambda,+}^{\nabla_0=0}$  as an  $R$ -module via the isomorphism  $R \xrightarrow{\sim} \Lambda_{R,+}^{\nabla_0=0}$ . Then  $M$  is finitely generated over  $R$  and we have a natural  $(\varphi, \nabla_0)$ -equivariant isomorphism*

$$\begin{aligned} \Lambda_{R,+} \otimes_R M &\xrightarrow{\sim} M_{\Lambda,+} \\ a \otimes x &\longmapsto ax. \end{aligned} \quad (4.29)$$

Moreover, the de Rham complex  $M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega_{\Lambda_{F,+}/O_F}^{\bullet}$  is acyclic in positive degrees. In particular, from Proposition 4.30, we have  $H^1(K_{M_{\Lambda,+}}(\nabla_{q,\tau})) = 0$ .

*Proof.* The proof works in exactly the same manner as the proof of Proposition 4.25 by changing the notations as follows: replace  $\Lambda_{R,0}$  with  $\Lambda_{R,+} = R[\tau^{[k]}, k \in \mathbb{N}]_p^{\wedge}$  and replace  $M_{\Lambda,0}$  with  $M_{\Lambda,+}$  equipped with a Frobenius endomorphism (after inverting  $p$ ) and a  $p$ -adically quasi-nilpotent flat connection as in Proposition 4.30, where in the definition of the connection we replace the parameter  $z$  with  $w$  and the operator  $\nabla_0 = \frac{1}{(p-1)z} \nabla_0^{\log}$  with  $\nabla_0 = \frac{1}{2w} \nabla_0^{\log}$ . Then the proof of the isomorphism (4.29) and the cyclicity of the de Rham complex  $M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega_{\Lambda_{F,+}/O_F}^{\bullet}$  in positive degrees follows by arguments similar to the ones given, respectively, for the isomorphism (4.19) and the acyclicity of the de Rham complex  $M_{\Lambda,+} \otimes_{\Lambda_{F,+}} \Omega_{\Lambda_{F,+}/O_F}^{\bullet}$  in positive degrees, in Proposition 4.25.  $\blacksquare$

**4.4. Proof of Theorem 4.5.** Let  $N$  be a Wach module over  $A_R$  as above and consider the  $A_R(1)$ -module  $A_R(1) \otimes_{p_2, A_R} N$  equipped with the tensor product Frobenius and tensor product action of  $\Gamma_R^2$ , where  $\Gamma_R^2$  acts on  $N$  via projection onto the second coordinate. Then, in Example 4.10, using the action of  $1 \times \Gamma'_R$ , we equipped  $A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$  with a  $q$ -connection, denoted as  $\nabla_q$ . Moreover, in Proposition 4.13, we equipped  $A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$  with a connection, denoted as  $\nabla$ , and showed that we have  $(A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla=0} = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla_q=0}$ . Since the action of  $1 \times \Gamma'_R$  on  $A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$  is continuous, we deduce that

$$(A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma'_R} = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla_q=0} = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla=0}.$$

Next, from Proposition 4.17, let us recall that  $M_{\Lambda} := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{\nabla=0}$  is a finitely generated  $\Lambda_R$ -module, equipped with an induced action of  $(\varphi, \Gamma_F)$ . Then from (4.6), we have that by  $A_R(1)/p_1(\mu)$ -linearly extending the natural inclusion  $M_{\Lambda} \subset A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$ , induces a  $(\varphi, \nabla, \Gamma_F)$ -equivariant, or equivalently, a  $(\varphi, \Gamma_R)$ -equivariant isomorphism

$$A_R(1)/p_1(\mu) \otimes_{\Lambda_R} M_{\Lambda} \xrightarrow{\sim} A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N.$$

Now, let us set  $M := M_\Lambda^{\Gamma_F}$  as an  $R$ -module. Then, by using Proposition 4.22 and Proposition 4.25 for  $p \geq 3$  and using Proposition 4.26 and Proposition 4.31 for  $p = 2$ , we see that  $M$  is a finitely generated  $R$ -module. Moreover, since  $A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N$  is  $p$ -torsion free by Lemma 4.11, therefore, we get that  $M$  is  $p$ -torsion free as well. Furthermore, for  $p \geq 3$ , using (4.13) in the proof of Proposition 4.22 and (4.19) in Proposition 4.25, and for  $p = 2$ , using (4.22) in Proposition 4.26 and (4.29) in Proposition 4.31, we see that by  $\Lambda_R$ -linearly extending the natural inclusion  $M \subset M_\Lambda$ , we obtain a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $\Lambda_R$ -modules

$$\begin{aligned} \Lambda_R \otimes_R M &\xrightarrow{\sim} M_\Lambda \\ a \otimes x &\longmapsto ax. \end{aligned} \quad (4.30)$$

Putting everything together from the discussion above, we have the following diagram with  $(\varphi, 1 \times \Gamma_R)$ -equivariant arrows:

$$\begin{array}{ccc} A_R(1)/p_1(\mu) \otimes_{p_1, R} M & \xrightarrow{(4.1)} & A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N \\ \downarrow \wr (4.30) & \nearrow \sim (4.6) & \\ A_R(1)/p_1(\mu) \otimes_{\Lambda_R} M_\Lambda & & \end{array}$$

The diagram commutes by the definition of the arrows. Hence, it follows that (4.1) is an isomorphism.

Finally, it remains to show that we have a  $\varphi$ -equivariant isomorphism of  $R$ -modules  $M \xrightarrow{\sim} N/\mu N$ . To show this claim, let us note that the multiplication map  $\Delta : A_R(1) \rightarrow A_R$  is  $(\varphi, \Gamma_R \times 1)$ -equivariant, where  $\Gamma_R \times 1$  acts on  $A_R$  via projection onto the first coordinate (see Lemma 3.78). Then the multiplication map  $\Delta$  induces a  $\varphi$ -equivariant map  $\Delta_N : A_R(1) \otimes_{p_2, A_R} N \rightarrow N$ . Reducing it modulo  $p_1(\mu)^n$  we obtain an  $A_R/\mu^n$ -linear and  $\varphi$ -equivariant map  $\Delta_N : A_R(1)/p_1(\mu)^n \otimes_{p_2, A_R} N \rightarrow N/\mu^n N$ . For  $n = 1$ , we claim the following:

**Proposition 4.32.** *Let  $M := (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R}$  as a finitely generated  $\varphi$ -module over  $R$ . Then the map  $\Delta_N$  restricts to a  $\varphi$ -equivariant  $R$ -linear isomorphism*

$$M = (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} \xrightarrow{\sim} N/\mu N. \quad (4.31)$$

*Proof.* By definition, we have that  $\Delta_N$  is  $\varphi$ -equivariant, so we only need to check the bijectivity of the map in claim. Let us note that the composition  $R \xrightarrow{p_1} A_R(1)/p_1(\mu) \xrightarrow{\Delta} R$  is identity. Moreover, the composition  $A_R \xrightarrow{p_2} A_R(1) \rightarrow A_R(1)/p_1(\mu) \xrightarrow{\Delta} R$  coincides with the map  $A_R \rightarrow A_R/\mu = R$ . So we have that,

$$\begin{aligned} \Delta_N : M &= R \otimes_{\Delta, A_R(1)/p_1(\mu)} (A_R(1)/p_1(\mu) \otimes_{p_1, R} M) \\ &\xrightarrow{\sim} R \otimes_{\Delta, A_R(1)/p_1(\mu)} (A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N) \xrightarrow{\sim} A_R/\mu \otimes_{A_R} N = N/\mu N, \end{aligned}$$

where the first isomorphism follows from the isomorphism in Theorem 4.5 and the second isomorphism follows from the discussion above. Hence, we get the claim.  $\blacksquare$

This completes the proof of Theorem 4.5.

**Remark 4.33.** Let  $N$  be a Wach module over  $A_R$ . Then we have a  $\varphi$ -equivariant commutative diagram,

$$\begin{array}{ccc} A_R(1)/p_1(\mu) \otimes_{p_1, A_R} N & \xrightarrow{\sim} & A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N \\ \parallel & & \uparrow \wr \\ A_R(1)/p_1(\mu) \otimes_{p_1, R} N/\mu N & \xrightarrow{\sim} & A_R(1)/p_1(\mu) \otimes_{p_1, R} M, \end{array} \quad (4.32)$$

where the bottom horizontal arrow is the extension along  $p_1 : R \rightarrow A_R(1)/p_1(\mu)$  of the inverse of the isomorphism  $\Delta_N$  in (4.31), the right vertical arrow is (4.1) and the top horizontal arrow is the composition of left vertical, bottom horizontal and left vertical arrows. Let us note that in Proposition 5.26, we will show that the map  $\Delta_N$  is equivariant for the action of  $1 \times \Gamma_R$  on the source and the action of  $\Gamma_R$  on the target. Then it will follow that the bottom arrow in the diagram (4.32) is  $(\varphi, 1 \times \Gamma_R)$ -equivariant and therefore, the diagram (4.32) is also  $(\varphi, 1 \times \Gamma_R)$ -equivariant.

## 5. PRISMATIC $F$ -CRYSTALS AND WACH MODULES

In this section, let  $X := \mathrm{Spf} R$  denote a  $p$ -adic formal scheme, where  $R$  is the ring from Subsection 1.6. Our main goal in this section is to relate analytic/completed prismatic  $F$ -crystals on the absolute prismatic site  $(\mathrm{Spf} R)_\Delta$  (see Subsection 2.3) to Wach modules over  $A_R$  (see Definition 4.1). We start with the following construction:

**Proposition 5.1.** *Let  $\mathcal{E}$  be an object of  $\mathrm{Vect}^{\mathrm{an}}(X_\Delta)$  and set  $N := \mathcal{E}(A_R, [p]_q)$ . Then  $N$  is a finitely generated  $A_R$ -module equipped with a continuous action of  $\Gamma_R$  such that the action of  $\Gamma_R$  is trivial on  $N/\mu N$ . Moreover, if  $\mathcal{E}$  is an object of  $\mathrm{Vect}^{\mathrm{an}, \varphi}(X_\Delta)$ , then  $N$  is a Wach module over  $A_R$ .*

*Proof.* First, by using Lemma 5.3, with  $\mathcal{E}(A_R, [p]_q)[1/p]$  as a finite projective  $A_R[1/p]$ -module and  $\mathcal{E}(A_R, [p]_q)[1/\mu]$  as a finite projective  $A_R[1/\mu]$ -module, we see that  $N := \mathcal{E}(A_R, [p]_q)$  is a finitely generated  $A_R$ -module and the sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular on  $N$ . Next, let  $g$  be any element of  $\Gamma_R$  and from Lemma 3.12 recall that  $g$  is an automorphism of  $(A_R, [p]_q)$  in  $X_\Delta$ , i.e. we have that  $g : (A_R, [p]_q) \xrightarrow{\sim} (A_R, [p]_q)$ . Since,  $\mathcal{E}$  is a crystal, it follows that base changing  $N = \mathcal{E}(A_R, [p]_q)$  along  $g$ , induces an  $A_R$ -semilinear isomorphism  $g : A_R \otimes_{g, A_R} N \xrightarrow{\sim} N$ , in particular, a semilinear action of  $\Gamma_R$ . Note that  $A_R/\mu \xrightarrow{\sim} R$ , the pair  $(R, p)$  is a prism and an object of  $(\mathrm{Spf} R)_\Delta$  and  $\Gamma_R$  acts trivially on  $R$ . Therefore, reducing the isomorphism  $g : A_R \otimes_{g, A_R} N \xrightarrow{\sim} N$  modulo  $\mu$ , we obtain an isomorphism  $g \bmod \mu : R \otimes_{g, R} N/\mu N \xrightarrow{\sim} N/\mu N$ , which is the identity, in particular, we see that the action of  $\Gamma_R$  is trivial on  $N/\mu N$ . Now, additionally assume that  $\mathcal{E}$  is an  $F$ -crystal, i.e. an object of  $\mathrm{Vect}^{\mathrm{an}, \varphi}(X_\Delta)$ . Then, from Definition 2.21 we have an  $A_R$ -linear isomorphism  $\varphi_N : (\varphi^* N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ . Again, using that  $\mathcal{E}$  is a crystal and any  $g$  in  $\Gamma_R$  is an automorphism of  $(A_R, [p]_q)$  in  $X_\Delta$ , it is easy to show that  $\varphi_N$  is equivariant for the action of  $\Gamma_R$  on  $N$  described above. So, we see that  $N = \mathcal{E}(A_R, [p]_q)$  satisfies all the axioms of Definition 4.1, i.e. it is a Wach module over  $A_R$ .  $\blacksquare$

**Remark 5.2.** A claim similar to Proposition 5.1 holds for completed prismatic  $F$ -crystals. More precisely, let  $\mathcal{E}$  be an object of  $\mathrm{CR}^\wedge(X_\Delta)$  and set  $N := \mathcal{E}(A_R, [p]_q)$ . Then,  $N$  is a finitely generated  $A_R$ -module equipped with a continuous action of  $\Gamma_R$  such that the action of  $\Gamma_R$  is trivial on  $N/\mu N$ . Moreover, if  $\mathcal{E}$  is an object of  $\mathrm{CR}^{\wedge, \varphi}(X_\Delta)$ , then  $N$  is a Wach module over  $A_R$ .

*Proof.* By Definition 2.16, note that  $N := \mathcal{E}(A_R, [p]_q)$  is a finitely generated  $(p, [p]_q) = (p, \mu)$ -adically complete  $A_R$ -module. Then, similar to the proof of Proposition 5.1, using that  $\mathcal{E}$  is a crystal, we see that any  $g$  in  $\Gamma_R$  is an automorphism of  $(A_R, [p]_q)$  in  $X_\Delta$  and the action of  $\Gamma_R$  is trivial on  $A_R/\mu \xrightarrow{\sim} R$ , in particular,  $N$  is equipped with a semilinear action of  $\Gamma_R$  such that the action of  $\Gamma_R$  is trivial on  $N/\mu N$ . Now additionally assume that  $\mathcal{E}$  is an  $F$ -crystal, i.e. an object of  $\mathrm{CR}^{\wedge, \varphi}(X_\Delta)$ . So from Definition 2.18, we have an  $A_R$ -linear isomorphism  $\varphi_N : (\varphi^* N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$  and similar to above, it is easy to see that  $\varphi_N$  is  $\Gamma_R$ -equivariant. It remains to show that the sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular on  $N$ . Now, let  $\mathfrak{S} := R[[u]]$ , then the Breuil-Kisin prism  $(\mathfrak{S}, u - p)$  from Subsection 2.16 is an object of  $X_\Delta$  and a cover of the final object of the topos  $\mathrm{Shv}(X_\Delta)$  (see [DLMS24, Subsection 3.3]). Since  $(\mathfrak{S}, u - p)$  is a cover of the final object of  $\mathrm{Shv}(X_\Delta)$ , there exists a prism  $(B, J)$  in  $X_\Delta$  so that it is  $(p, [p]_q)$ -completely faithfully flat over  $(A_R, [p]_q)$  and admits a map from  $(\mathfrak{S}, u - p)$  in  $X_\Delta$ . Moreover, as  $A_R$  is noetherian, it follows that the map  $A_R \rightarrow B$  is faithfully flat. Now using an argument similar to [DLMS24, Lemma 3.19] we have that  $B \otimes_{A_R} N \xrightarrow{\sim} \mathcal{E}(B, J)$ , therefore,

$$(B \otimes_{A_R} N)[1/p] \xrightarrow{\sim} \mathcal{E}(B, J)[1/p] \xleftarrow{\sim} B \otimes_{\mathfrak{S}} \mathcal{E}(\mathfrak{S}, u - p)[1/p],$$

where the last isomorphism follows from [DLMS24, Lemma 3.24]. As  $\mathcal{E}(\mathfrak{S}, u - p)[1/p]$  is finite projective over  $\mathfrak{S}[1/p]$  and  $A_R \rightarrow B$  is faithfully flat, it follows that  $N[1/p]$  is finite projective over  $A_R[1/p]$ . A similar argument shows that  $N[1/[p]_q]$  is finite projective over  $A_R[1/[p]_q]$  and from Remark 4.2 it follows that  $N[1/\mu]$  is finite projective over  $A_R[1/\mu]$ . Hence, using Lemma 5.3, we see that  $N = \mathcal{E}(A_R, [p]_q)$  satisfies all the axioms of Definition 4.1, in particular, it is a Wach module over  $A_R$ .  $\blacksquare$

The following observation was used above:

**Lemma 5.3.** *Let  $M$  be a finite projective  $A_R[1/p]$ -module and  $D$  a finite projective  $A_R[1/\mu]$ -module, equipped with an  $A_R[1/p, 1/\mu]$ -linear isomorphism  $f : A_R[1/p, 1/\mu] \otimes_{A_R[1/p]} M \xrightarrow{\sim} A_R[1/p, 1/\mu] \otimes_{A_R[1/\mu]} D$ . Then  $N := M \cap D \subset A_R[1/p, 1/\mu] \otimes_{A_R[1/\mu]} D$  is a finitely generated  $A_R$ -module and the sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular on  $N$ .*

*Proof.* Let  $S = \text{Spec}(A_R)$ ,  $Z = V(p, \mu) \subset S$  as the zero locus of the ideal  $(p, \mu) \subset A_R$  (the same as the zero locus of  $(p, [p]_q) \subset A_R$ ) and set  $j : U = S \setminus Z \subset S$ . Then, note that the natural map  $\text{Spec}(A_R[1/\mu]) \cup \text{Spec}(A_R[1/p]) \rightarrow \text{Spec}(A_R) \setminus V(p, \mu) = U$  is a flat cover. Therefore, from the data  $(M, D, f)$  and faithfully flat descent we obtain a vector bundle  $\mathcal{F}$  over  $U \subset S$ . Since  $\mathcal{F}$  is a vector bundle and  $S$  is irreducible, therefore, the associated point of  $\mathcal{F}$  is the generic point of  $U$ . By definition,  $Z$  is of codimension 2, so for any  $z \in Z$  and  $\mathfrak{p}$  the associated point of  $\mathcal{F}$  in  $\mathcal{O}_{S,z}^\wedge$ , we have that  $\dim(\mathcal{O}_{S,z}^\wedge/\mathfrak{p}) \geq 2$ . Then from [Sta23, Tag 0BK1] it follows that  $j_*\mathcal{F}$  is coherent, i.e.  $N = H^0(S, j_*\mathcal{F}) = H^0(U, \mathcal{F})$  is a finitely presented  $A_R$ -module. Moreover, [Sta23, Tag 0BK0] implies that  $j^*j_*\mathcal{F} \xrightarrow{\sim} \mathcal{F}$  on  $U$  and  $H_Z^0(N) = H_Z^1(N) = 0$ . Therefore, we get that  $N[1/p] \xrightarrow{\sim} M$  and  $N[1/\mu] \xrightarrow{\sim} D$  and using [Abh23b, Lemma 3.3] we conclude that the sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular on  $N$ . Furthermore, from [Abh23b, Lemma 3.5], it follows that  $N = N[1/p] \cap N[1/\mu] = M \cap D \subset A_R[1/p, 1/\mu] \otimes_{A_R[1/p]} M$ . This proves the lemma.  $\blacksquare$

**Remark 5.4.** Let  $N$  be a finitely generated  $A_R$ -module such that the sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular on  $N$ . Then, from [Abh23b, Lemma 3.5], we have that  $N := N[1/p] \cap N[1/\mu] \subset N[1/p, 1/\mu]$ . Moreover, if  $N$  is equipped with an  $A_R$ -linear isomorphism  $\varphi_N : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ , then, from [Abh23b, Proposition 3.11] we have that  $N[1/p]$  is finite projective over  $A_R[1/p]$  and  $N[1/\mu]$  is finite projective over  $A_R[1/\mu]$ .

