Crystalline representations and Wach modules in the imperfect residue field case

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Abstract. For an absolutely unramified field extension $L/\mathbb{Q}_p$ with imperfect residue field, we develop the notion of Wach modules in the setting of (φ, Γ)-modules for L. Our main result establishes a direct categorical equivalence between lattices inside crystalline representations of the absolute Galois group of L and integral Wach modules. Along the way, we generalise several comparison results from classical p-adic Hodge theory over $\mathbb{Q}_p$, to the case of L. In particular, we provide a direct relation between a rational Wach module equipped with the Nygaard filtration and the filtered φ-module of its associated crystalline representation.

1. Introduction

In [Fon90] Fontaine introduced the notion of *représentations de cr-hauteur finie* (finite crystalline-height representations) of the absolute Galois group of an unramified extension of $\mathbb{Q}_p$ with perfect residue field. This notion was further developed by Wach [Wac96; Wac97], Colmez [Col99] and Berger [Ber04]. In [Abh21] we studied a similar notion in the relative case, i.e. over formally étale algebras over a formal torus. In this article, we generalise the definition of Wach modules to characterise $\mathbb{Z}_p$-lattices inside crystalline representations of the absolute Galois group of an absolutely unramified extension of $\mathbb{Q}_p$ having imperfect residue field (see Theorem 1.5).

Before providing further motivations for our results, let us remark that recent developments in the theory of prismatic F-crystals [BS23; DLMS22; GR22] provide a new approach to the classification of lattices inside crystalline representations. While the prismatic point of view is an exciting new development, in the current paper, we employ techniques from the theory of (φ, Γ)-modules to obtain our results. Some advantages of our approach are as follows. Firstly, using the classical techniques enables us to establish new comparison results between objects appearing in the p-adic Hodge theory of complete discrete valuation fields with imperfect residue field from [Ked04; Bri06; Ohk13] (see Theorem 1.7, Proposition 3.14, Proposition 4.21 and Corollary 4.22). These results directly follow by the use of classical methods; however, their prismatic interpretation is not yet known. Secondly, in the relative case, to construct relative Wach modules associated to crystalline representations, a crucial input is the theory of Wach modules in the imperfect residue field case (see [Abh23a, Theorem 1.7]). Such results have interesting applications, for example, using loc. cit. and Theorem 1.5, we provide new criteria for checking crystallinity of p-adic representations in the relative case (see [Abh23a, Theorem 1.9 & Corollary 1.10]). We refer the reader to §1.2.3 for more details on relation of our results to the prismatic theory.

An additional motivation for considering Wach modules is to construct a deformation of crystalline cohomology, i.e. the functor $\mathbf{D}_{\mathrm{cris}}$ from classical p-adic Hodge theory, to better capture mixed characteristic information. In [Fon90, §B.2.3] Fontaine expressed similar expectations which were verified by Berger in [Ber04, Théorème III.4.4], and generalised to finer intergral conjectures in [Sch17, §6]. In this article, we provide a generalisation of [Ber04, Théorème III.4.4] to the imperfect residue field case.

Besides being a natural question of intrinsic p-adic Hodge theoretic interest, our motivation to study Wach modules also stems from its potential applications. Classically, Wach modules have witnessed applications in Iwasawa theory [Ben00], theory of p-adic L-functions [LZ13] and the p-adic

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local Langlands program [BB10]. The current paper is part of an overarching goal, the aim of which is to use Wach modules in the study of (crystalline/prismatic) syntomic complexes with coefficients.

Crystalline syntomic cohomology is a p-adic cohomology theory of algebraic varieties introduced by Fontaine and Messing [FM87], and has been classically used in the proof of p-adic comparison theorems in [Tsu99] and [CN17] (see [Abh23c, §1.1] for a survey). In [Abh23c, Theorem 1.15], for smooth (p-adic formal) schemes, we defined crystalline syntomic complex with coefficients, in relative Fontaine-Laffaille modules, and compared it to the complex of p-adic nearby cycles of associated crystalline local systems. Wach modules were central objects used in local computations for the proof of loc. cit. To generalise these results beyond the Fontaine-Laffaille case, it is therefore necessary to understand the relationship between general Wach modules and crystalline representations (cf. Theorem 1.5 and [Abh23a, Theorem 1.7]).