From Proposition 5.1, note that we have a well-defined evaluation functor for analytic  $F$ -crystals:

$$\begin{aligned} \text{ev}_{A_R}^\Delta : \text{Vect}^{\text{an}, \varphi}(X_\Delta) &\longrightarrow (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q} \\ \mathcal{E} &\longmapsto \mathcal{E}(A_R, [p]_q). \end{aligned} \quad (5.1)$$

Similarly, from Remark 5.2, we also obtain a well-defined evaluation functor for completed  $F$ -crystals:

$$\begin{aligned} \text{ev}_{A_R}^\Delta : \text{CR}^{\wedge, \varphi}(X_\Delta) &\longrightarrow (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q} \\ \mathcal{E} &\longmapsto \mathcal{E}(A_R, [p]_q). \end{aligned} \quad (5.2)$$

Recall that from Lemma 3.11 we have that  $(A_R, [p]_q)$  covers the final object of the topos  $\text{Shv}(X_\Delta)$ . Then we claim the following:

**Theorem 5.5.** *The evaluation functors in (5.1) and (5.2) induce equivalences of categories.*

*Proof.* The claim follows from Proposition 5.8 and Theorem 5.9 shown below.  $\blacksquare$

In the rest of this section we will build the theory needed to state and prove Proposition 5.8 and Theorem 5.9.

**5.1.  $A_R$ -modules with stratification.** In order to prove Theorem 5.5 we will interpret crystals in terms of modules with stratification. We begin with the definition of some cosimplicial objects in  $\text{Shv}(X_\Delta)$ .

**5.1.1. Stratifications.** Using the cosimplicial object  $A_R(\bullet)$  in  $(\text{Spf } R)_\Delta$  as described in Construction 3.13, we define stratifications as follows:

**Definition 5.6.** A *stratification* on an  $A_R$ -module  $N$  with respect to  $A_R(\bullet)$  is an  $A_R(1)$ -linear isomorphism  $\varepsilon : A_R(1) \otimes_{p_1, A_R} N \xrightarrow{\sim} A_R(1) \otimes_{p_2, A_R} N$  satisfying the following conditions:

- (1) The scalar extension  $\Delta^*(\varepsilon)$  of  $\varepsilon$  by  $\Delta : A_R(1) \rightarrow A_R$  is the identity map on  $N$ .

- (2) Cocycle condition, i.e. we have an isomorphism  $p_{23}^*(\varepsilon) \circ p_{12}^*(\varepsilon) = p_{13}^*(\varepsilon) : A_R(2) \otimes_{r_1, A_R} N \xrightarrow{\sim} A_R(2) \otimes_{r_3, A_R} N$ .

Let  $\text{Strat}(A_R(\bullet))$  denote the category of finitely generated  $A_R$ -modules equipped with a stratification with respect to  $A_R(\bullet)$ . Additionally, we will say that  $N$  is *analytic* if the following holds:

- (3) The sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular on  $N$ .

Let  $\text{Strat}^{\text{an}}(A_R(\bullet))$  denote the full subcategory of analytic objects in  $\text{Strat}(A_R(\bullet))$ . Furthermore, we will say that  $N$  is a  $\varphi$ -module over  $A_R$  equipped with a stratification if  $N$  is additionally equipped with a Frobenius, i.e.

- (4) An  $A_R$ -linear isomorphism  $\varphi_N : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$  compatible with the stratification  $\varepsilon$ .

Let  $\text{Strat}^{\text{an}, \varphi}(A_R(\bullet))$  denote the full subcategory of  $\varphi$ -modules over  $A_R$  equipped with a stratification.

Let  $\text{CR}(X_{\Delta}, \mathcal{O}_{\Delta})$  denote the category of crystals of  $\mathcal{O}_{\Delta}$ -modules. Then, from Lemma 3.11, Construction 3.13, and the well-known result describing the category of finitely generated crystals of  $\mathcal{O}_{\Delta}$ -module as modules equipped with a stratification (see [Ber74, Chapitre IV, Subsection 1.6]) and Lemma 5.3, we obtain a natural equivalence of categories,

$$\text{ev}_{A_R(\bullet)}^{\Delta} : \text{CR}(X_{\Delta}, \mathcal{O}_{\Delta}) \xrightarrow{\sim} \text{Strat}(A_R(\bullet)), \quad (5.3)$$

where the functor  $\text{ev}_{A_R(\bullet)}^{\Delta}$  is defined by sending a crystal  $\mathcal{E}$  to the finitely generated  $A_R$ -module  $N = \mathcal{E}(A_R(0))$ , equipped with a stratification given as the composition

$$\varepsilon : A_R(1) \otimes_{p_1, A_R} N \xrightarrow[\sim]{p_1} \mathcal{E}(A_R(1)) \xrightarrow[\sim]{p_2^{-1}} A_R(1) \otimes_{p_2, A_R} N.$$

Now, by using Lemma 3.15 and Remark 5.4, we note that the functor in (5.3) restricts to a natural equivalence of categories,

$$\text{ev}_{A_R(\bullet)}^{\Delta} : \text{Vect}^{\text{an}}(X_{\Delta}) \xrightarrow{\sim} \text{Strat}^{\text{an}}(A_R(\bullet)). \quad (5.4)$$

Furthermore, for  $F$ -crystals the functor in (5.4) restricts to the following natural equivalence of categories,

$$\text{ev}_{A_R(\bullet)}^{\Delta} : \text{Vect}^{\text{an}, \varphi}(X_{\Delta}) \xrightarrow{\sim} \text{Strat}^{\text{an}, \varphi}(A_R(\bullet)). \quad (5.5)$$

A similar argument shows that we have another natural equivalence of categories,

$$\text{ev}_{A_R(\bullet)}^{\Delta} : \text{CR}^{\wedge, \varphi}(X_{\Delta}) \xrightarrow{\sim} \text{Strat}^{\text{an}, \varphi}(A_R(\bullet)). \quad (5.6)$$

In the rest of this subsection, we will show that the functor  $\text{ev}_{A_R(\bullet)}^{\Delta}$  in (5.5) (resp. (5.6)) is suitably compatible with the functor  $\text{ev}_{A_R}^{\Delta}$  in (5.1) (resp. (5.2)), similar to [MT20, Lemma 3.16].

From Remark 3.20, recall that for  $n \in \mathbb{N}$ , the product  $\Gamma_R^{\times(n+1)}$  of  $n+1$  copies of  $\Gamma_R$  naturally acts on  $(A_R(n), [p]_q)$ . Moreover, from Proposition 3.17 we have that the action of the  $i^{\text{th}}$  component of  $\Gamma_R^{\times(n+1)}$  is trivial on  $A_R(n)/(n_i(\mu))$ .

**Construction 5.7.** Let  $(N, \varepsilon)$  be an object of  $\text{Strat}^{\text{an}, \varphi}(A_R(\bullet))$ , i.e.  $N$  is a finitely generated  $A_R$ -module on which the sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular,  $N$  is equipped with a stratification  $\varepsilon$  with respect to  $A_R(\bullet)$  and an  $A_R$ -linear isomorphism  $\varphi_N : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$ . We define a functor,

$$\text{ev}_{A_R}^{\text{Strat}} : \text{Strat}^{\text{an}, \varphi}(A_R(\bullet)) \longrightarrow (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q}, \quad (5.7)$$

by setting the underlying  $A_R$ -module to be  $N$  satisfying (1) and (3) of Definition 4.1. We equip  $N$  with an action of  $\Gamma_R$  as follows: Note that the  $\Gamma_R^2$ -action on  $A_R(1)$  induces a semilinear-action of  $\Gamma_R$

on  $N$ . Indeed, for each  $g$  in  $\Gamma_R$ , the base change of the stratification  $\varepsilon$  along  $A_R(1) \xrightarrow{(g,1)} A_R(1) \xrightarrow{\Delta} A_R$  defines an isomorphism  $A_R \otimes_{g,A_R} N \xrightarrow{\sim} N$ , i.e. a semilinear action of the element  $g$  on  $N$ . Moreover, for any  $g'$  in  $\Gamma_R$ , base change of the aforementioned isomorphism along  $g' : A_R \xrightarrow{\sim} A_R$  is the base change of  $\varepsilon$  along  $g' \circ \Delta \circ g = \Delta \circ (g'g, g')$ . So, from Definition 5.6 it follows that the isomorphisms  $A_R \otimes_{g,A_R} N \xrightarrow{\sim} N$ , for  $g$  in  $\Gamma_R$ , define a semilinear action of  $\Gamma_R$  on  $N$ . Finally, since the action of  $\Gamma_R \times 1$  is trivial on  $A_R(1)/(p_1(\mu))$ , therefore, it follows that the induced action of  $\Gamma_R$  on  $N/\mu N$  is also trivial, and therefore, continuous by [Abh23b, Lemma 3.7]. Hence,  $N$  is a Wach module over  $A_R$  in the sense of Definition 4.1.  $\blacksquare$

We have the following compatibility between the functors of (5.1), (5.5) and (5.7) (resp. (5.2), (5.6) and (5.7))

**Proposition 5.8.** *The following diagram is commutative up to canonical isomorphisms:*

$$\begin{array}{ccccc}
\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\underline{\Delta}}) & \xrightarrow[\sim]{\mathrm{ev}_{A_R(\bullet)}^{\Delta} \text{ (5.5)}} & \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)) & \xleftarrow[\sim]{\mathrm{ev}_{A_R(\bullet)}^{\Delta} \text{ (5.6)}} & \mathrm{CR}^{\wedge,\varphi}(X_{\underline{\Delta}}) \\
& \searrow^{\mathrm{ev}_{A_R}^{\Delta} \text{ (5.1)}} & \downarrow^{\text{(5.7)} \mathrm{ev}_{A_R}^{\mathrm{Strat}}} & \swarrow_{\mathrm{ev}_{A_R}^{\Delta} \text{ (5.1)}} & \\
& & (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q} & & 
\end{array}$$

*Proof.* We will only prove the commutativity of the left triangle; commutativity of the right triangle follows by a similar argument. Let  $\mathcal{E}$  be an object of  $\mathrm{Vect}^{\mathrm{an},\varphi}(X_{\underline{\Delta}})$  and set  $(N, \varepsilon) := \mathrm{ev}_{A_R(\bullet)}^{\Delta}(\mathcal{E})$ . Since  $N = \mathcal{E}(A_R(0))$  and  $A_R(0) = A_R$ , we note that the  $A_R$ -module obtained by composing the top horizontal arrow and the left vertical arrow is isomorphic (as an  $A_R$ -module) to the module obtained via the slanted vertical arrow. From Lemma 5.3, we know that the sequences  $\{p, \mu\}$  and  $\{\mu, p\}$  are regular on  $N$ . The conditions on  $\varphi_N$  as explained in Proposition 5.1 and Construction 5.7 ensure that the Frobenius on  $N$  is of finite  $[p]_q$ -height. So it remains to check the compatibility of the action of  $\Gamma_R$ . Note that for each  $g$  in  $\Gamma_R$ , the following diagrams commute:

$$\begin{array}{ccc}
A_R(1) & \xrightarrow{(g,1)} & A_R(1) \\
p_1 \uparrow & & \downarrow \Delta \\
A_R(0) & \xrightarrow{g} & A_R(0),
\end{array}
\qquad
\begin{array}{ccc}
A_R(1) & \xrightarrow{(g,1)} & A_R(1) \\
p_2 \uparrow & & \downarrow \Delta \\
A_R(0) & \xrightarrow{id} & A_R(0).
\end{array}$$

Therefore, the action of  $g$  on  $N = \mathcal{E}(A_R(0))$  induced by the stratification  $\varepsilon$  (see Construction 5.7) coincides with the action of  $g$  on  $\mathcal{E}(A_R(0))$  induced by the action on  $A_R(0)$  and crystal property of  $\mathcal{E}$  (see Proposition 5.1). Hence,  $\mathrm{ev}_{A_R}^{\mathrm{Strat}}(N, \varepsilon) \xrightarrow{\sim} \mathcal{E}(A_R(0)) = \mathcal{E}(A_R) = \mathrm{ev}_{A_R}^{\Delta}(\mathcal{E})$ , allowing us to conclude.  $\blacksquare$

From Proposition 5.8, we note that in order to show that the functor  $\mathrm{ev}_{A_R}^{\Delta}$  induces an equivalence of categories, it is enough to show that the functor  $\mathrm{ev}_{A_R}^{\mathrm{Strat}}$  induces an equivalence of categories.

**5.2. Constructing stratifications on Wach modules.** In this subsection we will show the following claim:

**Theorem 5.9.** *The functor in (5.7) induces a natural equivalence of categories*

$$\mathrm{ev}_{A_R}^{\mathrm{Strat}} : \mathrm{Strat}^{\mathrm{an},\varphi}(A_R(\bullet)) \xrightarrow{\sim} (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]_q}.$$

*Proof.* In (5.29), we will define a functor  $\mathrm{Strat}_{A_R(\bullet)}$  from the category of Wach modules to the category of  $\varphi$ -modules over  $A_R$  equipped with a stratification with respect to  $A_R(\bullet)$ . Moreover, in Proposition 5.28, we will show that  $\mathrm{Strat}_{A_R(\bullet)}$  is a quasi-inverse to the functor  $\mathrm{ev}_{A_R}^{\mathrm{Strat}}$ , thus establishing the desired categorical equivalence  $\blacksquare$

In the rest of this subsection, we will construct the promised quasi-inverse to the functor  $\mathrm{ev}_{A_R}^{\mathrm{Strat}}$ , where the non-trivial step is the construction of a stratification on a Wach module. Our strategy will be similar to that of [MT20, Subsection 3.2]. However, as we are working with the Galois group  $\Gamma_R$ , which has “arithmetic =  $\Gamma_F$ ” and “geometric =  $\Gamma'_R$ ” parts, instead of being a truly “geometric” Galois group as considered in loc. cit., therefore, our arguments are of different nature and require different computations. More precisely, recall that  $\Gamma_R$  fits into the following exact sequence:

$$1 \longrightarrow \Gamma'_R \longrightarrow \Gamma_R \longrightarrow \Gamma_F \longrightarrow 1.$$

Furthermore, from (1.6), recall that  $\Gamma_F$  fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\mathrm{tor}} \longrightarrow 1,$$

where, for  $p \geq 3$ , we have that  $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$  and for  $p = 2$ , we have that  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$ . Moreover, for  $p \geq 3$ , we have that  $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$  and the projection map  $\Gamma_F \rightarrow \Gamma_{\mathrm{tor}}$ , admits a section  $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$ , where the second map is given as  $a \mapsto [a]$ , the Teichmüller lift of  $a$ , and the final isomorphism is induced by the  $p$ -adic cyclotomic character. Finally, for  $p = 2$ , we have that  $\Gamma_{\mathrm{tor}} \xrightarrow{\sim} \{\pm 1\}$ , as groups.

Let us set  $N(1) := A_R(1) \otimes_{p_2, A_R} N$  equipped with the tensor product Frobenius and tensor product action of  $\Gamma_R^2$ , where  $\Gamma_R^2$  acts on  $N$  via projection to the second coordinate. We will start by proving some results on the cohomology of  $N(1)$  for the action of  $1 \times \Gamma_R$ . The steps will be similar to the “3-step” argument presented in Subsection 3.4. In particular, let us note that the results for the action of the geometric part of  $\Gamma_R$  in Subsection 5.2.1, are applicable for all primes  $p$ . However, for  $p = 2$ , since  $\mathbb{F}_p^\times$  is the trivial group, in Subsections 5.2.2 and 5.2.3, we will assume that  $p \geq 3$ . For  $p = 2$ , the arithmetic action of  $\Gamma_R$  will be handled in Subsection 5.2.4. Finally, we will put everything together in Subsection 5.2.5 to construct a stratification using the action of  $\Gamma_R$  on Wach modules (see Proposition 5.28).

**5.2.1. The action of  $\Gamma'_R$ .** In this subsection, our first goal is to show the following claim:

**Lemma 5.10.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant sequence is exact:*

$$0 \longrightarrow (N(1)/p_1(\mu))^{1 \times \Gamma'_R} \xrightarrow{p_1(\mu)^n} (N(1)/p_1(\mu)^{n+1})^{1 \times \Gamma'_R} \longrightarrow (N(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \longrightarrow 0. \quad (5.8)$$

*Proof.* The proof is similar to the proof of Lemma 3.43. To lighten notations, let us denote by  $A(1) := A_R(1)$ ,  $\bar{A}(1) := A_R(1)/p_1(\mu)$  and  $\bar{N}(1) := N(1)/p_1(\mu)$ . Instead of working with the action of  $1 \times \Gamma'_R$ , we will work with the  $q$ -connection arising from this action. More precisely, in the notation of Definition 3.31, take  $D$  to be  $\tilde{\Lambda}_R \xrightarrow{\sim} A(1)^{1 \times \Gamma'_R}$  (see Lemma 3.47), and  $A$  to be  $A(1)$  equipped with a  $\tilde{\Lambda}_R$ -linear action of  $1 \times \Gamma'_R$  and let  $\{\gamma_1, \dots, \gamma_d\}$  be the topological generators of  $\Gamma'_R$  (see Subsection 3.1). Then, by setting  $q = 1 + p_2(\mu)$  and  $U_i = p_2([X_i^b])$ , for  $1 \leq i \leq d$ , we know that  $A(1)$  satisfies the hypothesis of Definition 3.31 (see the proof of Lemma 3.43). In particular,  $A(1)$  is equipped with a  $\tilde{\Lambda}_R$ -linear  $q$ -connection  $\nabla_q : A(1) \rightarrow q\Omega_{A(1)/\tilde{\Lambda}_R}^1$ , given as  $f \mapsto \sum_{i=1}^d \frac{\gamma_i(f) - f}{p_2(\mu)} d\log(p_2([X_i^b]))$ .

Next, we have that  $N$  is a Wach module over  $A_R$  and  $N(1) = A(1) \otimes_{p_2, A_R} N$  is equipped with the tensor product Frobenius and the tensor product action of  $\Gamma_R^2$ . Note that for any  $f \otimes y$  in  $N(1)$  and  $g$  in  $1 \times \Gamma_R$ , we have that  $(g-1)(f \otimes y) = (g-1)y \otimes y + g(f) \otimes (g-1)y$  is in  $p_2(\mu)N(1)$ . Therefore, the operator

$$\begin{aligned} \nabla_q : N(1) &\longrightarrow N(1) \otimes_{A(1)} \Omega_{A(1)/\tilde{\Lambda}_R}^1 \\ x &\longmapsto \sum_{i=1}^d \frac{\gamma_i(x) - x}{p_2(\mu)} d\log([X_i^b]), \end{aligned}$$

satisfies the assumptions of Definition 4.7. Moreover, from the proof of Lemma 3.47 and Example 4.8, we see that the  $q$ -connection  $\nabla_q$  on  $N(1)$  is  $(p, \mu)$ -adically quasi-nilpotent, and it is flat because



$\gamma_i$  commute with each other. Let us also note that the action of  $1 \times \Gamma'_R$  is trivial on  $p_1(\mu)$  and  $\tilde{\Lambda}_R/p_1(\mu) \xrightarrow{\sim} \Lambda_R \xrightarrow{\sim} \overline{A}(1)^{1 \times \Gamma'_R}$  (see (3.23), Example 3.36 and the proof of Lemma 3.47). Therefore, we see that the  $q$ -connection on  $\overline{N}(1)$ , induced by reducing modulo  $p_1(\mu)$  the  $q$ -connection on  $N(1)$ , coincides with the  $q$ -connection on  $\overline{N}(1)$  described in Example 4.10. Now consider the following exact sequence of  $q$ -de Rham complexes:

$$\begin{aligned} 0 \longrightarrow \overline{N}(1) \otimes_{\overline{A}(1)} q\Omega_{\overline{A}(1)/\Lambda_R}^\bullet &\xrightarrow{p_1(\mu)^n} N(1)/p_1(\mu)^{n+1} \otimes_{A(1)} q\Omega_{A(1)/\tilde{\Lambda}_R}^\bullet \longrightarrow \\ &\longrightarrow N(1)/p_1(\mu)^n \otimes_{A(1)} q\Omega_{A(1)/\tilde{\Lambda}_R}^\bullet \longrightarrow 0. \end{aligned}$$

Since the action of  $1 \times \Gamma'_R$  is continuous for the  $(p, p_1(\mu))$ -adic topology on  $N(1)$ , therefore, we see that  $(N(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} = (N(1)/p_1(\mu)^n)^{\nabla_q=0}$ . In particular, showing that (5.8) is exact, is equivalent to showing that  $H^1(\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega_{\overline{A}(1)/\Lambda_R}^\bullet) = 0$ . Now, from Proposition 4.13, recall that the  $q$ -de Rham complex  $q\Omega_{\overline{A}(1)/\Lambda_R}^\bullet$  is naturally quasi-isomorphic to the de Rham complex  $\Omega_{\overline{A}(1)/\Lambda_R}^\bullet$ . Moreover, in Proposition 4.17, we have shown that the de Rham complex  $\overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^\bullet$  is acyclic in positive degrees, in particular, we see that  $H^1(\overline{N}(1) \otimes_{\overline{A}(1)} q\Omega_{\overline{A}(1)/\Lambda_R}^\bullet) = H^1(\overline{N}(1) \otimes_{\overline{A}(1)} \Omega_{\overline{A}(1)/\Lambda_R}^\bullet) = 0$ . Hence, it follows that (5.8) is exact.  $\blacksquare$

Let  $M_{\tilde{\Lambda}} := N(1)^{1 \times \Gamma'_R}$  as a module over  $A(1)^{1 \times \Gamma'_R} \xleftarrow{\sim} \tilde{\Lambda}_R$  (see Lemma 3.47), equipped with the induced Frobenius and an induced semilinear and continuous action of  $\Gamma_R \times \Gamma_F$ . Let us first note the following:

**Lemma 5.11.** *The sequence  $\{p_2(\mu), p\}$  is regular on  $M_{\tilde{\Lambda}}$ .*

*Proof.* From Definition 4.1 recall that  $\{\mu, p\}$  is a regular sequence on  $N$  and  $p_2 : A_R \rightarrow A_R(1)$  is faithfully flat from Lemma 3.15. Therefore, it follows that  $\{p_2(\mu), p\}$  is a regular sequence on  $A_R(1) \otimes_{p_2, A_R} N$ . Since  $p_2(\mu)$  is invariant under the action of  $1 \times \Gamma'_R$ , it follows that  $p_2(\mu)N(1) \cap M_{\tilde{\Lambda}} = p_2(\mu)M_{\tilde{\Lambda}}$ . In particular, the natural map  $M_{\tilde{\Lambda}}/p_2(\mu)M_{\tilde{\Lambda}} \rightarrow N(1)/p_2(\mu)N(1)$  is injective. Hence,  $M_{\tilde{\Lambda}}/p_2(\mu)M_{\tilde{\Lambda}}$  is  $p$ -torsion free, as claimed.  $\blacksquare$

Our next goal is to describe the action of  $1 \times \Gamma_F \subset \Gamma_R \times \Gamma_F$  on  $M_{\tilde{\Lambda}}$  more explicitly. So let us consider the following  $(\varphi, 1 \times \Gamma_F)$ -equivariant diagram

$$\begin{array}{ccc} A_R & \xrightarrow{p_2} & A_R(1) \\ \iota \uparrow & & \downarrow \Delta' \\ R[[\mu]] & \xrightarrow{\rho} & \tilde{\Lambda}_R, \end{array}$$

where the map  $\iota$  is described in Subsection 3.1, the map  $p_2$  is described in Subsection 3.2.2, the map  $\Delta'$  is described in Subsection 3.4.1 and we set  $\rho := \Delta' \circ p_2 \circ \iota$ . In particular, for the bottom horizontal map we have that  $\rho(\mu) = p_2(\mu)$  and  $\rho(X_i) = p_1([X_i^?])$ , for all  $1 \leq i \leq d$ . Tensoring the right vertical arrow with the Wach module  $N$  over  $A_R$ , we obtain the following  $(\varphi, 1 \times \Gamma_F)$ -equivariant map

$$\Delta'_N : N(1) = A_R(1) \otimes_{p_2, A_R} N \longrightarrow \tilde{\Lambda}_R \otimes_{\Delta' \circ p_2, A_R} N = \tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N,$$

where we consider  $N$  as an  $R[[\mu]]$ -module via the isomorphism  $\iota : R[[\mu]] \xrightarrow{\sim} A_R$ , equipped with a  $(\varphi, 1 \times \Gamma_F)$ -action (see Subsection 4.3.1). For each  $n \geq 1$ , reducing the map  $\Delta'_N$  modulo  $p_1(\mu)^n$  and taking  $(1 \times \Gamma'_R)$ -invariants of the source, we have the following  $(\varphi, 1 \times \Gamma_F)$ -equivariant composition:

$$\Delta'_N : (N(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \longrightarrow (\tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N)/p_1(\mu)^n. \quad (5.9)$$

Then we claim the following:

**Lemma 5.12.** *For each  $n \geq 1$ , the map  $\Delta'_N$  in (5.9) is a  $(\varphi, 1 \times \Gamma_F)$ -equivariant isomorphism of  $\tilde{\Lambda}_R$ -modules. Moreover, (5.9) induces a  $\tilde{\Lambda}_R$ -linear and  $(\varphi, 1 \times \Gamma_F)$ -equivariant isomorphism*

$$\Delta'_N : M_{\tilde{\Lambda}} := N(1)^{1 \times \Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N. \quad (5.10)$$

*In particular, for each  $n \geq 1$ , we have natural  $\tilde{\Lambda}_R$ -linear and  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphisms*

$$M_{\tilde{\Lambda}}/p_1(\mu)^n \xrightarrow{\sim} (N(1)/p_1(\mu)^n)^{1 \times \Gamma'_R}. \quad (5.11)$$

*Proof.* Note that for  $n = 1$ , from (5.9), we have the following  $\Lambda_R$ -linear and  $(\varphi, \Gamma_F)$ -equivariant composition

$$M_{\Lambda} = \overline{N}(1)^{1 \times \Gamma'_R} \longrightarrow \Lambda_R \otimes_{\rho, R[[\mu]]} N,$$

where  $\overline{N}(1) := N(1)/p_1(\mu)$ , and we have used that  $\overline{N}(1)^{1 \times \Gamma'_R} = \overline{N}(1)^{\nabla_q=0} = \overline{N}(1)^{\nabla=0} = M_{\Lambda}$  (in the notations of Proposition 4.13 and Proposition 4.17), since the action of  $1 \times \Gamma'_R$  is continuous on  $\overline{N}(1)$ . Now, from Proposition 4.17 and Remark 4.18, we have that the composition above is precisely the isomorphism in (4.7), in particular, for  $n = 1$ , (5.9) is an isomorphism. Now consider the following  $\tilde{\Lambda}_R$ -linear and  $(\varphi, 1 \times \Gamma_F)$ -equivariant diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_{\Lambda} & \xrightarrow{p_1(\mu)^n} & (N(1)/p_1(\mu)^{n+1})^{1 \times \Gamma'_R} & \longrightarrow & (N(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \longrightarrow 0 \\ & & \downarrow (5.9) & & \downarrow (5.9) & & \downarrow (5.9) \\ 0 & \longrightarrow & \Lambda_R \otimes_{\rho, R[[\mu]]} N & \xrightarrow{p_1(\mu)^n} & (\tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N)/p_1(\mu)^{n+1} & \longrightarrow & (\tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N)/p_1(\mu)^n \longrightarrow 0, \end{array}$$

where the first exact sequence is from (5.8). Using the diagram, an easy induction on  $n \geq 1$ , shows that the natural  $\tilde{\Lambda}_R$ -linear and  $(\varphi, 1 \times \Gamma_F)$ -equivariant map in (5.9) is an isomorphism, i.e.  $\Delta'_N : M_{\tilde{\Lambda}}/p_1(\mu)^n \xrightarrow{\sim} (\tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N)/p_1(\mu)^n$ . Moreover, as both  $N(1)$  and  $(\tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N)$  are  $p_1(\mu)$ -adically complete, taking the limit over  $n \geq 1$  and noting that limit commutes with right adjoint functors, in particular, taking  $(1 \times \Gamma'_R)$ -invariants, we obtain the  $\tilde{\Lambda}_R$ -linear and  $(\varphi, 1 \times \Gamma_F)$ -equivariant isomorphism in (5.10), i.e.  $\Delta'_N : M_{\tilde{\Lambda}} = N(1)^{1 \times \Gamma'_R} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N$ . Finally, note that for each  $n \geq 1$ , it is clear that we have a natural  $\tilde{\Lambda}_R$ -linear and  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant inclusions  $M_{\tilde{\Lambda}}/p_1(\mu)^n \subset (N(1)/p_1(\mu)^n)^{1 \times \Gamma'_R}$ , in particular, the map in (5.11) is injective. Now consider the following  $\tilde{\Lambda}_R$ -linear and  $(\varphi, 1 \times \Gamma_F)$ -equivariant morphisms

$$M_{\tilde{\Lambda}}/p_1(\mu)^n \xrightarrow{(5.11)} (N(1)/p_1(\mu)^n)^{1 \times \Gamma'_R} \xrightarrow{\sim} (\tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N)/p_1(\mu)^n.$$

where it is easy to see that the composition is reduction modulo  $p_1(\mu)^n$  of the isomorphism  $M_{\tilde{\Lambda}} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N$ . Since the last arrow is bijective, therefore, it follows that (5.11) is bijective as well. This allows us to conclude.  $\blacksquare$

**5.2.2. The action of  $\mathbb{F}_p^\times$ .** In this subsection, we will assume  $p \geq 3$  and consider the invariants of the exact sequence (5.8), for the action of  $1 \times \mathbb{F}_p^\times$ . More precisely, we claim the following:

**Lemma 5.13.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant sequence is exact:*

$$0 \longrightarrow M_{\Lambda}^{\mathbb{F}_p^\times} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1 \times \mathbb{F}_p^\times} \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1 \times \mathbb{F}_p^\times} \longrightarrow 0. \quad (5.12)$$

*Proof.* Using the  $\tilde{\Lambda}_R$ -linear and  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism (5.11), the exact sequence (5.8) can be written as the following  $\tilde{\Lambda}_R$ -linear and  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant exact sequence:

$$0 \longrightarrow M_{\Lambda} \xrightarrow{p_1(\mu)^n} M_{\tilde{\Lambda}}/p_1(\mu)^{n+1} \longrightarrow M_{\tilde{\Lambda}}/p_1(\mu)^n \longrightarrow 0.$$

By considering the associated long exact sequence for the cohomology of  $(1 \times \mathbb{F}_p^\times)$ -action and noting that  $H^1(1 \times \mathbb{F}_p^\times, M_{\Lambda}) = 0$ , since  $p - 1$  is invertible in  $\mathbb{Z}_p$ , we obtain that the sequence in (5.12) is exact.  $\blacksquare$

Let us describe the  $\tilde{\Lambda}_R$ -modules in Lemma 5.13 more explicitly. Recall that, from Construction 3.52, we have the ring  $\tilde{\Lambda}_{R,0} = \tilde{\Lambda}_R^{1 \times \mathbb{F}_p^\times}$  equipped with an induced Frobenius and an induced continuous action of  $\Gamma_R \times \Gamma_F$ . Moreover,  $M_{\tilde{\Lambda}} = N(1)^{1 \times \Gamma'_R}$  is a  $\tilde{\Lambda}_R$ -module equipped with an induced continuous action of  $(\varphi, \Gamma_R \times \Gamma_F)$ . Set  $M_{\tilde{\Lambda},0} := M_{\tilde{\Lambda}}^{1 \times \mathbb{F}_p^\times}$  as a  $\tilde{\Lambda}_{R,0}$ -module, equipped with an induced Frobenius and an induced semilinear and continuous action of  $\Gamma_R \times \Gamma_0$ . Furthermore, from (5.11) in Lemma 5.12, we have a natural  $\Lambda_R$ -linear and  $(\varphi, 1 \times \Gamma_F)$ -equivariant isomorphism  $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} M_\Lambda$ , and we set  $M_{\Lambda,0} := M_\Lambda^{1 \times \mathbb{F}_p^\times}$  as a module over  $\Lambda_{R,0} \xleftarrow{\sim} \tilde{\Lambda}_{R,0}/p_1(\mu)$  (see (3.29)), equipped with an induced Frobenius and an induced semilinear and continuous action of  $\Gamma_0$ . Then from the discussion before (4.13), we have that  $M_{\Lambda,0}$  is a finitely generated  $\Lambda_{R,0}$ -module and  $\Lambda_R$ -linearly extending the natural inclusion  $M_{\Lambda,0} \subset M_\Lambda$ , induces a  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $\Lambda_R$ -modules  $\Lambda_R \otimes_{\Lambda_{R,0}} M_{\Lambda,0} \xrightarrow{\sim} M_\Lambda$ . More generally, we have the following:

**Lemma 5.14.** *The  $\tilde{\Lambda}_{R,0}$ -module  $M_{\tilde{\Lambda},0}$  is finitely generated. Moreover, by  $\tilde{\Lambda}_R$ -linearly extending the natural inclusion  $M_{\tilde{\Lambda},0} \subset M_{\tilde{\Lambda}}$ , we obtain a  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism of  $\tilde{\Lambda}_R$ -modules*

$$\begin{aligned} \tilde{\Lambda}_R \otimes_{\tilde{\Lambda}_{R,0}} M_{\tilde{\Lambda},0} &\xrightarrow{\sim} M_{\tilde{\Lambda}} \\ a \otimes x &\longmapsto ax. \end{aligned} \tag{5.13}$$

Furthermore, we have a natural  $(\varphi, \Gamma_0)$ -equivariant isomorphism  $M_{\tilde{\Lambda},0}/p_1(\mu) = M_{\tilde{\Lambda}}^{1 \times \mathbb{F}_p^\times}/p_1(\mu) \xrightarrow{\sim} M_\Lambda^{1 \times \mathbb{F}_p^\times} = M_{\Lambda,0}$ .

*Proof.* From (5.10) in Lemma 5.12, we have a  $(\varphi, 1 \times \Gamma_F)$ -equivariant isomorphism of  $\tilde{\Lambda}_R$ -modules  $\Delta'_N : M_{\tilde{\Lambda}} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N$ , in particular,  $M_{\tilde{\Lambda}}$  is finitely generated over  $\tilde{\Lambda}_R$ . Moreover, from Proposition 4.19, recall that  $N_0 := N^{\mathbb{F}_p^\times}$  is a finitely generated module over  $R[[\mu_0]] = R[[\mu]]^{\mathbb{F}_p^\times}$  (see Lemma 3.4), equipped with the induced action of  $(\varphi, \Gamma_0)$ , and we have a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $R[[\mu]]$ -modules  $R[[\mu]] \otimes_{R[[\mu_0]]} N_0 \xrightarrow{\sim} N$ . Then, taking invariants of (5.10) under the action of  $1 \times \mathbb{F}_p^\times$ , induces a  $(\varphi, 1 \times \Gamma_0)$ -equivariant isomorphism of  $\tilde{\Lambda}_{R,0}$ -modules,

$$M_{\tilde{\Lambda},0} \xrightarrow{\sim} \tilde{\Lambda}_{R,0} \otimes_{\rho, R[[\mu_0]]} N_0. \tag{5.14}$$

In particular, it follows that  $M_{\tilde{\Lambda},0}$  is finitely generated over  $\tilde{\Lambda}_{R,0}$ . Now consider the following natural  $\tilde{\Lambda}_R$ -linear diagram

$$\begin{array}{ccc} \tilde{\Lambda}_R \otimes_{\tilde{\Lambda}_{R,0}} M_{\tilde{\Lambda},0} & \xrightarrow{(5.13)} & M_{\tilde{\Lambda}} \\ \downarrow (5.14) & & \downarrow (5.9) \\ \tilde{\Lambda}_R \otimes_{\rho, R[[\mu_0]]} N_0 & \xrightarrow{\sim} & \tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N, \end{array}$$

where the bottom horizontal arrow is the extension along the  $(\varphi, 1 \times \Gamma_F)$ -equivariant map  $\rho : R[[\mu]] \rightarrow \tilde{\Lambda}_R$  (see the discussion before Lemma 5.12) of the  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $R[[\mu]] \otimes_{R[[\mu_0]]} N_0 \xrightarrow{\sim} N$  from Proposition 4.19. The diagram commutes by definition and it follows that the top horizontal arrow, i.e. (5.13) is also an isomorphism. Finally, it is easy to see that the isomorphism (5.13) induces a natural  $(\varphi, \Gamma_0)$ -equivariant isomorphism  $M_{\tilde{\Lambda},0}/p_1(\mu) = M_{\tilde{\Lambda}}^{1 \times \mathbb{F}_p^\times}/p_1(\mu) \xrightarrow{\sim} M_\Lambda^{1 \times \mathbb{F}_p^\times} = M_{\Lambda,0}$ .  $\blacksquare$

Let us note an important observation for the action of  $\Gamma_R \times \Gamma_F$  on  $M_{\tilde{\Lambda}}$ .