Analogously, in [BMS19] for smooth p-adic formal schemes the authors defined a prismatic syntomic complex, and compared it to p-adic nearby cycles integrally (see [BM23, §1] for a survey). So it is natural to contemplate a generalisation of [BMS19, Theorem 10.1] to non-trivial coefficients, i.e. prismatic F-crystals. Also, note that the proof of loc. cit. relies on local computation of prismatic cohomology using q-de Rham complex [BS22].

A Wach module can naturally be seen as a q-de Rham complex and as q-deformation of crystalline cohomology (cf. Theorem 1.7 and Remark 1.8 for a partial explanation and [Abh23a, Theorem 1.14] for more details). In addition, a Wach module can also be regarded as the evaluation of a prismatic F-crystal over a covering (by a q-de Rham prism) of the final object of the prismatic topos (cf. [Abh23b]), similar to the Breuil-Kisin case (cf. [DLMS22]). But, while Breuil-Kisin modules are well-suited for studying p-adic crystalline representations, Wach modules have the additional feature of being well-suited for explicit geometric calculations via the theory of q-de Rham cohomology. Given these tight connections between Wach modules and the prismatic and crystalline world, we expect that these objects will play an important role in the study of prismatic syntomic complex with coefficients (for smooth formal schemes), and its comparison with p-adic nearby cycles of associated crystalline $\mathbb{Z}_p$-local systems.

As indicated earlier, the theory of Wach modules in the imperfect residue field case is crucial for the analogous theory in the relative setting (see [Abh23a, Theorem 1.7]). So the current paper can be seen as a first step towards realising the goals of our program outlined above. Before introducing our main results, we begin by recalling the main result on arithmetic Wach modules.

1.1. The arithmetic case. Let p be a fixed prime number and let κ denote a perfect field of characteristic p; set $O_F := W(\kappa)$ to be the ring of p-typical Witt vectors with coefficients in κ and $F := \mathrm{Frac}(O_F)$. Let $\overline{F}$ denote a fixed algebraic closure of F, $\mathbb{C}_p := \widehat{\overline{F}}$ the p-adic completion, and $G_F := \mathrm{Gal}(\overline{F}/F)$ the absolute Galois group of F. Moreover, let $F_\infty := \cup_n F(\mu_{p^n})$ with $\Gamma_F := \mathrm{Gal}(F_\infty/F)$ and $H_F := \mathrm{Gal}(\overline{F}/F_\infty)$. Furthermore, let $F_\infty^\flat$ denote the tilt of F_∞ (see §1.3) and fix $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in O_{F_\infty}^\flat$, $\mu := [\varepsilon] - 1$, $[p]_q := \tilde{\xi} := \varphi(\mu)/\mu \in \mathbf{A}_{\mathrm{inf}}(O_{F_\infty}) := W(O_{F_\infty}^\flat)$, the ring of p-typical Witt vectors with coefficients in $O_{F_\infty}^\flat$.

In [Fon90] Fontaine estalished an equivalence of categories between $\mathbb{Z}_p$-representations of G_F and étale (φ, Γ_F)-modules over a certain period ring $\mathbf{A}_F := O_F[\![\mu]\!][1/\mu]^\wedge \subset W(F_\infty^\flat)$, where $^\wedge$ denotes the p-adic completion and $\mathbf{A}_F$ is stable under the (φ, Γ_F)-action on $W(F_\infty^\flat)$. For a fixed finite free $\mathbb{Z}_p$-representation T of G_F, the associated finite free étale (φ, Γ_F)-module over $\mathbf{A}_F$ is given by $\mathbf{D}_F(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_F}$, where $\mathbf{A} \subset W(\mathbb{C}_p^\flat)$ is the maximal unramified extension of $\mathbf{A}_F$ inside $W(\mathbb{C}_p^\flat)$. In loc. cit. Fontaine conjectured that if $V := T[1/p]$ is crystalline then there exists a lattice inside $\mathbf{D}_F(V) := \mathbf{D}_F(T)[1/p]$ over which the action of Γ_F admits a simpler form. Denote by $\mathbf{A}_F^+ := O_F[\![\mu]\!] \subset \mathbf{A}_F$, which is stable under the (φ, Γ_F)-action, and note the following:

Definition 1.1. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}_F^+$ with weights in the interval $[a, b]$ is a finite free $\mathbf{A}_F^+$-module N equipped with a continuous and semilinear action of Γ_F satisfying the following assumptions:

(1) The action of Γ_F on $N/\mu N$ is trivial.