**Lemma 5.15.** *The action of  $1 \times \Gamma_F$  is trivial on  $M_{\tilde{\Lambda}}/p_2(\mu)$  and the action of  $\Gamma_R \times 1$  is trivial on  $M_{\tilde{\Lambda}}/p_1(\mu)$ .*

*Proof.* From (5.10) in Lemma 5.12, recall that we have a  $(\varphi, 1 \times \Gamma_F)$ -equivariant isomorphism  $\Delta'_N : M_{\tilde{\Lambda}} \xrightarrow{\sim} \tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N$ . Now let  $g$  be any element of  $\Gamma_F = 1 \times \Gamma_F$ , then for any  $f \otimes y$  in  $\tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N$ , we have that

$$(g - 1)(f \otimes y) = ((g - 1)f) \otimes y + g(f) \otimes (g - 1)y.$$

Since the action of  $1 \times \Gamma_F$  is trivial on  $\tilde{\Lambda}_R/p_2(\mu)$  from Lemma 3.48 as well as on  $N/\mu N$  by definition, it follows that  $(g-1)(f \otimes x)$  is an element of  $p_2(\mu)\tilde{\Lambda}_R \otimes_{\rho, R[[\mu]]} N$ . Then using the  $(\varphi, 1 \times \Gamma_F)$ -equivariant isomorphism (5.10), it follows that for any  $x$  in  $M_{\tilde{\Lambda}}$ , we have that  $(g-1)x$  is an element of  $p_2(\mu)M_{\tilde{\Lambda}}$ , in particular, the action of  $1 \times \Gamma_F$  is trivial on  $M_{\tilde{\Lambda}}/p_2(\mu)$ . Next, since the action of  $\Gamma_R \times 1$  is trivial on  $N(1)/p_1(\mu)$  (see the proof of Lemma 5.10), therefore, it easily follows that the induced action of  $\Gamma_R \times 1$  is trivial on  $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} (N(1)/p_1(\mu))^{1 \times \Gamma'_R}$  (see (5.11) in Lemma 5.12). This concludes our proof.  $\blacksquare$

**5.2.3. The action of  $1 + p\mathbb{Z}_p$ .** In this subsection, we will assume that  $p \geq 3$  and consider the invariants of the exact sequence (5.12), for the action of  $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + p\mathbb{Z}_p)$ , and show the following:

**Lemma 5.16.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times 1)$ -equivariant sequence is exact:*

$$0 \longrightarrow M_{\tilde{\Lambda}}^{1 \times \Gamma_F} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1 \times \Gamma_F} \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1 \times \Gamma_F} \longrightarrow 0. \quad (5.15)$$

For each  $n \geq 1$ , note that by reducing modulo  $p_1(\mu)^n$  the  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism in (5.13) and taking its  $(1 \times \mathbb{F}_p^\times)$ -invariants, we obtain a  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism  $M_{\tilde{\Lambda},0}/p_1(\mu)^n \xrightarrow{\sim} (M_{\tilde{\Lambda},0}/p_1(\mu)^n)^{1 \times \mathbb{F}_p^\times}$ , because  $p_1(\mu)$  is invariant under the action of  $1 \times \Gamma_F$ . Consequently, the sequence in (3.31) can be rewritten as the following  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant sequence:

$$0 \longrightarrow M_{\tilde{\Lambda},0}^{\Gamma_0} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1})^{1 \times \Gamma_0} \longrightarrow (M_{\tilde{\Lambda},0}/p_1(\mu)^n)^{1 \times \Gamma_0} \longrightarrow 0. \quad (5.16)$$

In order to prove that (5.16) is exact, we will now look at the action of  $\Gamma_R \times \Gamma_F$  on  $M_{\tilde{\Lambda}}$  and  $M_{\tilde{\Lambda},0}$ , respectively. We start with the following observation:

**Lemma 5.17.** *The action of  $1 \times \Gamma_0$  is trivial on  $M_{\tilde{\Lambda},0}/p_2(\mu_0)$  and the action of  $\Gamma_R \times 1$  is trivial on  $M_{\tilde{\Lambda},0}/p_1(\mu)$ .*

*Proof.* Using the triviality of the action of  $1 \times \Gamma_F$  on  $M_{\tilde{\Lambda}}/p_2(\mu)$  from Lemma 5.15, the  $(\varphi, 1 \times \Gamma_0)$ -equivariant isomorphism  $M_{\tilde{\Lambda},0} \xrightarrow{\sim} \tilde{\Lambda}_{R,0} \otimes_{\rho, R[[\mu_0]]} N_0$  (see (5.13) in Lemma 5.13) and the fact that the action of  $1 \times \Gamma_0$  is trivial on  $\tilde{\Lambda}_{R,0}/p_2(\mu_0)$  (see Lemma 3.55) as well as on  $N_0/\mu_0 N_0$  (see Proposition 4.19), it follows that the action of  $1 \times \Gamma_0$  is trivial on  $M_{\tilde{\Lambda},0}/p_2(\mu_0)$ . Next, since the action of  $\Gamma_R \times 1$  is trivial on  $N(1)/p_1(\mu)$  (see the proof of Lemma 5.10), therefore, it easily follows that the induced action of  $\Gamma_R \times 1$  is trivial on  $M_{\tilde{\Lambda},0}/p_1(\mu) \xrightarrow{\sim} M_{\tilde{\Lambda}}^{1 \times \mathbb{F}_p^\times}$  (see Lemma 5.14).  $\blacksquare$

**Remark 5.18.** From Lemma 5.17, note that the action of  $1 \times \Gamma_0$  is trivial on  $M_{\tilde{\Lambda},0}/p_2(\mu_0)$  and the element  $p_1(\mu)$  is invariant under the action of  $1 \times \Gamma_0$  on  $\tilde{\Lambda}_{R,0}$ . Therefore, it follows that for any  $g$  in  $\Gamma_0$  and any  $x$  in  $M_{\tilde{\Lambda},0}/p_1(\mu)^n$  we have that  $(g-1)x$  is an element of  $p_2(\mu_0)M_{\tilde{\Lambda},0}/p_1(\mu)^n$ . In particular, for  $n = 1$ , using the isomorphism  $M_{\tilde{\Lambda},0}/p_1(\mu) \xleftarrow{\sim} M_{\tilde{\Lambda},0}$  from Lemma 5.14, we get that for any  $g$  in  $\Gamma_0$  and any  $x$  in  $M_{\tilde{\Lambda},0}$  the element  $(g-1)x$  belongs to  $\mu_0 M_{\tilde{\Lambda},0}$ .

Using the action of  $1 \times \Gamma_0$  on  $M_{\tilde{\Lambda},0}$ , we will define a  $q$ -connection (see Definition 4.7). Recall that in Subsection 3.4.3, in order to define a  $q$ -de Rham complex over  $\tilde{\Lambda}_{R,0}$ , we fixed the following element in  $\tilde{\Lambda}_{R,0}$  as a parameter (see (3.33)):

$$\tilde{s} = \frac{1 \otimes \tilde{p} - \tilde{p} \otimes 1}{\tilde{p} \otimes 1} = \frac{p_2(\tilde{p}) - p_1(\tilde{p})}{p_1(\tilde{p})}$$

Moreover, if  $\gamma_0$  is any element of  $1 \times \Gamma_0$ , then from Lemma 3.57 we have that  $(\gamma_0 - 1)\tilde{s} = u p_2(\mu_0)$ , for some unit  $u$  in  $\tilde{\Lambda}_{R,0}$ .

In the rest of this subsection, we will fix the choice of a topological generator  $\gamma_0$  of  $1 \times \Gamma_0$  such that  $\chi(\gamma_0) = 1 + pa$ , for a unit  $a$  in  $\mathbb{Z}_p$ . Let us now consider the following operator on  $M_{\tilde{\Lambda},0}$ :

$$\begin{aligned} \nabla_{q, \tilde{s}} : M_{\tilde{\Lambda},0} &\longrightarrow M_{\tilde{\Lambda},0} \\ x &\mapsto \frac{(\gamma_0 - 1)x}{(\gamma_0 - 1)\tilde{s}}. \end{aligned} \quad (5.17)$$

From the triviality of the action of  $1 \times \Gamma_0$  on  $M_{\tilde{\Lambda},0}/p_2(\mu_0)$  (see Lemma 5.17) and from Lemma 3.57, it follows that the operator  $\nabla_{q,\tilde{s}}$  is well-defined. For each  $n \in \mathbb{N}$ , using Remark 5.18, the operator in (5.17), induces well-defined operators  $\nabla_{q,\tilde{s}} : M_{\tilde{\Lambda},0}/p_1(\mu)^n \rightarrow M_{\tilde{\Lambda},0}/p_1(\mu)^n$ . As the operator  $\nabla_{q,\tilde{s}}$  is an endomorphism of  $M_{\tilde{\Lambda},0}/p_1(\mu)^n$ , we can define the following two term Koszul complex:

$$K_{M_{\tilde{\Lambda},0}/p_1(\mu)^n}(\nabla_{q,\tilde{s}}) : [M_{\tilde{\Lambda},0}/p_1(\mu)^n \xrightarrow{\nabla_{q,\tilde{s}}} M_{\tilde{\Lambda},0}/p_1(\mu)^n]. \quad (5.18)$$

In particular, for  $n = 1$ , we set  $s := \mu_0/p$  in  $\Lambda_{R,0}$ , then using Remark 5.18 and the fact that  $(\gamma_0 - 1)s = v\mu_0$ , for some unit  $v$  in  $\Lambda_{F,0}$  (see Lemma 3.58), we have a well-defined operator

$$\begin{aligned} \nabla_{q,s} : M_{\Lambda,0} &\longrightarrow M_{\Lambda,0} \\ x &\longmapsto \frac{(\gamma_0 - 1)x}{(\gamma_0 - 1)s}. \end{aligned}$$

Note that the operator above coincides with the operator defined in (4.14) and the complex from (5.18) for  $n = 1$ , coincides with the complex from (4.15). Therefore, from Proposition 4.25, we have that the cohomology of the Koszul complex  $K_{M_{\Lambda,0}}(\nabla_{q,s})$  vanishes in degree 1, i.e.  $H^1(K_{M_{\Lambda,0}}(\nabla_{q,s})) = 0$ .

**Remark 5.19.** Considering  $\tilde{s}$  as a parameter, similar to Remark 3.59, the operator  $\nabla_{q,\tilde{s}}$  in (5.17), may be considered as a  $q$ -connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32. Then, (4.15) is the  $q$ -de Rham complex arising from such a  $q$ -connection. Similarly, considering  $s$  as a parameter, the operator  $\nabla_{q,s}$  on  $M_{\Lambda,0}$ , may be also considered as a  $q$ -connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32.

*Proof of Lemma 5.16.* Note that using the  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism  $M_{\Lambda,0}/p_1(\mu)^n \xrightarrow{\sim} (M_{\Lambda}/p_1(\mu)^n)^{1 \times \mathbb{F}_p^\times}$  (see the discussion before (5.16)), we can rewrite the exact sequence in (5.12) as follows:

$$0 \longrightarrow M_{\Lambda,0} \xrightarrow{p_1(\mu)^n} M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1} \longrightarrow M_{\tilde{\Lambda},0}/p_1(\mu)^n \longrightarrow 0.$$

Then, using the operator  $\nabla_{q,\tilde{s}}$  in (5.17) and the Koszul complex defined in (5.18), we obtain an exact sequence of Koszul complexes:

$$0 \longrightarrow K_{M_{\Lambda,0}}(\nabla_{q,s}) \xrightarrow{p_1(\mu)^n} K_{M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1}}(\nabla_{q,\tilde{s}}) \longrightarrow K_{M_{\tilde{\Lambda},0}/p_1(\mu)^n}(\nabla_{q,\tilde{s}}) \longrightarrow 0.$$

Considering the associated long exact sequence, and noting that  $H^1(K_{M_{\Lambda,0}}(\nabla_{q,s})) = 0$  from Proposition 4.25, we obtain the following exact sequence:

$$0 \longrightarrow M_{\Lambda,0}^{\nabla_{q,s}=0} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{s}}=0} \longrightarrow (M_{\tilde{\Lambda},0}/p_1(\mu)^n)^{\nabla_{q,\tilde{s}}=0} \longrightarrow 0.$$

Since the action of  $1 \times \Gamma_0$  is continuous on  $M_{\tilde{\Lambda},0}$  for the  $(p, p_1(\mu))$ -adic topology, therefore, we see that  $(M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{s}}=0} = (M_{\tilde{\Lambda},0}/p_1(\mu)^{n+1})^{1 \times \Gamma_0}$ , for each  $n \in \mathbb{N}$ . Hence, from the preceding exact sequence we obtain that the sequence in (5.16) is exact, and therefore, the sequence (5.15) is exact as well.  $\blacksquare$

**5.2.4. The case  $p = 2$ .** In this subsection, our goal is to prove a statement similar to Lemma 5.16, for  $p = 2$ . From (1.6), recall that  $\Gamma_F$  fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\text{tor}} \longrightarrow 1,$$

where, we have that  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$  and  $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$ , as groups.

We will first look at the action of  $\Gamma_{\text{tor}}$  on  $M_{\tilde{\Lambda}}$ . Let  $\sigma$  denote a generator of  $\Gamma_{\text{tor}}$ . Then, from (A.2), recall that by setting  $M_{\tilde{\Lambda},+} := \{x \in M_{\tilde{\Lambda}} \text{ such that } \sigma(x) = x\}$  and  $M_{\tilde{\Lambda},-} := \{x \in M_{\tilde{\Lambda}} \text{ such that } \sigma(x) = -x\}$ , we have a natural injective map of  $\tilde{\Lambda}_{R,+}$ -modules

$$M_{\tilde{\Lambda},+} \oplus M_{\tilde{\Lambda},-} \longrightarrow M_{\tilde{\Lambda}}, \quad (5.19)$$

given as  $(x, y) \mapsto x + y$ . Note that the action of  $1 \times \Gamma_F$  is continuous for the  $(p, p_1(\mu))$ -adic topology on  $M_{\tilde{\Lambda}}$ , so it follows that  $M_{\tilde{\Lambda},+}$  is a  $(p, p_1(\mu))$ -adically complete  $\tilde{\Lambda}_{R,+}$ -submodule of  $M_{\tilde{\Lambda}}$ , stable under the action of  $(\varphi, \Gamma_R \times \Gamma_F)$  on  $M_{\tilde{\Lambda}}$ , and similarly,  $M_{\tilde{\Lambda},-}$  is a complete  $\tilde{\Lambda}_{R,+}$ -submodule of  $M_{\tilde{\Lambda}}$ , stable under the action of  $(\varphi, \Gamma_R \times \Gamma_F)$ . Equipping  $M_{\tilde{\Lambda},+}$  and  $M_{\tilde{\Lambda},-}$  with induced structures, we see that (5.19) is  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant.

Now, from (5.11), recall that  $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} (N(1)/p_1(\mu))^{1 \times \Gamma'_R} = M_{\Lambda}$ . Similar to above, in Subsection 4.3.3, we defined  $\Lambda_{R,+}$ -modules  $M_{\Lambda,+}$  and  $M_{\Lambda,-}$  and showed that their natural inclusion in  $M_{\Lambda}$  induces a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism of  $\Lambda_{R,+}$ -modules  $M_{\Lambda,+} \oplus M_{\Lambda,-} \xrightarrow{\sim} M_{\Lambda,+}$  (see (4.21) in Lemma 4.26).

**Lemma 5.20.** *For each  $n \geq 1$ , reduction modulo  $p_1(\mu)^n$  of (5.19), induces a natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism*

$$M_{\tilde{\Lambda},+}/p_1(\mu)^n \xrightarrow{\sim} (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}}. \quad (5.20)$$

Moreover, for  $n = 1$ , the  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} M_{\Lambda}$  from (5.11), induces a natural  $(\varphi, \Gamma_F)$ -equivariant isomorphism

$$M_{\tilde{\Lambda},+}/p_1(\mu) \xrightarrow{\sim} (M_{\tilde{\Lambda}}/p_1(\mu))^{1 \times \Gamma_{\text{tor}}} \xrightarrow{\sim} M_{\Lambda,+}. \quad (5.21)$$

*Proof.* Let us consider the following natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant commutative diagram:

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_{\tilde{\Lambda},+}/p_1(\mu) & \xrightarrow{p_1(\mu)^n} & M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1} & \longrightarrow & M_{\tilde{\Lambda},+}/p_1(\mu)^n \longrightarrow 0 \\ & & \downarrow (5.19) & & \downarrow (5.19) & & \downarrow (5.19) \\ 0 & \longrightarrow & (M_{\tilde{\Lambda}}/p_1(\mu))^{1 \times \Gamma_{\text{tor}}} & \xrightarrow{p_1(\mu)^n} & (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1 \times \Gamma_{\text{tor}}} & \longrightarrow & (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}} \longrightarrow 0, \end{array} \quad (5.22)$$

where the top row is exact and the vertical maps are injective because we have  $p_1(\mu)^n M_{\tilde{\Lambda}} \cap M_{\tilde{\Lambda},+} = p_1(\mu)^n M_{\tilde{\Lambda},+}$ , as  $p_1(\mu)$  is invariant under the action of  $1 \times \Gamma_F$ . As the action of  $1 \times \Gamma_{\text{tor}} \subset 1 \times \Gamma_F$  is trivial on  $M_{\tilde{\Lambda}}/p_2(\mu)$  from Lemma 5.15, therefore, for each  $n \geq 1$  and  $x$  in  $M_{\tilde{\Lambda}}/p_1(\mu)^n$ , we see that  $(\sigma - 1)x$  is an element of  $p_2(\mu)(M_{\tilde{\Lambda}}/p_1(\mu)^n)$ . Then from Lemma A.11, it follows that the bottom right horizontal arrow in (5.22) is surjective, in particular, the bottom row is exact.