(2) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ commuting with the action of Γ_F such that $\varphi(\mu^b N) \subset \mu^b N$ and the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) := \mathbf{A}_F^+ \otimes_{\varphi, \mathbf{A}_F^+} \mu^b N \to \mu^b N$ is injective and its cokernel is killed by $[p]_q^{b-a}$.

Denote the category of Wach modules over $\mathbf{A}_F^+$ as $(\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_F^+$-linear, Γ_F-equivariant and φ-equivariant (after inverting μ) morphisms. Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_F. To any T in $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F)$ using [Wac96] and [Col99], Berger functorially attaches a Wach module $\mathbf{N}_F(T)$ over $\mathbf{A}_F^+$ in [Ber04]. The main result in the arithmetic case is as follows (see [Ber04]):

Theorem 1.2. *The Wach module functor induces an equivalence of $\otimes$-catgeories*

$$\begin{aligned} \mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_F) &\xrightarrow{\sim} (\varphi, \Gamma_F)\text{-Mod}_{\mathbf{A}_F^+}^{[p]_q} \\ T &\longmapsto \mathbf{N}_F(T), \end{aligned}$$

with a quasi-inverse $\otimes$-functor given by $N \mapsto (W(\mathbb{C}_p^\flat) \otimes_{\mathbf{A}_F^+} N)^{\varphi=1}$.

1.2. The imperfect residue field case. Let $d \in \mathbb{N}$ and $X_1, X_2, \ldots, X_d$ be indeterminates and let $O_{L^\square} := (O_F[X_1^{\pm 1}, \ldots, X_d^{\pm d}]_{(p)})^\wedge$, where $^\wedge$ denotes the p-adic completion. It is a complete discrete valuation ring with uniformiser p, imperfect residue field $\kappa(X_1, \ldots, X_d)$ and fraction field $L^\square := O_{L^\square}[1/p]$. Let O_L denote a finite étale extension of $O_{L^\square}$ such that it is a complete discrete valuation ring with uniformiser p, imperfect residue field a finite étale extension of $\kappa(X_1, \ldots, X_d)$ and fraction field $L := O_L[1/p]$. Let G_L denote the absolute Galois group of L for a fixed algebraic closure $\overline{L}/L$; let Γ_L denote the Galois group of the L_∞ over L where L_∞ is the fraction field of O_{L_∞} obtained by adjoining to O_L all p-power roots of unity and all p-power roots of X_i for all $1 \leq i \leq d$. In this setting, we have the theory of crystalline representations of G_L [Bri06] and étale (φ, Γ)-modules [And06; AB08].

1.2.1. Wach modules. For $1 \leq i \leq d$, fix $X_i^\flat := (X_i, X_i^{1/p}, \ldots) \in O_{L_\infty}^\flat$ and take $[X_i^\flat] \in \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = W(O_{L_\infty}^\flat)$ as the Teichmüller representative of $X_i^\flat$. Let $\mathbf{A}_L^+$ denote the unique finite étale extension (along the finite étale map $O_{L^\square} \to O_L$) of the (p, μ)-adic completion of the localisation $O_F[\![\mu]\!][[X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]_{(p,\mu)}$ (see §1.3 and §2.1). The ring $\mathbf{A}_L^+$ is equipped with a Frobenius endomorphism φ and a continuous action of Γ_L.

Definition 1.3. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}_L^+$ with weights in the interval $[a, b]$ is a finite free $\mathbf{A}_L^+$-module N equipped with a continuous and semilinear action of Γ_L satisfying the following assumptions:

(1) The action of Γ_L on $N/\mu N$ is trivial.

(2) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ commuting with the action of Γ_L such that $\varphi(\mu^b N) \subset \mu^b N$ and the map $(1 \otimes \varphi) : \varphi^*(\mu^b N) := \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mu^b N \to \mu^b N$ is injective and its cokernel is killed by $[p]_q^{b-a}$.

Say that N is *effective* if one can take $b = 0$ and $a \leq 0$. Denote the category of Wach modules over $\mathbf{A}_L^+$ as $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_L^+$-linear, Γ_L-equivariant and φ-equivariant (after inverting μ) morphisms.