Next, by composing the left vertical arrow in (5.22) with the  $(\varphi, \Gamma_F)$ -equivariant isomorphism  $M_{\tilde{\Lambda}}/p_1(\mu) \xrightarrow{\sim} M_{\Lambda}$  from (5.11), we obtain a natural  $(\varphi, \Gamma_F)$ -equivariant injective map  $M_{\tilde{\Lambda},+}/p_1(\mu) \rightarrow M_{\Lambda,+}$ , and we will show that it is surjective as well. Indeed, since the action of  $1 \times \Gamma_F$  is trivial on  $M_{\tilde{\Lambda}}/p_2(\mu)$  from Lemma 5.15 and the action of  $\Gamma_F$  is trivial on  $M_{\Lambda}/\mu M_{\Lambda}$  using (4.7) in Proposition 4.17, therefore, by using Lemma A.11 it follows that the  $\tilde{\Lambda}_R$ -linear and  $(\varphi, 1 \times \Gamma_F)$ -equivariant surjective map  $M_{\tilde{\Lambda}} \twoheadrightarrow M_{\Lambda}$  from (5.11), induces a  $\tilde{\Lambda}_{R,+}$ -linear and  $(\varphi, 1 \times \Gamma_0)$ -equivariant surjective map  $M_{\tilde{\Lambda},+} \twoheadrightarrow M_{\Lambda,+}$ , which factors through (5.21). In particular, we get that the composition in (5.21) is bijective, therefore, the left vertical arrow in (5.22) is also bijective. Now, using the diagram (5.22), an easy induction on  $n \geq 1$ , gives that for each  $n \geq 1$ , the right vertical arrow is bijective. Hence, it follows that the natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant map  $M_{\tilde{\Lambda},+}/p_1(\mu)^n \rightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}}$ , induced by (5.19), is bijective for each  $n \geq 1$ .  $\blacksquare$

From Lemma 5.20, we obtain the following:

**Lemma 5.21.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant sequence is exact:*

$$0 \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu))^{1 \times \Gamma_{\text{tor}}} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1 \times \Gamma_{\text{tor}}} \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}} \longrightarrow 0. \quad (5.23)$$

*Proof.* The sequence (5.23) is the same as the second row of the diagram (5.20), which was shown to be exact in the proof of Lemma 5.20.  $\blacksquare$

Next, we will look at the action of  $1 \times \Gamma_0 \xrightarrow{\sim} 1 \times (1 + 4\mathbb{Z}_2)$  on  $M_{\tilde{\Lambda},+}$  and show the following:

**Lemma 5.22.** *For each  $n \geq 1$ , the following natural  $(\varphi, \Gamma_R \times 1)$ -equivariant sequence is exact:*

$$0 \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu))^{1 \times \Gamma_F} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda}}/p_1(\mu)^{n+1})^{1 \times \Gamma_F} \longrightarrow (M_{\tilde{\Lambda}}/p_1(\mu)^n)^{1 \times \Gamma_F} \longrightarrow 0. \quad (5.24)$$

For each  $n \geq 1$ , note that by reducing modulo  $p_1(\mu)^n$  the  $(\varphi, \Gamma_R \times \Gamma_F)$ -equivariant isomorphism in (5.13) and taking its  $(1 \times \Gamma_{\text{tor}})$ -invariants, we obtain a  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism  $M_{\Lambda,+}/p_1(\mu)^n \xrightarrow{\sim} (M_{\Lambda,+}/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}}$ , because  $p_1(\mu)$  is invariant under the action of  $1 \times \Gamma_F$ . Consequently, the sequence in (5.15) can be rewritten as the following  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant sequence:

$$0 \longrightarrow M_{\tilde{\Lambda},+}^{\Gamma_0} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1})^{1 \times \Gamma_0} \longrightarrow (M_{\tilde{\Lambda},+}/p_1(\mu)^n)^{1 \times \Gamma_0} \longrightarrow 0. \quad (5.25)$$

In order to prove that (5.25) is exact, we will now look at the action of  $\Gamma_R \times \Gamma_F$  on  $M_{\tilde{\Lambda}}$  and  $M_{\tilde{\Lambda},+}$ , respectively. From (3.40) recall that we set  $\nu = \frac{\mu^2}{1+\mu}$  in  $A_F$ , and we make the following observation:

**Lemma 5.23.** *The action of  $1 \times \Gamma_0$  is trivial on  $M_{\tilde{\Lambda},+}/p_2(\nu)$  and the action of  $\Gamma_R \times 1$  is trivial on  $M_{\tilde{\Lambda},+}/p_1(\mu)$ .*

*Proof.* For the first claim, note that  $\nu$  is invariant under the action of  $\Gamma_{\text{tor}}$  and we have that  $p_2(\nu)M_{\tilde{\Lambda}} \cap M_{\tilde{\Lambda},+} = p_2(\nu)M_{\tilde{\Lambda},+}$ . So, if  $x$  is an element of  $M_{\tilde{\Lambda},+}$  and  $g$  any element of  $1 \times \Gamma_0$ , then it is enough to show that  $(g-1)x$  is an element of  $p_2(\nu)M_{\tilde{\Lambda}}$ . Moreover, note that  $\nu$  and  $\mu^2$  differ by a unit in  $A_F$ . Therefore, we are reduced to showing that  $(g-1)x$  is an element of  $p_2(\mu)^2M_{\tilde{\Lambda}}$ . Now, using Lemma 5.15, we can write  $(g-1)x = p_2(\mu)y$ , for some  $y$  in  $M_{\tilde{\Lambda}}$ . Let  $\sigma$  be a generator of  $1 \times \Gamma_{\text{tor}}$  and note that  $\sigma(x) = x$ . Then, we have that  $\sigma(p_2(\mu))\sigma(y) = p_2(\mu)y$ , in particular,  $(\sigma-1)y = -(2+p_2(\mu))y$ . Again using Lemma 5.15, we can write  $-p_2([p]_q)y = (\sigma-1)y = p_2(\mu)z$ , for some  $z$  in  $M_{\tilde{\Lambda}}$ . So we get that  $-py = 0 \pmod{p_2(\mu)M_{\tilde{\Lambda}}}$ . Note that  $(p_2(\mu), p)$  is a regular sequence on  $M_{\tilde{\Lambda}}$  (see Lemma 5.11). Therefore, we conclude that  $y = 0 \pmod{p_2(\mu)\tilde{\Lambda}_R}$ , i.e.  $y$  is an element of  $p_2(\mu)\tilde{\Lambda}_R$  and  $(g-1)x = p_2(\mu)y$  is an element of  $p_2(\mu)^2\tilde{\Lambda}_R$ , as claimed. The second claim easily follows from Lemma 3.48. ■

**Remark 5.24.** From Lemma 5.23, note that the action of  $1 \times \Gamma_0$  is trivial on  $M_{\tilde{\Lambda},+}/p_2(\nu)$  and the element  $p_1(\mu)$  is invariant under this action. Therefore, it follows that for any  $g$  in  $1 \times \Gamma_0$  and any  $x$  in  $M_{\tilde{\Lambda},+}/p_1(\mu)^n$ , we have that  $(g-1)x$  is an element of  $p_2(\nu)M_{\tilde{\Lambda},+}/p_1(\mu)^n$ . In particular, for  $n=1$ , using the isomorphism  $M_{\tilde{\Lambda},+}/p_1(\mu) \xrightarrow{\sim} M_{\Lambda,+}$  from (5.21), we get that for any  $g$  in  $\Gamma_0$  and any  $x$  in  $M_{\Lambda,+}$ , the element  $(g-1)x$  belongs to  $\nu M_{\Lambda,+}$ .

Using the action of  $1 \times \Gamma_0$  on  $M_{\tilde{\Lambda},+}$ , we will define a  $q$ -connection (see Definition 4.7). Recall that in Subsection 3.4.4, in order to define a  $q$ -de Rham complex over  $\tilde{\Lambda}_R$ , we fixed the following element in  $\tilde{\Lambda}_{R,+}$  as a parameter (see (3.42)):

$$\tilde{\tau} = \frac{1}{p_2([p]_q)} \delta \left( \frac{p_2([p]_q)}{p_1([p]_q)} \right).$$

Moreover, if  $\gamma_0$  is any element of  $1 \times \Gamma_0$ , then from Lemma 3.71 we have that  $(\gamma_0-1)\tilde{\tau} = up_2(\nu)$ , for some unit  $u$  in  $\tilde{\Lambda}_{R,+}$ .

In the rest of this subsection, we will fix the choice of a topological generator  $\gamma_0$  of  $1 \times \Gamma_0$  such that  $\chi(\gamma_0) = 1 + 4a$ , for a unit  $a$  in  $\mathbb{Z}_2$ . Let us now consider the following operator on  $M_{\tilde{\Lambda},+}$ :

$$\begin{aligned} \nabla_{q,\tilde{\tau}} : M_{\tilde{\Lambda},+} &\longrightarrow M_{\tilde{\Lambda},+} \\ x &\mapsto \frac{(\gamma_0-1)x}{(\gamma_0-1)\tilde{\tau}}. \end{aligned} \quad (5.26)$$

From the triviality of the action of  $1 \times \Gamma_0$  on  $M_{\tilde{\Lambda},+}/p_2(\nu)$  (see Lemma 5.23) and from Lemma 3.71, it follows that the operator  $\nabla_{q,\tilde{\tau}}$  is well-defined. For each  $n \in \mathbb{N}$ , using Remark 5.24, the operator in (5.26), induces well-defined operators  $\nabla_{q,\tilde{\tau}} : M_{\tilde{\Lambda},+}/p_1(\mu)^n \longrightarrow M_{\tilde{\Lambda},+}/p_1(\mu)^n$ . As the operator  $\nabla_{q,\tilde{\tau}}$  is an endomorphism of  $M_{\tilde{\Lambda},+}/p_1(\mu)^n$ , we can define the following two term Koszul complex:

$$K_{M_{\tilde{\Lambda},+}/p_1(\mu)^n}(\nabla_{q,\tilde{\tau}}) : [M_{\tilde{\Lambda},+}/p_1(\mu)^n \xrightarrow{\nabla_{q,\tilde{\tau}}} M_{\tilde{\Lambda},+}/p_1(\mu)^n]. \quad (5.27)$$

In particular, for  $n = 1$ , we set  $\tau := \nu/8$  in  $\Lambda_{R,+}$ , then using Remark 5.24 and the fact that  $(\gamma_0 - 1)s = \nu\nu$ , for some unit  $\nu$  in  $\Lambda_{F,+}$  (see Lemma 3.72), we have a well-defined operator

$$\begin{aligned} \nabla_{q,\tau} : M_{\Lambda,+} &\longrightarrow M_{\Lambda,+} \\ x &\longmapsto \frac{(\gamma_0 - 1)x}{(\gamma_0 - 1)\tau}. \end{aligned}$$

Note that the operator above coincides with the operator defined in (4.24) and the complex from (5.27) for  $n = 1$ , coincides with the complex from (4.25). Therefore, from Proposition 4.31, we have that the cohomology of the Koszul complex  $K_{M_{\Lambda,+}}(\nabla_{q,s})$  vanishes in degree 1, i.e.  $H^1(K_{M_{\Lambda,+}}(\nabla_{q,\tau})) = 0$ .

**Remark 5.25.** Considering  $\tilde{\tau}$  as a parameter, similar to Remark 3.73, the operator  $\nabla_{q,\tilde{\tau}}$  in (5.26), may be considered as a  $q$ -connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32. Then, (4.25) is the  $q$ -de Rham complex arising from such a  $q$ -connection. Similarly, considering  $s$  as a parameter, the operator  $\nabla_{q,\tau}$  on  $M_{\Lambda,+}$ , may be also considered as a  $q$ -connection in non-logarithmic coordinates, in the sense of Definition 4.7 and Remark 3.32.

*Proof of Lemma 5.22.* Note that by using the  $(\varphi, \Gamma_R \times \Gamma_0)$ -equivariant isomorphism  $M_{\Lambda,+}/p_1(\mu)^n \xrightarrow{\sim} (M_{\Lambda}/p_1(\mu)^n)^{1 \times \Gamma_{\text{tor}}}$  (see the discussion before (5.25)), we can rewrite the exact sequence in (5.23) as follows:

$$0 \longrightarrow M_{\Lambda,+} \xrightarrow{p_1(\mu)^n} M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1} \longrightarrow M_{\tilde{\Lambda},+}/p_1(\mu)^n \longrightarrow 0.$$

Then, using the operator  $\nabla_{q,\tilde{\tau}}$  in (5.26) and the Koszul complex defined in (5.27), we obtain an exact sequence of Koszul complexes:

$$0 \longrightarrow K_{M_{\Lambda,+}}(\nabla_{q,\tau}) \xrightarrow{p_1(\mu)^n} K_{M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1}}(\nabla_{q,\tilde{\tau}}) \longrightarrow K_{M_{\tilde{\Lambda},+}/p_1(\mu)^n}(\nabla_{q,\tilde{\tau}}) \longrightarrow 0.$$

Considering the associated long exact sequence, and noting that  $H^1(K_{M_{\Lambda,+}}(\nabla_{q,\tau})) = 0$  from Proposition 4.31, we obtain the following exact sequence:

$$0 \longrightarrow M_{\Lambda,+}^{\nabla_{q,\tau}=0} \xrightarrow{p_1(\mu)^n} (M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{\tau}}=0} \longrightarrow (M_{\tilde{\Lambda},+}/p_1(\mu)^n)^{\nabla_{q,\tilde{\tau}}=0} \longrightarrow 0.$$

Since the action of  $1 \times \Gamma_0$  is continuous on  $M_{\tilde{\Lambda},+}$  for the  $(p, p_1(\mu))$ -adic topology, therefore, we see that  $(M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1})^{\nabla_{q,\tilde{\tau}}=0} = (M_{\tilde{\Lambda},+}/p_1(\mu)^{n+1})^{1 \times \Gamma_0}$ , for each  $n \in \mathbb{N}$ . Hence, from the preceding exact sequence we obtain that the sequence in (5.25) is exact, and therefore, the sequence (5.24) is exact as well.  $\blacksquare$

**5.2.5. Stratification on Wach modules.** In this subsection we will construct a stratification on a Wach module over  $A_R$  and prove Proposition 5.28 stated below. The most important input for our arguments is Theorem 4.5.

Let  $N$  be the Wach module over  $A_R$  and set recall that  $N(1) = A_R(1) \otimes_{p_2, A_R} N$  is equipped with a  $(\varphi, \Gamma_R^2)$ -action (see the discussion before Subsection 5.2.1). Moreover, from the discussion before Proposition 4.32, recall that we have a  $\varphi$ -equivariant homomorphism  $\Delta_N : N(1) \rightarrow N$  induced by tensoring  $\Delta : A_R(1) \rightarrow A_R$  with  $N$ . After reducing  $\Delta_N$  modulo  $p_1(\mu)^n$ , we claim the following:

**Proposition 5.26.** *Let  $n \in \mathbb{N}_{\geq 1}$  then  $\Delta_N$  modulo  $p_1(\mu)^n$  restricts to a  $(\varphi, \Gamma_R \times 1)$ -equivariant isomorphism of  $A_R/\mu^n$ -modules*

$$\Delta_N : (N(1)/p_1(\mu)^n)^{1 \times \Gamma_R} \xrightarrow{\sim} N/\mu^n N.$$

where the  $A_R$ -module structure on the source is defined via the map  $p_1 : A_R \rightarrow A_R(1)$ .

*Proof.* Let us first consider the following  $(\varphi, 1 \times \Gamma_R)$ -equivariant short exact sequence of  $A_R(1)$ -modules:

$$0 \longrightarrow N(1)/p_1(\mu) \xrightarrow{p_1(\mu)^n} N(1)/p_1(\mu)^{n+1} \longrightarrow N(1)/p_1(\mu)^n \longrightarrow 0$$



Then for  $p \geq 3$  using Lemma 5.10, Lemma 5.13 and Lemma 5.16, and for  $p = 2$  using Lemma 5.10, Lemma 5.21 and Lemma 5.22, it follows that the following  $(\varphi, \Gamma_R \times 1)$ -equivariant sequence is exact:

$$0 \longrightarrow (N(1)/p_1(\mu))^{1 \times \Gamma_R} \xrightarrow{p_1(\mu)^n} (N(1)/p_1(\mu)^{n+1})^{1 \times \Gamma_R} \longrightarrow (N(1)/p_1(\mu)^n)^{1 \times \Gamma_R} \longrightarrow 0. \quad (5.28)$$

Now, consider the following  $\varphi$ -equivariant commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & (N(1)/p_1(\mu))^{1 \times \Gamma_R} & \xrightarrow{p_1(\mu)^n} & (N(1)/p_1(\mu)^{n+1})^{1 \times \Gamma_R} & \longrightarrow & (N(1)/p_1(\mu)^n)^{1 \times \Gamma_R} \longrightarrow 0 \\ & & \downarrow \Delta_N & & \downarrow \Delta_N & & \downarrow \Delta_N \\ 0 & \longrightarrow & N/\mu N & \xrightarrow{\mu^n} & N/\mu^{n+1} N & \longrightarrow & N/\mu^n N \longrightarrow 0, \end{array}$$

where the top row is the exact sequence in (5.28). For  $n = 1$ , from Theorem 4.5, in particular, from Proposition 4.32, recall that we have the isomorphism  $\Delta_N : (N(1)/p_1(\mu))^{1 \times \Gamma_R} \xrightarrow{\sim} N/\mu N$ . Then, by using the diagram, an easy induction on  $n \geq 1$ , gives the  $\varphi$ -equivariant isomorphism

$$\Delta_N : (A_R(1)/p_1(\mu)^{n+1} \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} \xrightarrow{\sim} N/\mu^{n+1} N.$$

Finally, we need to check the  $(\Gamma_R \times 1)$ -equivariance of  $\Delta_N$ , the proof of which is similar to that of [MT20, Lemma 3.19]. From Lemma 3.78 recall that for any  $g$  in  $\Gamma_R$  and  $a$  in  $A_R(1)/p_1(\mu)^{n+1}$ , we have that  $\Delta((g, g)a) = g(a)$ , which implies that  $\Delta_N((g, g)(x)) = g(\Delta_N(x))$ , for any  $g$  in  $\Gamma_R$  and  $x$  in  $(N(1)/p_1(\mu)^{n+1})^{1 \times \Gamma_R}$ . So if  $x$  is  $(1 \times \Gamma_R)$ -invariant, then for  $g_1, g_2$  in  $\Gamma_R$ , we have that  $\Delta_N((g_1, g_2)x) = \Delta_N((g_1, g_1)x) = g_1(\Delta_N(x))$ . This concludes our proof.  $\blacksquare$

An immediate consequence of Proposition 5.26 is the following:

**Proposition 5.27.** *Let  $N$  be a Wach module over  $A_R$ . Then the  $(\varphi, \Gamma_R)$ -equivariant homomorphism  $\Delta_N : A_R(1) \otimes_{p_2, A_R} N \rightarrow N$  induced by  $\Delta : A_R(1) \rightarrow A_R$  restricts to a  $(\varphi, \Gamma_R \times 1)$ -equivariant isomorphism of  $A_R$ -modules*

$$\Delta_N : (A_R(1) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} \xrightarrow{\sim} N,$$

where the  $A_R$ -module structure on the source is defined via the map  $p_1 : A_R \rightarrow A_R(1)$ .