Set $\mathbf{A}_L := \mathbf{A}_L^+[1/\mu]^\wedge$ as the p-adic completion, equipped with a Frobenius endomorphism φ and a continuous action of Γ_L. Let T be a finite free $\mathbb{Z}_p$-module equipped with a continuous action of G_L, then one can functorially attach to T a finite free étale (φ, Γ_L)-module $\mathbf{D}_L(T)$ over $\mathbf{A}_L$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ equipped with a Frobenius-semilinear operator φ and a semilinear and continuous action of Γ_L. In fact, the preceding functor induces an equialence between finite free $\mathbb{Z}_p$-representations of G_L and finite free étale (φ, Γ_L)-modules over $\mathbf{A}_L$ (see §2.2).

Remark 1.4. The category of Wach modules over $\mathbf{A}_L^+$ can be realized as a full subcategory of étale (φ, Γ)-modules over $\mathbf{A}_L$ (see Proposition 3.3).

1.2.2. Main results. Let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_L. The main result of this article is as follows:

Theorem 1.5 (Corollary 4.2)**.** *The Wach module functor induces an equivalence of $\otimes$-categories*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L) \xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L^+}^{[p]_q}$$
$$T \longmapsto \mathbf{N}_L(T),$$

with a quasi-inverse given as $N \mapsto \mathbf{T}_L(N) := \big(W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} N\big)^{\varphi=1}$*, where* $\mathbb{C}_L := \widehat{\overline{L}}$*.*

To prove the theorem, starting with a Wach module N over $\mathbf{A}_L^+$, we use ideas developed in [Abh21] to show that $\mathbf{T}_L(N)[1/p]$ is crystalline (see Theorem 3.12). In the opposite direction, starting with a $\mathbb{Z}_p$-lattice T inside a p-adic crystalline representation of G_L, we use the result in perfect residue field case (see Theorem 1.2) to construct a finite free $\mathbf{A}_L^+$-module $\mathbf{N}_L(T)$ of finite $[p]_q$-height. However, the existence of a non-trivial Γ_L-action on N does not follow from this construction. So we use the Galois action on $T[1/p]$ and its crystallinity together with (φ, Γ)-module theory to construct a natural action of Γ_L on $\mathbf{N}_L(T)$ (see Theorem 4.1).

Remark 1.6. In Theorem 1.5, we do not expect the functor $\mathbf{N}_L$ to be exact (cf. [CD11, Example 7.1] for an example in the arithmetic case). However, after inverting p, the Wach module functor induces an exact equivalence of $\otimes$-categories $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L) \xrightarrow{\sim} (\varphi, \Gamma)\text{-}\mathrm{Mod}_{\mathbf{B}_L^+}^{[p]_q}$ via $V \mapsto \mathbf{N}_L(V)$, with an exact quasi-inverse $\otimes$-functor given as $M \mapsto \mathbf{V}_L(M) := \big(W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} M\big)^{\varphi=1}$ (see Corollary 4.3).

As indicated earlier, proof of Theorem 1.5 is based on techniques in the theory of (φ, Γ)-modules. One of the advantages of using this approach is that it enables us to establish new comparison results between objects appearing in classical p-adic Hodge theory over L (see Proposition 3.14, Proposition 4.21, Corollary 4.22 and Corollary 3.16). In order to keep the introduction light on notations, we will only mention one of the comparison results here and refer the reader to the main body of this article for the rest.

Let N be a Wach module over $\mathbf{A}_L^+$. We equip N with a Nygaard filtration defined as $\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}$. Then we note that $(N/\mu N)[1/p]$ is a φ-module over L since $[p]_q = p \bmod \mu\mathbf{A}_L^+$ and $N/\mu N$ is equipped with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $(N/\mu N)[1/p]$ with induced filtration, in particular, it is a filtered φ-module over L. Moreover, let $V := \mathbf{T}_L(N)[1/p]$ denote the associated crystalline representation of G_L from Theorem 1.5. Then we can functorially attach to V a filtered (φ, ∂)-module over L denoted by $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ (see §2.3). We show the following:

Theorem 1.7 (Corollary 3.16)**.** *Let N be a Wach module over $\mathbf{A}_L^+$ and $V := \mathbf{T}_L(N)[1/p]$ the associated crystalline representation from Theorem 1.5. Then we have $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ as filtered φ-modules over L.*

The proof of Theorem 1.7 is obtained by utilising the computations done in the proof of Theorem 3.12 (based on ideas developed in the proof of [Abh21, Theorem 4.25 & Proposition 4.28]).