*Proof.* Note that by construction  $\Delta_N$  is  $(\varphi, \Gamma_R \times 1)$ -equivariant. Moreover,  $A_R$  is  $\mu$ -adically complete and  $N$  is a finite  $A_R$ -module, in particular, we have that  $N = \lim_n N/\mu^n N$ . Similarly,  $A_R(1)$  is  $p_1(\mu)$ -adically complete, so we have that  $(A_R(1) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} = \lim_n (A_R(1)/p_1(\mu)^n \otimes_{p_2, A_R} N)^{1 \times \Gamma_R}$ . Now, recall that inverse limit commutes with right adjoint functors, in particular, with taking  $(1 \times \Gamma_R)$ -invariants. So we get that

$$\begin{aligned} (A_R(1) \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} &= \left( \lim_n A_R(1)/p_1(\mu)^n \otimes_{p_2, A_R} N \right)^{1 \times \Gamma_R} \\ &= \lim_n (A_R(1)/p_1(\mu)^n \otimes_{p_2, A_R} N)^{1 \times \Gamma_R} \xrightarrow[\Delta]{\sim} \lim_n N/\mu^n N = N, \end{aligned}$$

where the isomorphism follows from Proposition 5.26. This concludes our proof.  $\blacksquare$

Using Proposition 5.27 we can define a natural stratification on a Wach module over  $A_R$  as follows:

**Proposition 5.28.** *Let  $N$  be a Wach module over  $A_R$  and let  $\varepsilon : A_R(1) \otimes_{p_1, A_R} N \rightarrow A_R(1) \otimes_{p_2, A_R} N$  be the  $A_R(1)$ -linear homomorphism induced by the inverse of the isomorphism in Proposition 5.27. Then we have the following:*

- (1) *The homomorphism  $\varepsilon$  is a stratification on  $N$  with respect to  $A_R(\bullet)$ .*
- (2) *The action of  $\Gamma_R$  on  $\mathrm{ev}_{A_R}^{\mathrm{Strat}}(N, \varepsilon)$ , whose underlying  $A_R$ -module is  $N$ , coincides with the original action of  $\Gamma_R$  on  $N$ , i.e.  $\mathrm{ev}_{A_R}^{\mathrm{Strat}}(N, \varepsilon) \xrightarrow{\sim} N$  as Wach modules over  $A_R$ .*

Let us first note that the construction of the stratification  $\varepsilon$  in Proposition 5.28 is functorial in  $N$ . In particular, we have defined a functor

$$\text{Strat}_{A_R(\bullet)} : (\varphi, \Gamma_R)\text{-Mod}_{A_R}^{[p]q} \longrightarrow \text{Strat}^{\text{an}, \varphi}(A_R(\bullet)). \quad (5.29)$$

Moreover, from the statement of Proposition 5.28 it is clear that the functor  $\text{Strat}_{A_R(\bullet)}$  in (5.29) is a quasi-inverse to the functor  $\text{ev}_{A_R}^{\text{Strat}}$  in (5.7). We note the following:

**Lemma 5.29.** *The  $A_R(1)$ -linear homomorphism  $\varepsilon$  in Proposition 5.28 is  $\Gamma_R^2$ -equivariant.*

*Proof.* It is enough to show that  $\varepsilon(1 \otimes g_1(y)) = (g_1, g_2)(\varepsilon(1 \otimes y))$ , for all  $y$  in  $N$  and  $g_1, g_2$  in  $\Gamma_R$ . Similar to the last part of the proof of Proposition 5.26 we note that for any  $x$  in  $A_R(1) \otimes_{p_1, A_R} N$ , we have that  $\Delta_N((g_1, g_2)x) = g_1(\Delta_N(x))$ . Then, by setting  $y = \Delta_N(x)$  we get the claim.  $\blacksquare$

The goal of the rest of this subsection is to prove Proposition 5.28. To this end, we need a result analogous to Proposition 5.27 over  $A_R(2)$ . Recall that we have  $(\varphi, \Gamma_R^3)$ -equivariant maps  $r_i : A_R \rightarrow A_R(2)$ , for  $i = 1, 2, 3$ , where  $A_R(2)$  is equipped with an action of  $\Gamma_R^3$  as discussed before Construction 5.7 and  $A_R$  is equipped with an action of  $\Gamma_R^3$  via projection onto the  $i^{\text{th}}$ -coordinate. Similarly, we have natural  $(\varphi, \Gamma_R^3)$ -equivariant maps  $p_{ij} : A_R(1) \rightarrow A_R(2)$  for  $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$  and where  $A_R(2)$  is equipped with an action of  $\Gamma_R^3$  via projection onto the  $(i, j)^{\text{th}}$ -coordinate. Now, let  $N$  be a Wach module over  $A_R$  as above and consider the  $A_R(2)$ -module  $A_R(2) \otimes_{r_3, A_R} N$  equipped with the tensor product Frobenius and the tensor product action of  $\Gamma_R^3$ , where  $\Gamma_R^3$  acts on  $N$  via projection onto the third coordinate. Note that the multiplication map  $\Delta : A_R(2) \rightarrow A_R$  is  $(\varphi, \Gamma_R \times 1 \times 1)$ -equivariant, where  $\Gamma_R \times 1 \times 1$  acts on  $A_R$  via projection onto the first coordinate. The multiplication map  $\Delta$  induces a  $(\varphi, \Gamma_R \times 1 \times 1)$ -equivariant map  $\Delta_N : A_R(2) \otimes_{r_3, A_R} N \rightarrow N$ , where  $\Gamma_R \times 1 \times 1$  acts on  $N$  (in the target) via projection onto the first coordinate. Then we claim the following:

**Proposition 5.30.** *Let  $N$  be a Wach module over  $A_R$ . Then the homomorphism  $\Delta_N : A_R(2) \otimes_{r_3, A_R} N \rightarrow N$  induced by  $\Delta : A_R(2) \rightarrow A_R$  restricts to an injective map*

$$\Delta_N : (A_R(2) \otimes_{r_3, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} \longrightarrow N.$$

*Proof.* Recall that inverse limit commutes with right adjoint functors, in particular, with taking  $(1 \times \Gamma_R \times \Gamma_R)$ -invariants. Moreover,  $A_R$  is  $\mu$ -adically complete,  $N$  is a finite  $A_R$ -module and  $A_R(2)$  is  $p_1(\mu)$ -adically complete. Therefore, we have that,

$$\begin{aligned} (A_R(2) \otimes_{r_3, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} &= (\lim_n A_R(2)/p_1(\mu)^n \otimes_{r_3, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} \\ &= \lim_n (A_R(2)/p_1(\mu)^n \otimes_{r_3, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} \\ &\xrightarrow{\Delta_N} \lim_n N/\mu^n N = N. \end{aligned}$$

Since limit is a left exact functor, to show the claim, it is enough to show that the following map is injective

$$(A_R(2)/p_1(\mu)^n \otimes_{r_3, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} \longrightarrow N/\mu^n N.$$

We will show this by induction on  $n \in \mathbb{N}_{\geq 1}$ . For  $n = 1$ , from Theorem 4.5, in particular, from Remark 4.33, recall that we have a  $(\varphi, \Gamma_R^2)$ -equivariant isomorphism (see (4.32)),

$$A_R(1)/p_1(\mu) \otimes_{p_1, A_R} N \xrightarrow{\sim} A_R(1)/p_1(\mu) \otimes_{p_2, A_R} N,$$

where  $\Gamma_R^2$  acts on  $N$  in the left hand term via projection onto the first coordinate and on  $N$  in the right hand term via projection onto the second coordinate. Moreover, the composition  $A_R \xrightarrow{p_1} A_R(1) \xrightarrow{p_{13}} A_R(2)$  coincides with the composition  $A_R \xrightarrow{r_1} A_R(2)$ . Similarly, the composition  $A_R \xrightarrow{p_2} A_R(1) \xrightarrow{p_{13}} A_R(2)$  coincides with the composition  $A_R \xrightarrow{r_3} A_R(2)$ . So, by base changing the top horizontal isomorphism in (4.32) along  $A_R(1) \xrightarrow{p_{13}} A_R(2)$ , we obtain a  $(\varphi, \Gamma_R^3)$ -equivariant isomorphism

$$A_R(2)/p_1(\mu) \otimes_{r_1, A_R} N \xrightarrow{\sim} A_R(2)/p_1(\mu) \otimes_{r_3, A_R} N, \quad (5.30)$$

where  $\Gamma_R^3$  acts on  $N$  in the source via projection onto the first coordinate and on  $N$  in the target via projection onto the third coordinate. Now consider the following diagram,

$$\begin{array}{ccc}
(A_R(2)/p_1(\mu) \otimes_{r_1, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} & \xrightarrow{\sim} & (A_R(2)/p_1(\mu) \otimes_{r_3, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} \\
\uparrow \wr_{r_1} & & \downarrow \Delta_N \\
N/\mu N & \xrightarrow[\sim]{id} & N/\mu N,
\end{array}$$

where the top horizontal arrow is obtained as  $(1 \times \Gamma_R \times \Gamma_R)$ -invariant of (5.30) and the left vertical arrow is the natural isomorphism from Lemma 5.31. The commutativity of the diagram follows from (4.32) and the observation that the composition  $A_R \xrightarrow{r_1} A_R(2) \xrightarrow{\Delta} A_R$  is the identity. Therefore, the right vertical arrow is bijective as well, i.e. we obtain a  $(\varphi, \Gamma_R)$ -equivariant isomorphism

$$(A_R(2)/p_1(\mu) \otimes_{r_3, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} \xrightarrow{\sim} N/\mu N.$$

To prove our claim, we will now proceed by induction on  $n \geq 1$ . From the discussion above, we see that the claim is true for  $n = 1$ , so let  $N(2) := A_R(2)/p_1(\mu) \otimes_{r_3, A_R} N$  and assume that  $\Delta_N \bmod p_1(\mu)^n$  is injective for some  $n \in \mathbb{N}_{\geq 1}$ . Now consider the following diagram with exact rows

$$\begin{array}{ccccccc}
0 & \longrightarrow & (N(2)/p_1(\mu))^{1 \times \Gamma_R \times \Gamma_R} & \xrightarrow{p_1(\mu)^n} & (N(2)/p_1(\mu)^{n+1})^{1 \times \Gamma_R \times \Gamma_R} & \longrightarrow & Q \longrightarrow 0 \\
& & \downarrow \wr_{\Delta_N} & & \downarrow \Delta_N & & \downarrow \Delta_N \\
0 & \longrightarrow & N/\mu N & \longrightarrow & N/\mu^{n+1} N & \longrightarrow & N/\mu^n N \longrightarrow 0,
\end{array}$$

where  $Q$  is the cokernel of the top left horizontal arrow. It is easy to see that we have an injective map  $Q \rightarrow (N(2)/p_1(\mu)^n)^{1 \times \Gamma_R \times \Gamma_R} \xrightarrow{\Delta_N} N/\mu^n N$ , where the injectivity of the first map is obtained by considering the long exact sequence for the  $(1 \times \Gamma_R \times \Gamma_R)$ -cohomology of the short exact sequence  $0 \rightarrow N(2)/p_1(\mu) \xrightarrow{p_1(\mu)^n} N(2)/p_1(\mu)^{n+1} \rightarrow N(2)/p_1(\mu)^n$ , and the second map is injective by the induction assumption. Therefore, it follows that the middle vertical arrow in the diagram above is injective as well. This proves the claim, allowing us to conclude.  $\blacksquare$

The following result was used above:

**Lemma 5.31.** *Extending scalars along  $r_1 : A_R \rightarrow A_R(1)$ , gives a  $(\varphi, \Gamma_R^3)$ -equivariant map  $r_1 : N \rightarrow A_R(2) \otimes_{r_1, A_R} N$ . Then, reduction modulo  $\mu$ , restricts  $r_1$  to an  $R$ -linear  $\varphi$ -equivariant isomorphism  $r_1 : N/\mu N \xrightarrow{\sim} (A_R(2)/p_1(\mu) \otimes_{r_1, A_R} N)^{1 \times \Gamma_R \times \Gamma_R}$ .*

*Proof.* Note that the map  $r_1 : R \xrightarrow{\sim} (A_R(2)/p_1(\mu))^{1 \times \Gamma_R \times \Gamma_R}$  is a  $\varphi$ -equivariant isomorphism by Remark 3.76. Moreover, we have that,

$$(A_R(2)/p_1(\mu) \otimes_{r_1, A_R} N)^{1 \times \Gamma_R \times \Gamma_R} = (A_R(2)/p_1(\mu))^{1 \times \Gamma_R \times \Gamma_R} \otimes_{r_1, A_R} N.$$

Hence, we get the claimed isomorphism  $r_1 : N/\mu N \xrightarrow{\sim} (A_R(2)/p_1(\mu) \otimes_{r_1, A_R} N)^{1 \times \Gamma_R \times \Gamma_R}$ .  $\blacksquare$

We have all the necessary input to prove Proposition 5.28 similar to that the proof of [MT20, Proposition 3.18].

*Proof of Proposition 5.28.* For the first claim, note that from the definition of  $\varepsilon$  it is clear that its base change along  $\Delta : A_R(1) \rightarrow A_R$  is the identity. Moreover, by using Lemma 5.29 we get that the base changes  $p_{ij}^*(\varepsilon) : A_R(2) \otimes_{r_i, A_R} N \rightarrow A_R(2) \otimes_{r_j, A_R} N$ , for  $(i, j) \in \{(1, 2), (2, 3), (1, 3)\}$  are  $\Gamma_R^3$ -equivariant. Therefore, restrictions of  $p_{13}^*(\varepsilon)$  and  $p_{23}^*(\varepsilon) \circ p_{12}^*(\varepsilon)$  to  $N$  have images in  $(A_R(2) \otimes_{r_3, A_R} N)^{1 \times \Gamma_R \times \Gamma_R}$ , and their composition with the injective map  $\Delta_N$  in Proposition 5.30 is the identity. So it follows that  $p_{13}^*(\varepsilon) = p_{23}^*(\varepsilon) \circ p_{12}^*(\varepsilon)$ , since both sides are  $A_R(2)$ -linear. We also get that  $\varepsilon$  is an isomorphism since we can give its inverse as its base change along the involution  $A_R(1) \xrightarrow{\sim} A_R(1)$  swapping the two factors. This proves the first claim.

To show the second claim, let  $g$  in  $\Gamma_R$ . From Lemma 5.29 we have the following diagram

$$\begin{array}{ccc}
A_R(1) \otimes_{(\gamma,1),A_R(1)} (A_R(1) \otimes_{p_1,A_R} N) & \xrightarrow[\sim]{1 \otimes \varepsilon} & A_R(1) \otimes_{(\gamma,1),A_R(1)} (A_R(1) \otimes_{p_2,A_R} N) \\
\downarrow \wr 1 \otimes (g,1) & & \downarrow \wr 1 \otimes (g,1) \\
A_R(1) \otimes_{p_1,A_R} N & \xrightarrow[\sim]{\varepsilon} & A_R(1) \otimes_{p_2,A_R} N.
\end{array}$$

From the diagram it is clear that the image of any  $x$  in  $N$  under the left vertical arrow is  $g(x)$  while its image under the right vertical arrow is  $x$ . Base changing the diagram along the  $(\Gamma_R \times 1)$ -equivariant map  $\Delta : A_R(1) \rightarrow A_R$ , we obtain the following diagram:

$$\begin{array}{ccc}
A_R \otimes_{g,A_R} N & \xrightarrow[\sim]{\Delta^*((g,1)^*(1 \otimes \varepsilon))} & N \\
\downarrow \wr 1 \otimes g & & \downarrow \wr id \\
N & \xrightarrow[\sim]{id} & N,
\end{array}$$

where the left vertical arrow is the action of  $g$  on  $N$ . This concludes our proof.  $\blacksquare$

## A. SOME BASIC DEFINITIONS AND LEMMAS

In this section, we collect some standard definitions to enhance the readability of the text. For more details, the reader should look at the cited references. Let  $p$  be a fixed prime number.

**A.1. Basic definitions.** Let  $R$  be a commutative ring and let  $D(R)$  denote the derived  $\infty$ -category of  $R$ -modules.

**Definition A.1.** Let  $S$  denote a commutative  $R$ -algebra. Take  $P_\bullet \rightarrow S$  to be a simplicial resolution of  $S$  by polynomial  $R$ -algebras. Define the *cotangent complex* of  $R \rightarrow S$  to be the simplicial  $S$ -module  $L_{S/R} := \Omega_{P_\bullet/R}^1 \otimes_{P_\bullet} S$ . Its wedge powers will be denoted by  $\wedge_S^i L_{S/R} = \Omega_{P_\bullet/R}^i \otimes_{P_\bullet} R$  for  $i \geq 1$ . This definition is independent of the choice of the resolution  $P_\bullet$ .

**Remark A.2.** The object in  $D(R)$  defined by  $L_{S/R}$  coincides with that attached to the simplicial  $R$ -module  $\Omega_{P_\bullet/A}^1$  via the Dold-Kan correspondence (and similarly for the wedge powers). One may also obtain  $L_{S/R}$  by left Kan extending the functor  $\Omega_{-/R}^1$  on polynomial  $R$ -algebras to all simplicial commutative  $R$ -algebras. Restricting to commutative  $R$ -algebras one obtains the cotangent complex (upto isomorphism) described in Definition A.1.

**Definition A.3.** Let  $a, b \in \mathbb{Z} \cup \{\pm\infty\}$ .

- (1) An object  $M$  in  $D(R)$  has  *$p$ -complete Tor amplitude* in  $[a, b]$  if  $M \otimes_R^{\mathbb{L}} R/pR$  in  $D(R/pR)$  has Tor amplitude in  $[a, b]$ .
- (2) An object  $M$  in  $D(R)$  is  *$p$ -completely (faithfully) flat* if  $M \otimes_R^{\mathbb{L}} R/pR$  in  $D(R/pR)$  is concentrated in degree 0 and it is a (faithfully) flat  $R/pR$ -module.

**Definition A.4** (Quasisyntomic site, [BMS19, Definition 4.10]). We define the following:

- (1) A ring  $R$  is called *quasisyntomic* if it is  $p$ -complete, has bounded  $p^\infty$ -torsion and  $L_{R/\mathbb{Z}_p}$  in  $D(R)$  has  $p$ -complete Tor amplitude in  $[-1, 0]$ . Denote by  $\text{Qsyn}$  the category of quasisyntomic rings.
- (2) A map  $R \rightarrow S$  of  $p$ -complete rings with bounded  $p^\infty$ -torsion is said to be *quasisyntomic* (resp. *quasisyntomic cover*) if  $S$  is  $p$ -completely flat (resp.  $p$ -completely faithfully flat) over  $R$  and  $L_{S/R}$  in  $D(S)$  has  $p$ -complete Tor amplitude in  $[-1, 0]$ . Endow  $\text{Qsyn}^{\text{op}}$  with the structure of a site via the quasisyntomic covers (see [BMS19, Lemma 4.17]).

**Definition A.5** (Perfectoid rings, [BMS18, Definition 3.5]). A ring  $R$  is called *perfectoid* if it is  $p$ -adically complete and there is some  $\pi$  in  $R$  such that  $\pi^p = pu$  for some unit  $u$  in  $R^\times$ , the ring  $R/p$  is semiperfect, i.e. the absolute Frobenius map on  $R/p$  is surjective, and the kernel of the map  $\theta : A_{\text{inf}}(R) \rightarrow R$  is principal.

**Definition A.6** (Quasiregular semiperfectoid rings, [BMS19, Definition 4.20]). A ring  $S$  is called *quasiregular semiperfectoid* if  $S$  is quasisyntomic in the sense of Definition A.4 (1), there exists a map  $R \rightarrow S$  with  $R$  perfectoid in the sense of Definition A.5 and  $S/pS$  is semiperfect, i.e. the Frobenius on  $S/pS$  is surjective. Denote by  $\text{Qrsp}$  the category of quasiregular semiperfectoid rings and endow  $\text{Qrsp}^{\text{op}}$  with the topology determined by quasisyntomic covers.

**Definition A.7.** Let  $(X, \mathcal{O})$  denote a ringed topos. A *vector bundle* on  $(X, \mathcal{O})$  is an  $\mathcal{O}$ -module  $\mathcal{E}$  such that there exists a cover  $\{U_i\}$  of  $X$  and finite projective  $\mathcal{O}(U_i)$ -modules  $\mathcal{P}_i$  such that  $\mathcal{E}|_{U_i} \xrightarrow{\sim} \mathcal{P}_i \otimes_{\mathcal{O}(U_i)} \mathcal{O}_{U_i}$ . Denote by  $\text{Vect}(X, \mathcal{O})$  the category of all vector bundles and note that it can also be defined as the global sections over  $X$  of the sheafification of the presheaf of categories on  $X$  given by  $U \mapsto \text{Vect}(\mathcal{O}(U))$ .