Remark 1.8. Based on the expectation put forth in [Abh21, Remark 4.48], it is reasonable to expect that the L-vector space $(N/\mu N)[1/p]$ may be equipped with a connection by defining a q-connection on N using the action of geometric part of Γ_L, i.e. Γ_L' (see §2), and inducing a connection via $N \xrightarrow{q \mapsto 1} N/\mu N$. Moreover, the isomorphism $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ in Theorem 1.7 should be further compatible with connections. These expectations will be verified in [Abh23a].

1.2.3. Relation to other works. Our first main result, Theorem 1.5, is a direct generalisation of Theorem 1.2 from [Wac96; Col99; Ber04]. As indicated after Theorem 1.5, starting with a Wach module N over $\mathbf{A}_L^+$, we use ideas from [Abh21] to show that $\mathbf{T}_L(N)[1/p]$ is crystalline (see §3.3). For the converse, starting with a crystalline $\mathbb{Z}_p$-representation T of G_L, the construction of a finite $[p]_q$-height module $\mathbf{N}_L(T)$ uses classical Wach modules and its compatibility with results of [Kis06; KR09] (see §4.2). A similar construction in the case of Breuil-Kisin modules was carried out in [BT08]. However, unlike the Breuil-Kisin case handled in [BT08], the mere definition of the Wach module functor in the imperfect residue field case is quite subtle. Namely, equipping $\mathbf{N}_L(T)$ with a natural action of Γ_L is highly non-trivial, in particular, it does not follow from previous works. In fact, furnishing a natural action of Γ_L on $\mathbf{N}_L(T)$ is the technical heart of the current paper (see §4.3, §4.4 and §4.5).

For expert readers, let us note that using the unpublished results of Tsuji in [Tsu] and the use of [BT08] in [DLMS22], it is easy to see that the current paper is a crucial input to the construction of relative Wach modules in [Abh23a, Theorem 1.7]. Moreover, recent developments in the theory of prismatic F-crystals [BS23; DLMS22; GR22], would suggest that there is a categorical equivalence between the category of Wach modules over $\mathbf{A}_L^+$ and the category of prismatic F-crystals on the absolute prismatic site $(\mathrm{Spf}\ O_L)_{\Delta}$. At this point, let us remark that unlike the case of Breuil-Kisin modules [DLMS22, Proposition 3.25], obtaining the aforementioned equivalence directly is a difficult question, in particular, it is highly non-trivial to directly show that the natural functor from prismatic F-crystals to Wach modules is essentially surjective. This point will be explored in our work [Abh23b]. Furthermore, let us remark that using the results of [DLMS22], together with Theorem 3.12, it might be possible to obtain Theorem 1.5. However, such an approach has some drawbacks in obtaining all the results proven in the current paper. Namely, the comparison results that we prove in this paper (see below), do not seem to follow easily using prismatic methods since the aforementioned approach, based on previous results in the prismatic theory, is indirect. In contrast, these results easily follow from our approach because we can give direct proofs generalising classical methods, in particular, our proofs are independent of the results in the prismatic theory.

As indicated previously, the motivation for interpreting a Wach module as a q-de Rham complex and as q-deformation of crystalline cohomology, i.e. $\mathcal{O}\mathbf{D}_{\mathrm{cris}}$, comes from [Fon90, §B.2.3], [Ber04, Théorème III.4.4] and [Sch17, §6]. Our second main result, Theorem 1.7, is an important step towards verifying such expectations. In addition, we note that our proof of Theorem 1.7 is entirely independent to that of [Ber04, Théorème III.4.4], thus providing an alternative proof (as well as a generalisation) of the important classical result in loc. cit. Furthermore, generalising [Ber02, Proposition 3.5 & Théorème 3.6] from the perfect residue field case, in Proposition 4.21, for a p-adic crystalline representation V of G_L, we prove a comparison isomorphism between the associated (φ, Γ_L)-module over the Robba ring and $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Moreover, in Proposition 4.19 and Corollary 4.22 (see Remark 4.23) and V as above, we generalise [Ber02, Proposition 3.7] to obtain a comparison between the associated Wach module, the overconvergent étale (φ, Γ_L)-module and $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$.

Finally, let us remark that using the theory of Breuil-Kisin modules in the imperfect residue field case from [BT08], in [Gao20], Gao studied lattices inside crystalline (more generally, semistable) representations using Breuil-Kisin G_L-modules. However, such objects are very different from Wach modules considered in this paper. More specifically, Breuil-Kisin G_L-modules of loc. cit. are defined using the "Kummer tower" and admit an action of the big Galois group G_L. In contrast, Wach modules are defined using the "cyclotomic tower", as in the theory of étale (φ, Γ)-modules, and admit an action of Γ_L, which is much smaller than G_L. Moreover, [Gao20, Theorem 1.1.3] only proves a full faithfulness result, whereas Theorem 1.5 proves a categorical equivalence. As indicated above, construction of the Wach module functor is quite subtle. Furthermore, the proof of the essential surjectivity of the Wach module functor requires some work (see Theorem 3.12).

1.3. Setup and notations. We will work under the convention that $0 \in \mathbb{N}$, the set of natural numbers. Let p be a fixed prime number, κ a perfect field of characteristic p, $O_F := W(\kappa)$ the ring of p-typical Witt vectors with coefficients in κ and $F := O_F[1/p]$, the fraction field of W. In particular, F is an unramified extension of $\mathbb{Q}_p$ with ring of integers O_F. Let $\overline{F}$ be a fixed algebraic closure of F so that its residue field, denoted as $\overline{\kappa}$, is an algebraic closure of κ. Further, we denote by $G_F := \mathrm{Gal}(\overline{F}/F)$, the absolute Galois group of F.

We fix $d \in \mathbb{N}$ and let $X_1, X_2, \ldots, X_d$ be indeterminates. Set $R^{\square}$ to be p-adic completion of $O_F[X_1^{\pm 1}, \ldots, X_d^{\pm 1}]$. Let $\varphi : R^{\square} \to R^{\square}$ denote a morphism extending the natural Frobenius on O_F by setting $\varphi(X_i) = X_i^p$ for all $1 \leq i \leq d$. The endomorphism φ of $R^{\square}$ is flat by [Bri08, Lemma 7.1.5] and faithfully flat since $\varphi(\mathfrak{m}) \subset \mathfrak{m}$ for any maximal ideal $\mathfrak{m} \subset R^{\square}$. Moreover, it is finite of degree p^d using Nakayama Lemma and the fact that φ modulo p is evidently of degree p^d. Let $O_{L^{\square}} := (R^{\square}_{(p)})^{\wedge}$, where $^{\wedge}$ denotes the p-adic completion. It is a complete discrete valuation ring with uniformiser p, imperfect residue field $\kappa(X_1, \ldots, X_d)$ and fraction field $L^{\square} := O_{L^{\square}}[1/p]$. The Frobenius on $R^{\square}$ extends to a unique finite and faithfuly flat of degree p^d Frobenius endomorphism $\varphi : O_{L^{\square}} \to O_{L^{\square}}$ lifting the absolute Frobenius on $O_{L^{\square}}/pO_{L^{\square}}$.

Let O_L denote a finite étale extension of $O_{L^{\square}}$ such that it is a domain. Then O_L is a complete discrete valuation ring with uniformiser p, imperfect residue field a finite étale extension of $\kappa(X_1, \ldots, X_d)$ and fraction field $L := O_L[1/p]$. Fix an algebraic closure $\overline{L}/L$ and let $G_L := \mathrm{Gal}(\overline{L}/L)$ denote the absolute Galois group. The Frobenius on $O_{L^{\square}}$ extends to a unique finite and faithfuly flat of degree p^d Frobenius endomorphism $\varphi : O_L \to O_L$ lifting the absolute Frobenius on O_L/pO_L (see [CN17, Proposition 2.1]). For $k \in \mathbb{N}$, let $\Omega^k_{O_L}$ denote the p-adic completion of module of k-differentials of O_L relative to $\mathbb{Z}$. Then we have $\Omega^1_{O_L} = \oplus_{i=1}^d O_L d\log X_i$ and $\Omega^k_{O_L} = \wedge^k_{O_L} \Omega^1_{O_L}$.