**Definition A.8** (Koszul complex, [BMS18, Definition 7.1]). Let  $M$  be an abelian group and for  $i = 1, \dots, d$ , let  $f_i : M \rightarrow M$  denote  $d$  commuting endomorphisms of  $M$ . The Koszul complex  $K_M(f_1, \dots, f_d)$  is defined as the complex

$$M \xrightarrow{(f_1, \dots, f_d)} \bigoplus_{1 \leq i \leq d} M \longrightarrow \bigoplus_{1 \leq i_1 < i_2 \leq d} M \longrightarrow \dots \longrightarrow \bigoplus_{1 \leq i_1 < \dots < i_k \leq d} M \longrightarrow \dots,$$

where the differential from  $M$  at the index  $i_1 < \dots < i_k$  to  $M$  at the index  $j_1 < \dots < j_{k+1}$  is nonzero if and only if  $\{i_1, \dots, i_k\}$  is contained in  $\{j_1, \dots, j_{k+1}\}$ , and in that case it is given as  $(-1)^n f_{j_n}$ , where  $n$  is the unique integer in  $\{1, \dots, k+1\}$  such that  $j_n$  does not belong to  $\{i_1, \dots, i_k\}$ .

We note the following fact from the proof of [BMS18, Corollary 12.5]:

**Lemma A.9.** *If  $f_i$ , for  $1 \leq i \leq d$ , are commuting endomorphisms of an abelian group  $M$  and  $h_i$ , for  $1 \leq i \leq d$ , are automorphisms of  $M$  commuting with each other and  $f_i$ 's, then we have a natural quasi-isomorphism of complexes*

$$K_M(f_1 h_1, \dots, f_d h_d) \xrightarrow{\sim} K_M(f_1, \dots, f_d).$$

**A.2. Modules with  $\mathbb{Z}_p^\times$ -action.** In this subsection, we will consider objects admitting a continuous action of  $\Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$ . From (1.6), recall that  $\Gamma_F \xrightarrow{\sim} \mathbb{Z}_p^\times$ , via the  $p$ -adic cyclotomic character, fits into the following exact sequence:

$$1 \longrightarrow \Gamma_0 \longrightarrow \Gamma_F \longrightarrow \Gamma_{\text{tor}} \longrightarrow 1,$$

where, for  $p \geq 3$ , we have  $\Gamma_0 \xrightarrow{\sim} 1 + p\mathbb{Z}_p$  and for  $p = 2$ , we have  $\Gamma_0 \xrightarrow{\sim} 1 + 4\mathbb{Z}_2$ . Moreover, for  $p \geq 3$ , we have that  $\omega : \Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times$  and the projection map  $\Gamma_F \rightarrow \Gamma_{\text{tor}}$ , admits a section  $\Gamma_{\text{tor}} \xrightarrow{\sim} \mathbb{F}_p^\times \rightarrow \mathbb{Z}_p^\times \xleftarrow{\sim} \Gamma_F$ , where the second map is given as  $a \mapsto [a]$ , the Teichmüller lift of  $a$ . Finally, for  $p = 2$ , we have  $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$ , as groups.

**A.2.1. The action of  $\mathbb{F}_p^\times$ .** In this subsubsection, we will assume that  $p \geq 3$  and recall the definition of  $\mathbb{F}_p^\times$ -decomposition from [Iwa59, §3] ( $\Delta$ -decomposition in loc. cit.). For  $0 \leq i \leq p-2$ , let  $\epsilon_i$  denote the following element in the group ring  $\mathbb{Z}_p[[\Gamma_F]]$ :

$$\epsilon_i = \frac{1}{p-1} \sum_{a \in \mathbb{F}_p} \omega(a)^{-i} a.$$

Then it is easy to check that we have  $\epsilon_i^2 = \epsilon_i$ ,  $\epsilon_i \epsilon_j = 0$  for  $i \neq j$ ,  $\sum_{i=0}^{p-2} \epsilon_i = 1$ .

Now, let  $M$  be a compact  $\mathbb{Z}_p$ -module admitting a continuous action of  $\Gamma_F$ . Set  $M_i := \epsilon_i(M)$  for  $0 \leq i \leq p-2$ . Then we have a canonical decomposition of  $M$  as follows:

$$M = \bigoplus_{i=0}^{p-1} M_i. \tag{A.1}$$

Here, each  $M_i$  can also be characterised as the submodule of all  $x$  in  $M$  such that  $ax = \omega(a)^i x$  for all  $a$  in  $\mathbb{F}_p^\times$ . We will refer to the decomposition of  $M$  in (A.1) as the  $\mathbb{F}_p^\times$ -decomposition of  $M$ . Moreover, since the action of  $\mathbb{F}_p^\times$  and  $\Gamma_0$  commute with each other, we see that each  $M_i$  admits a continuous action of  $\Gamma_0$ .

**A.2.2. The action of  $\{\pm 1\}$ .** In this subsection, we will assume that  $p = 2$  and recall the following construction: Let  $M$  be a compact  $\mathbb{Z}_p$ -module admitting a continuous action of  $\Gamma_F$  and let  $\sigma$  be a generator of  $\Gamma_{\text{tor}} \xrightarrow{\sim} \{\pm 1\}$ . Then we set  $M_+ := \{x \in M \text{ such that } \sigma(x) = x\}$  and  $M_- := \{x \in M \text{ such that } \sigma(x) = -x\}$ . Using these notations, we have a canonical injective map of  $\mathbb{Z}_p$ -modules

$$M_+ \oplus M_- \longrightarrow M, \quad (\text{A.2})$$

sending  $(x, y) \mapsto x + y$ . Note that the map (A.2) need not be surjective. If  $M = A$  is a  $\mathbb{Z}_p$ -algebra, then it is easy to verify that  $A_+$  is a  $\mathbb{Z}_p$ -algebra as well,  $A_-$  is an  $A_+$ -module and the map in (A.2) is  $A_+$ -linear.

Let us consider the ring  $\mathbb{Z}_p[[q-1]]$  and equip it with a  $\mathbb{Z}_p$ -linear action of  $\Gamma_{\text{tor}}$  given as  $\sigma(q) = q^{-1}$ . Let  $M$  be a topological  $\mathbb{Z}_p[[q-1]]$ -module admitting an action of  $\Gamma_{\text{tor}}$  such that the induced action of  $\Gamma_{\text{tor}}$  is trivial on  $M/(q-1)M$ , in particular,  $\mathbb{Z}_p[[q-1]]$  satisfies these conditions. Then we note that the operator  $\nabla_\sigma := \frac{\sigma-1}{q-1}$  is well-defined on  $M$  and we claim the following:

**Lemma A.10.** *The  $\mathbb{Z}_p[[q-1]]$ -module  $M$  admits an  $\mathbb{Z}_p[[q-1]]^{\nabla_\sigma=0}$ -linear decomposition  $M = M^{\nabla_\sigma=0} \oplus M^{\nabla_\sigma=1}$ .*

*Proof.* Let us first note that  $\nabla_\sigma^2 = \nabla_\sigma$ . Indeed,

$$\begin{aligned} \nabla_\sigma^2 &= \frac{\sigma-1}{q-1} \circ \frac{\sigma-1}{q-1} = \frac{1}{q-1} \left( \frac{\sigma(\sigma-1)}{\sigma(q)-1} - \frac{\sigma-1}{q-1} \right) = \frac{1}{q-1} \left( \frac{1-\sigma}{q^{-1}-1} - \frac{\sigma-1}{q-1} \right) \\ &= \frac{1}{q-1} \left( \frac{q(\sigma-1)}{q-1} - \frac{\sigma-1}{q-1} \right) = \frac{\sigma-1}{q-1} = \nabla_\sigma. \end{aligned}$$

As  $\nabla_\sigma$  is an idempotent operator on  $M$ , it follows that  $M$  admits a decomposition  $M = \nabla_\sigma(M) \oplus (\nabla_\sigma - 1)M = M^{\nabla_\sigma=1} \oplus M^{\nabla_\sigma=0}$ . It is easy to see that the decomposition is  $\mathbb{Z}_p[[q-1]]^{\nabla_\sigma=0}$ -linear. ■

Let  $M$  and  $N$  be two  $\mathbb{Z}_p[[q-1]]$ -modules admitting actions of  $\Gamma_{\text{tor}}$  such that the induced action of  $\Gamma_{\text{tor}}$  is trivial modulo  $(q-1)$  and let  $M \rightarrow N$  be a  $\mathbb{Z}_p[[q-1]]$ -linear map compatible with the action of  $\Gamma_{\text{tor}}$ . Then we claim the following:

**Lemma A.11.** *The  $\mathbb{Z}_p[[q-1]]^{\Gamma_{\text{tor}}}$ -linear map  $M^{\Gamma_{\text{tor}}} \rightarrow N^{\Gamma_{\text{tor}}}$  is surjective.*

*Proof.* Note that we have  $M^{\Gamma_{\text{tor}}} = M^{\nabla_\sigma=0}$  and similarly for  $N$ . We get the claim by using the decomposition in Lemma A.10. ■

## B. $\delta$ -RINGS AND DIVIDED POWER ALGEBRAS

The content of this section has been adapted from some notes of Takeshi Tsuji on prismatic envelopes; we are thankful to him for sharing his computations.

In this section, we will describe certain prismatic envelopes explicitly. We begin this section by fixing some terminology. Let  $p$  be a fixed prime and  $A$  a commutative ring. A  $\delta$ -ring is a pair  $(A, \delta)$  where  $A$  is a commutative ring and  $\delta : A \rightarrow A$  is a map of sets with  $\delta(0) = \delta(1) = 0$  and satisfying:

$$\begin{aligned} \delta(x+y) &= \delta(x) + \delta(y) + \frac{x^p + y^p - (x+y)^p}{p}, \\ \delta(xy) &= x^p \delta(y) + y^p \delta(x) + p\delta(x)\delta(y). \end{aligned} \quad (\text{B.1})$$

Given a  $\delta$ -ring  $(A, \delta)$  define an endomorphism  $\varphi : A \rightarrow A$  by the formula  $\varphi(x) = x^p + p\delta(x)$ , for any  $x$  in  $A$ . This determines a *lifting of (the absolute) Frobenius* on  $A/pA$ . Then the product formula in (B.1) can be rewritten as

$$\delta(xy) = \varphi(x)\delta(y) + \delta(x)y^p. \quad (\text{B.2})$$

If  $A$  is  $p$ -torsion-free then any lift  $\varphi : A \rightarrow A$  of the absolute Frobenius on  $A/pA$  determines a unique  $\delta$ -structure on  $A$  given as  $\delta(x) = (\varphi(x) - x)/p$  for all  $x$  in  $A$ .

A  $\delta$ -homomorphism  $f : A \rightarrow B$  between  $\delta$ -rings is a homomorphism of the underlying rings compatible with the respective  $\delta$ -structures, i.e.  $f \circ \delta(x) = \delta \circ f(x)$ , for all  $x$  in  $A$ . For a  $\delta$ -ring  $A$ , a

$\delta$ -algebra over  $A$  is a  $\delta$ -ring  $B$  equipped with a  $\delta$ -homomorphism  $A \rightarrow B$ . A  $\delta$ -ideal of a  $\delta$ -ring  $A$  is an ideal  $I$  of the underlying ring, stable under  $\delta$ , i.e.  $\delta(I) \subset I$ . If  $I$  is a  $\delta$ -ideal of a  $\delta$ -ring  $A$ , then the quotient ring  $A/I$  is equipped with a unique  $\delta$ -structure over  $A$ . Now, let  $S$  be a subset of a  $\delta$ -ring  $A$ . Then, we see that the ideal of  $A$  generated by  $\{\delta^n(S)\}_{n \geq 1}$  is stable under  $\delta$  and it is the smallest  $\delta$ -ideal of  $A$  containing  $S$ . (Note that for any ideal  $J$  of  $A$ , the map  $J \rightarrow A/J$ , given by  $x \mapsto \delta(x) \bmod J$ , is a  $\varphi$ -semilinear homomorphism of  $A$ -modules by (B.1) and (B.2).) We write this ideal as  $(S)_\delta$  and call it the  $\delta$ -ideal generated by  $S$ . If  $I$  is an ideal of  $A$ , then we write  $I_\delta$  for  $(I)_\delta$ . If the ideal  $I$  is generated by a subset  $S$  of  $I$ , then we have  $I_\delta = (S)_\delta$  because the latter contains  $I$ . This implies that if  $B$  is a  $\delta$ -algebra over a  $\delta$ -ring  $A$  and  $I$  a  $\delta$ -ideal of  $A$ , then the ideal  $IB$  of  $B$  generated by  $I$  is a  $\delta$ -ideal of the  $\delta$ -ring  $B$ .

A  $\delta$ -subring of a  $\delta$ -ring  $A$  is a subring  $A' \subset A$ , stable under the  $\delta$ -structure on  $A$ , i.e.  $\delta(A') \subset A'$ . Similarly, a  $\delta$ -subalgebra of a  $\delta$ -algebra  $B$  over a  $\delta$ -ring  $A$  is an  $A$ -subalgebra  $B' \subset B$ , stable under the  $\delta$ -structure on  $B$ , i.e.  $\delta(B') \subset B'$ . Let  $S$  be a subset of a  $\delta$ -algebra  $B$  over a  $\delta$ -ring  $A$  and note that  $\delta(A[\delta^k(S), 1 \leq k \leq n]) \subset A[\delta^k(S), 1 \leq k \leq n+1]$ , for each  $n \in \mathbb{N}_{\geq 1}$ . (Note that for any  $A$ -subalgebra  $B'$  of  $B$ , the subset  $\{x \text{ in } B' \text{ such that } \delta(x) \text{ in } B'\}$  of  $B'$  is an  $A$ -subalgebra by (B.1) and (B.2)). This implies that the  $A$ -subalgebra of  $B$  generated by  $\{\delta^n(S)\}_{n \geq 1}$  is stable under  $\delta$  and it is the smallest  $\delta$ -subalgebra of  $B$  over  $A$  containing  $S$ ; we denote this  $\delta$ -subalgebra by  $A[S]_\delta$  and call it the  $\delta$ -subalgebra of  $B$  over  $A$  generated by  $S$ .

**Assumption B.1.** Let  $A$  be a  $\delta$ -ring equipped with an element  $q$  such that  $\varphi(q) = q^p$ . Set  $\mu := q - 1$  and assume that  $\varphi^n(\mu)$  is a nonzerodivisor on  $A$ , for each  $n \in \mathbb{N}$ . Moreover, assume that  $A$  and  $A/\mu A$  are  $p$ -torsion free.

In the rest of this section we will consider a  $\delta$ -ring  $A$  satisfying Assumption B.1. Set  $[p]_q = \frac{q^p - 1}{q - 1} = 1 + q + \cdots + q^{p-1}$ . Then, by Assumption B.1, we see that for each  $n \in \mathbb{N}$ , the element  $\varphi^n([p]_q)$  is a nonzerodivisor on  $A$ . Before proceeding further, we note a simple lemma on regular sequences.

**Lemma B.2.** *Let  $R$  be a ring.*

- (1) *If a sequence  $\{x, y\}$  is regular on  $R$ , then  $x$  is regular on  $R/yR$ .*
- (2) *Let  $x, y_1, \dots, y_d$  be elements of  $R$ . If  $x$  is regular on  $R$  and the sequence  $y_1, \dots, y_d$  is regular on  $R/xR$ , then  $x$  is regular on  $R/\sum_{i=1}^d y_i R$ .*

*Proof.* For (1), note that we obtain the claim by applying the snake lemma to the multiplication by  $y$  on the exact sequence  $0 \rightarrow R \xrightarrow{x} R \rightarrow R/xR \rightarrow 0$ . We will prove the claim in (2) by induction on  $d$ . The case  $d = 1$  is clear from (1), so let  $d \geq 2$  and assume that  $x$  is regular on  $R' = R/\sum_{i=1}^{d-1} y_i R$ . Now, since  $y_d$  is regular on  $R'$  by assumption, the claim in (1) implies that  $x$  is regular on  $R'/y_d R' = R/\sum_{i=1}^d y_i R$ .  $\blacksquare$

**Remark B.3.** By using Lemma B.2 (1), we see that for the ring  $A$  as in Assumption B.1, the quotient  $A/pA$  is  $\mu$ -torsion free. Moreover, since  $\varphi^n(\mu) = \mu^{p^n} \bmod pA$ , for each  $n \in \mathbb{N}$ , and  $A$  is  $p$ -torsion free, therefore, by using Lemma B.2 (1), we get that  $A/\varphi^n(\mu)A$  is  $p$ -torsion free.

Let  $\overline{A}$  denote the  $p$ -torsion free algebra  $A/\mu A$  and note that image of  $\varphi^n([p]_q)$  in  $\overline{A}$  is  $p$ , for every  $n \in \mathbb{N}$ . Moreover, the lifting of Frobenius on  $A$  induces a lifting of Frobenius on  $\overline{A}$ . Furthermore, as  $A/\mu A$  is  $p$ -torsion free, we see that the image of  $\delta([p]_q)$  in  $A$  is  $\delta(p)$  in  $A/pA$ , which can be computed as  $(\varphi(p) - p^p)/p = 1 - p^{p-1}$ . Hence, it follows that  $\delta([p]_q)$  is a unit modulo any power of the ideal  $(p, \mu) \subset A$ .

**Assumption B.4.** Let  $B$  be a  $\delta$ -algebra over  $A$ . Assume that  $B$  and  $B/\mu B$  are  $p$ -torsion free and  $\varphi^n(\mu)$  is a nonzerodivisor on  $B$ , for each  $n \in \mathbb{N}$ . Let  $Y_0, Y_1, \dots, Y_d$  be elements of  $B$  such that the sequence  $\{Y_1, \dots, Y_d\}$  is regular on  $B$  and the sequence  $\{Y_0, Y_1, \dots, Y_d\}$  is regular on  $B/[p]_q B$  and  $B/(p, \mu)B$ .

**Remark B.5.** Similar to the case of  $A$  as in Remark B.3, by using Lemma B.2 (1), we see that  $B/pB$  is  $\mu$ -torsion free and  $B/\varphi^n(\mu)B$  is  $p$ -torsion free, for each  $n \in \mathbb{N}$ .

Let  $I$  denote the set of natural numbers  $\{0, 1, \dots, d\}$ . Let  $C_0 := B[Y_{0,0}, \dots, Y_{d,0}]$  denote a polynomial ring over  $B$  in  $d + 1$  variables  $Y_{0,0}, \dots, Y_{d,0}$ , and let  $D_0 := C_0/([p]_q Y_{i,0} - Y_i, i \in I)$ . Moreover, set  $E_0$  to be the  $B$ -subalgebra of  $B[1/[p]_q]$  generated by  $y_i := Y_i/[p]_q$ , for  $i \in I$ . Then the surjective homomorphism of  $B$ -algebras  $C_0 \rightarrow E_0$  via  $Y_{i,0} \mapsto y_i$  for  $i \in I$ , induces a surjective homomorphism of  $B$ -algebras,

$$D_0 \longrightarrow E_0. \quad (\text{B.3})$$

Set  $\bar{B} := B/\mu B$ ,  $\bar{C}_0 := C_0/\mu C_0$ ,  $\bar{D}_0 := D_0/\mu D_0$  and  $\bar{E}_0 := E_0/\mu E_0$ , and write  $\bar{Y}_{i,0}$  (resp.  $\bar{y}_i$ ) for the image of  $Y_{i,0}$  (resp.  $y_i$ ) in  $\bar{C}_0$  (resp.  $\bar{E}_0$ ). We claim the following:

**Lemma B.6.** *With notations as above, we have the following:*

- (1) *The homomorphism (B.3) is an isomorphism.*
- (2) *The algebra  $\bar{E}_0$  is  $p$ -torsion free.*

*Proof.* Note that we have  $D_0[1/[p]_q] \xrightarrow{\sim} B[1/[p]_q][Y_{i,0}, i \in I]/(Y_{i,0} - y_i, i \in I) \xrightarrow{\sim} B[1/[p]_q]$ . Therefore, to show (1), it suffices to show that  $D_0$  is  $[p]_q$ -torsion free. Now, note that by Assumption B.4, the element  $[p]_q$  is regular on  $C_0$  and the sequence  $\{[p]_q Y_{0,0} - Y_0, \dots, [p]_q Y_{d,0} - Y_d\}$  is regular on  $C_0/[p]_q C_0$ . Hence, from Lemma B.2 (2), it follows that  $[p]_q$  is regular on  $D_0$ . For (2), note that using claim (1), we are reduced to showing that  $\bar{D}_0$  is  $p$ -torsion free. Moreover, as  $\bar{C}_0$  is  $p$ -torsion free, by using Lemma B.2 (2), we see that it is enough to show that the sequence  $\{[p]_q Y_{0,0} - Y_0, \dots, [p]_q Y_{d,0} - Y_d\}$  is regular on  $\bar{C}_0/p\bar{C}_0$ . Now note that  $\bar{C}_0/p\bar{C}_0 = C_0/(p, \mu)C_0$  and since  $[p]_q$  is in the ideal  $(p, \mu) \subset A$ , therefore, by Assumption B.4, we get that the sequence  $\{[p]_q Y_{0,0} - Y_0, \dots, [p]_q Y_{d,0} - Y_d\}$  is regular on  $\bar{C}_0/p\bar{C}_0$ . This concludes our proof.  $\blacksquare$

The lifting of Frobenius on  $B$  naturally extends to  $B[1/p, 1/\varphi^n([p]_q), n \in \mathbb{N}]$  and its  $E_0$ -subalgebra  $E_0[1/p, 1/\varphi^n([p]_q), n \in \mathbb{N}]$  is stable under  $\varphi$ . Let  $E$  be the  $\delta$ -subalgebra of  $E_0[1/p, 1/\varphi^n([p]_q), n \in \mathbb{N}]$  generated over  $B$  by  $y_i$ , for all  $i \in I$ . Note that we have  $E_0 \subset E$ . Now, recall that  $\varphi^n([p]_q) = p \bmod \mu B$ , for each  $n \in \mathbb{N}$ , therefore, we get that  $E_0[1/p, 1/\varphi^n([p]_q), n \geq 1]/\mu \xrightarrow{\sim} \bar{E}_0[1/p]$ . Via the preceding isomorphism, the lifting of Frobenius on  $E_0[1/p, 1/\varphi^n([p]_q), n \geq 1]$  induces a lifting of Frobenius on  $\bar{E}_0[1/p]$ . Define  $\bar{E}$  to be  $\delta$ -subalgebra of  $\bar{E}_0[1/p]$  generated over  $\bar{B}$  by  $\bar{y}_i$ , for all  $i \in I$ . Note that we have  $\bar{E}_0 \subset \bar{E}$  by Lemma B.6 (2) and the natural ring homomorphism  $E_0[1/p, 1/\varphi^n([p]_q), n \geq 1] \rightarrow \bar{E}_0[1/p]$  induces the following surjective ring homomorphism compatible with  $\varphi$ ,

$$E \longrightarrow \bar{E}. \quad (\text{B.4})$$

We will study the  $\delta$ -rings  $E$  and  $\bar{E}$  by comparing them to the  $\delta$ -rings obtained by universally adjoining  $Y_i/[p]_q$ , for all  $i \in I$ , to the  $\delta$ -rings  $B$  and  $\bar{B}$ , respectively. Let  $C$  denote a polynomial ring over  $B$  in variables  $Y_{i,n}$ , for  $i \in I$  and  $n \in \mathbb{N}$ . Equip  $C$  with a lifting of Frobenius  $\varphi$  compatible with that on  $A$  and defined on the variables as  $\varphi(Y_{i,n}) = Y_{i,n}^p + pY_{i,n+1}$ , for each  $i \in I$  and  $n \in \mathbb{N}$ . Since,  $C$  is  $p$ -torsion free, therefore, for  $\delta$ -structure associated to  $\varphi$ , we have  $\delta(Y_{i,n}) = Y_{i,n+1}$ , for each  $i \in I$  and  $n \in \mathbb{N}$ . Set  $\bar{C} := C/\mu C$  equipped with a lifting of Frobenius induced by that of  $C$ . Denote by  $\bar{Y}_i$  (resp.  $\bar{Y}_{i,n}$ ) the image of  $Y_i$  (resp.  $Y_{i,n}$ ) in  $\bar{C}$ . We define  $D$  (resp.  $\bar{D}$ ) to be the quotient of  $C$  (resp.  $\bar{C}$ ) by the  $\delta$ -ideal generated by  $[p]_q Y_{i,0} - Y_i$  (resp.  $[p]_q \bar{Y}_{i,0} - \bar{Y}_i$ ), for all  $i \in I$ . Then we have a  $\varphi$ -compatible (in particular,  $\delta$ -compatible) surjective homomorphism of  $B$ -algebras  $C \rightarrow E$  (resp. surjective homomorphism of  $\bar{B}$ -algebras  $\bar{C} \rightarrow \bar{E}$ ) defined by sending  $Y_{i,n}$  to  $\delta^n(y_i)$  (resp. by sending  $\bar{Y}_{i,n}$  to  $\delta^n(\bar{y}_i)$ ), for each  $i \in I$  and  $n \in \mathbb{N}$ . The preceding ring homomorphisms induce surjective horizontal arrows in the following diagram:

$$\begin{array}{ccc} D = C/([p]_q Y_{i,0} - Y_i)_\delta & \longrightarrow & E, \\ \downarrow & & \downarrow (\text{B.4}) \\ \bar{D} = \bar{C}/(p\bar{Y}_{i,0} - \bar{Y}_i)_\delta & \longrightarrow & \bar{E}, \end{array} \quad (\text{B.5})$$

where the left vertical arrow is induced by the isomorphism  $D/\mu D \xrightarrow{\sim} \bar{D}$  and the diagram commutes by definition. In particular, all arrows in the diagram (B.5) are surjective.



For each  $i \in I$  and  $n \in \mathbb{N}$ , let  $C_n^{(i)}$  be the  $B$ -subalgebra of  $C$  generated by  $Y_{i,m}$  for  $0 \leq m \leq n$ . Set  $C_{-1}^{(i)} := B$  and note that we have  $\delta(C_{n-1}^{(i)}) \subset C_n^{(i)}$ , for each  $n \in \mathbb{N}$ .

**Lemma B.7.** *For each  $i \in I$  and  $n \in \mathbb{N}_{\geq 1}$ , the element  $\delta^n([p]_q Y_{i,0} - Y_i)$  in  $C_n^{(i)}$  can be written as*

$$\delta^n([p]_q Y_{i,0} - Y_i) = \varphi^n([p]_q) Y_{i,n} + c_0 Y_{i,n-1}^p + \sum_{k=0}^{p-1} c_k Y_{i,n-1}^k,$$

where  $c_0 = \varphi^{n-1}(\delta([p]_q)) \sum_{m=0}^{n-1} p^{m(p-1)}$  is in  $A$  and  $c_k$  is in  $C_{n-2}^{(i)}$ , for each  $0 \leq k \leq p-1$ .

*Proof.* We will prove the claim by induction on  $n$ . For  $n = 1$ , we have

$$\begin{aligned} \delta([p]_q Y_{i,0} - Y_i) &= \delta([p]_q Y_{i,0}) + \delta(-Y_i) - \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} ([p]_q Y_{i,0})^k (-Y_i)^{p-k} \\ &= \varphi([p]_q) Y_{i,1} + \delta([p]_q) Y_{i,0}^p + \delta(-Y_i) - \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} [p]_q^k (-Y_i)^{p-k} Y_{i,0}^k, \end{aligned}$$

which is of the desired form because  $\delta(-Y_i)$ ,  $[p]_q$  and  $Y_i$  belong to  $C_{-1}^{(i)} = B$ . Now, let  $n \geq 1$  and assume that the claim is true for  $\delta^n([p]_q Y_{i,0} - Y_i)$ , with  $c_0$  in  $A$  and  $c_k$  in  $C_{n-2}^{(i)}$ . Set  $b = c_0 Y_{i,n-1}^p + \sum_{k=0}^{p-1} c_k Y_{i,n-1}^k$  in  $C_{n-1}^{(i)}$ . Then, we have

$$\begin{aligned} \delta^{n+1}([p]_q Y_{i,0} - Y_i) &= \delta(\varphi^n([p]_q) Y_{i,n} + b) \\ &= \delta(\varphi^n([p]_q) Y_{i,n}) + \delta(b) - \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} (\varphi^n([p]_q) Y_{i,n})^k b^{p-k} \\ &= \varphi^{n+1}([p]_q) Y_{i,n+1} + \delta(\varphi^n([p]_q)) Y_{i,n}^p + \delta(b) - \sum_{k=1}^{p-1} \frac{1}{p} \binom{p}{k} \varphi^n([p]_q)^k b^{p-k} Y_{i,n}^k. \end{aligned}$$

Since,  $\varphi^n([p]_q)$  and  $b$  are in  $C_{n-1}^{(i)}$ , it remains to show that  $\delta(b)$  is of the form  $p^{p-1} c_0 + \sum_{k=0}^{p-1} d_k Y_{i,n}^k$ , for some  $d_k$  in  $C_{n-1}^{(i)}$ . From (B.1), note that for any  $x$  and  $y$  in  $C_{n-1}^{(i)}$ , we have that  $\delta(x+y) = \delta(x) + \delta(y) \pmod{C_{n-1}^{(i)}}$  and from (B.2), for  $x$  in  $C_{n-2}^{(i)}$  and  $y$  in  $C_{n-1}^{(i)}$ , we have that  $\delta(xy) = \varphi(x)\delta(y) \pmod{C_{n-1}^{(i)}}$ . Therefore, we can write

$$\delta(b) = \varphi(c_0)\delta(Y_{i,n-1}^p) + \sum_{k=0}^{p-1} \varphi(c_k)\delta(Y_{i,n-1}^k) \pmod{C_{n-1}^{(i)}}.$$

Now, we have that  $\delta(1) = 0$  on  $A$  because  $A$  is  $p$ -torsion free, and by using Lemma B.8, for each  $1 \leq k \leq p$ , we have

$$\delta(Y_{i,n-1}^k) = \sum_{j=1}^k \binom{k}{j} p^{j-1} Y_{i,n-1}^{p(k-j)} Y_{i,n}^j.$$

Note that  $Y_{i,n-1}$  is in  $C_{n-1}^{(i)}$  and  $Y_{i,n}^p$  appears only when  $k = p$ , in which case the coefficient of  $Y_{i,n}^p$  is  $p^{p-1}$ . This allows us to conclude.  $\blacksquare$

The following fact was used above:

**Lemma B.8.** *Let  $R$  be a  $\delta$ -ring and  $x$  any element of  $R$ . Then for each  $n \in \mathbb{N}_{\geq 1}$ , we have*

$$\delta(x^n) = \sum_{j=1}^n \binom{n}{j} p^{j-1} x^{p(n-j)} \delta(x)^j.$$

*Proof.* We will prove the claim by induction on  $n$ . The case  $n = 1$  is obvious, so assume that the claim holds for  $\delta(x^n)$  for some  $n \geq 1$ . Then, we have that

$$\begin{aligned} \delta(x^{n+1}) &= \delta(x^n) x^p + x^{np} \delta(x) + p \delta(x) \delta(x^n) \\ &= \sum_{j=1}^n \binom{n}{j} p^{j-1} x^{p(n+1-j)} \delta(x)^j + x^{np} \delta(x) + \sum_{j=1}^n \binom{n}{j} p^j x^{p(n-j)} \delta(x)^{j+1}. \end{aligned}$$

By changing the parameter  $j$  in the last term by  $j = k - 1$  with  $2 \leq k \leq n + 1$  and expanding, it is easy to see that the expression thus obtained is the sum  $\sum_{j=1}^{n+1} \binom{n+1}{j} p^{j-1} x^{p(n+1-j)} \delta(x)^j$ .  $\blacksquare$

Let  $\overline{C}_n$  be the  $\overline{B}$ -subalgebra of  $\overline{C}$  generated by  $\overline{Y}_{i,m}$ , for all  $i \in I$  and  $0 \leq m \leq n$ . Set  $\overline{C}_{-1} := \overline{B}$  and note that we have  $\delta(\overline{C}_{n-1}) \subset \overline{C}_n$ , for each  $n \in \mathbb{N}$ . Define  $\overline{D}_n$  to be the quotient of  $\overline{C}_n$  by the ideal generated by  $\delta^m(p\overline{Y}_{i,0} - \overline{Y}_i)$ , for all  $i \in I$  and  $0 \leq m \leq n$ . Note that the  $\overline{B}$ -algebras  $\overline{C}_0$  and  $\overline{D}_0$  coincide with those defined before Lemma B.6. For each  $i \in I$  and  $n \in \mathbb{N}_{\geq 1}$ , let us set

$$X_n^{(i)} := \delta^n(p\overline{Y}_{i,0} - \overline{Y}_i) - p\overline{Y}_{i,n} \in \overline{C}, \quad (\text{B.6})$$

and by Lemma B.7 note that it is contained in  $\overline{u}_n \overline{Y}_{i,n-1}^p + \sum_{k=0}^{p-1} \overline{Y}_{i,n-1}^k \overline{C}_{n-2} \subset \overline{C}_{n-1}$ , where  $\overline{u}_n = \varphi^{n-1}(\delta(p)) \sum_{m=0}^{n-1} p^{m(p-1)}$  is in  $\overline{A}$ . In particular, we see that for each  $n \in \mathbb{N}$ , we have

$$\overline{D}_{n+1} = \overline{D}_n[\overline{Y}_{i,n+1}, i \in I]/(p\overline{Y}_{i,n+1} + X_{n+1}^{(i)}, i \in I). \quad (\text{B.7})$$

The inclusion maps  $\overline{C}_n \hookrightarrow \overline{C}_m \hookrightarrow \overline{C}$ , for  $0 \leq n < m$ , induce maps  $\overline{D}_n \rightarrow \overline{D}_m \rightarrow \overline{D}$  and an isomorphism

$$\operatorname{colim}_n \overline{D}_n \xrightarrow{\sim} \overline{D}. \quad (\text{B.8})$$

**Lemma B.9.** *With notations as above, we have the following:*

- (1) *The homomorphism  $\overline{D}_n \rightarrow \overline{D}_{n+1}$  induces an isomorphism  $\overline{D}_n[1/p] \xrightarrow{\sim} \overline{D}_{n+1}[1/p]$ , for each  $n \in \mathbb{N}$ .*
- (2) *Let  $S_n$  denote the image of  $\overline{D}_n/p\overline{D}_n$  in  $\overline{D}_{n+1}/p\overline{D}_{n+1}$ , for each  $n \in \mathbb{N}$ . Then, we have  $S_n = (\overline{D}_n/p\overline{D}_n)/(X_{n+1}^{(i)}, i \in I)$  and  $\overline{D}_{n+1}/p\overline{D}_{n+1} = S_n[\overline{Y}_{i,n+1}, i \in I]$ , for each  $n \in \mathbb{N}$ .*
- (3) *For each  $n \in \mathbb{N}$ , the algebra  $\overline{D}_n$  is  $p$ -torsion free.*

*Proof.* The claim in (1) follows immediately from (B.7) and the fact that  $X_{n+1}^{(i)}$  is in  $\overline{C}_n$ . For (2), note that by taking the reduction modulo  $p$  of (B.8), we get that

$$\overline{D}_{n+1}/p\overline{D}_{n+1} = (\overline{D}_n/p\overline{D}_n)/(X_{n+1}^{(i)}, i \in I)[\overline{Y}_{i,n+1}, i \in I].$$

To prove (3), let us first recall that  $\overline{D}_0$  is  $p$ -torsion free by Lemma B.6. So, for each  $n \in \mathbb{N}$ , by Lemma B.2 (2) and (B.7), it suffices to show that the sequence  $\{X_{n+1}^{(i)}, i \in I\}$ , where  $X_{n+1}^{(i)}$  is in  $\overline{C}_n$ , is regular on  $(\overline{D}_n/p\overline{D}_n)[\overline{Y}_{i,n+1}, i \in I]$ , i.e. it is regular on  $\overline{D}_n/p\overline{D}_n$ . For  $n \geq 1$ , the claim follows because we have  $\overline{D}_n/p\overline{D}_n = S_{n-1}[\overline{Y}_{i,n}, i \in I]$  and  $\overline{u}_{n+1}$  modulo  $p$  is a unit in  $\overline{A}/p\overline{A}$  (see the discussions after (B.6) and before Assumption B.4). For  $n = 0$ , it suffices to show that the sequence  $\{[p]_q Y_{i,0} - Y_i, i \in I\}, \{X_1^{(i)}, i \in I\}$  is regular on  $\overline{C}_0/p\overline{C}_0$ . As  $[p]_q$  is contained in the ideal  $(p, \mu) \subset B$ , we are reduced to showing that the sequence  $\{-Y_i, i \in I\}, \{X_1^{(i)}, i \in I\}$  is regular on  $\overline{C}_0/p\overline{C}_0 = B/(p, \mu)[\overline{Y}_{i,0}, i \in I]$ . This is obvious since  $\overline{u}_1$  modulo  $p$  is a unit in  $\overline{A}/p\overline{A}$  (see the discussions after (B.6) and before Assumption B.4). Hence, the lemma is proved.  $\blacksquare$

Now, note that reducing the top arrow in (B.5) modulo  $\mu$ , we obtain the following commutative diagram with surjective arrows:

$$\begin{array}{ccc} D/\mu D & \longrightarrow & E/\mu E, \\ \downarrow \wr & & \downarrow (\text{B.4}) \\ \overline{D} & \longrightarrow & \overline{E}. \end{array} \quad (\text{B.9})$$

**Proposition B.10.** *All arrows in the diagram (B.9) are isomorphisms.*

*Proof.* By definition, the left vertical arrow in (B.9) is an isomorphism and all arrows are surjective. Hence, it suffices to show that the lower horizontal arrow is injective. Note that we have the following commutative diagram:

$$\begin{array}{ccc} \overline{D}_0[1/p] & \longrightarrow & \overline{D}[1/p] \\ \wr \downarrow (\text{B.3}) & & \downarrow \\ \overline{E}_0[1/p] & \longrightarrow & \overline{E}[1/p]. \end{array}$$

The top horizontal arrow is a bijection by (B.8) and Lemma B.9 (1), the left vertical arrow is a bijection by Lemma B.6 and the bottom horizontal arrow is a bijection by the definition of  $\bar{E}$ . Therefore, it follows that the right vertical arrow is a bijection as well, i.e.  $\bar{D}[1/p] \xrightarrow{\sim} \bar{E}[1/p]$ . Finally, from (B.8) and Lemma B.9 (3), note that  $\bar{D}$  is  $p$ -torsion free. Hence, it follows that the map  $\bar{D} \rightarrow \bar{E}$  in (B.9) is injective. This completes our proof. ■

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