Next, let O_K be one of O_{F_∞}, O_{L_∞}, $O_{\overline{F}}$ or $O_{\overline{L}}$ and $K := \mathrm{Frac}(O_K)$. Then the tilt of O_K is defined as $O_K^{\flat} := \lim_{\varphi} O_K/p$ and the tilt of K is defined as $K^{\flat} := \mathrm{Frac}(O_K^{\flat})$ (see [Fon77, Chapitre V, §1.4]). Finally, let A be a $\mathbb{Z}_p$-algebra equipped with a Frobenius endomorphism φ lifting the absolute Frobenius on A/p, then for any A-module M we write $\varphi^*(M) := A \otimes_{\varphi, A} M$.

Outline of the paper. This article consists of three main sections. In §2 we collect relevant results on p-adic Hodge theory in the imperfect residue field case. In §2.1 we define several period rings, in particular, we recall crystalline period rings, (φ, Γ)-module theory rings, overconvergent rings and Robba rings and prove several important technical results to be used in our main proofs in §4. In §2.2 we quickly recall the relation between p-adic representations and (φ, Γ)-module theory over the period rings described in previous section. In §2.3 we focus on crystalline representations and prove some results relating Galois action on a crystalline representation to its associated filtered (φ, ∂)-module. The goal of §3 is to define Wach modules in the imperfect residue field case and study the associated representation of G_L. In §3.1 we give the definition of Wach modules and relate it to étale (φ, Γ)-modules (see Proposition 3.3). Then given a Wach module, we functorially attach to it a $\mathbb{Z}_p$-representation of G_L and in §3.2 we show that these are related to finite $[p]_q$-height representations studied in [Abh21]. Finally, in §3.3 we show that the $\mathbb{Z}_p$-representation of G_L associated to a Wach module is a lattice inside a p-adic crystalline representation of G_L (see Theorem 3.12) and prove the filtered isomorphism claimed in Theorem 1.7. In §4 we prove our main result, i.e. Theorem 1.5. In §4.1 we collect important properties of classical Wach modules, i.e. the perfect residue field case. In §4.2 we use ideas from [Kis06; KR09] to construct a finite $[p]_q$-height module on the open unit disk over L. On the module thus obtained, we use results of §2.3 to construct an action of Γ_L and study its properties in §4.3. Then in §4.4 we check that our construction is compatible with the theory of étale (φ, Γ_L)-modules. Finally, in §4.5 we construct the promised Wach module and prove Theorem 1.5.

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2. Period rings and p-adic representations

We will use the setup and notations from §1.3. Recall that O_L is a finite étale algebra over $O_{L^{\square}}$. Set $L_\infty := \cup_{i=1}^d L(\mu_{p^\infty}, X_i^{1/p^\infty})$ and for $1 \leq i \leq d$, we fix $X_i^{\flat} := (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots) \in O_{L_\infty}^{\flat}$. Then we have the following Galois groups (see [Hyo86, §1.1] for details):

$$G_L := \mathrm{Gal}(\overline{L}/L),\ H_L := \mathrm{Gal}(\overline{L}/L_\infty),\ \Gamma_L := G_L/H_L = \mathrm{Gal}(L_\infty/L) \xrightarrow{\sim} \mathbb{Z}_p(1)^d \rtimes \mathbb{Z}_p^\times,$$
$$\Gamma'_L := \mathrm{Gal}(L_\infty/L(\mu_{p^\infty})) \xrightarrow{\sim} \mathbb{Z}_p(1)^d,\ \mathrm{Gal}(L(\mu_{p^\infty})/L) = \Gamma_L/\Gamma'_L \xrightarrow{\sim} \mathbb{Z}_p^\times.$$

Let $O_{\breve{L}} := (\cup_{i=1}^d O_L[X_i^{1/p^\infty}])^\wedge$, where $^\wedge$ denotes the p-adic completion. The O_L-algebra $O_{\breve{L}}$ is a complete discrete valuation ring with perfect residue field, uniformiser p and fraction field $\breve{L} := O_{\breve{L}}[1/p]$. The Witt vector Frobenius on $O_{\breve{L}}$ is given by the Frobenius on O_L described in §1.3 and setting $\varphi(X_i^{1/p^n}) = X_i^{1/p^{n-1}}$ for all $1 \leq i \leq d$ and $n \in \mathbb{N}$. Let $\breve{L}_\infty := \breve{L}(\mu_{p^\infty})$ and let $\overline{\breve{L}} \supset \overline{L}$ denote a fixed algebraic closure of $\breve{L}$. We have the following Galois groups:

$$G_{\breve{L}} := \mathrm{Gal}(\overline{\breve{L}}/\breve{L}) \xrightarrow{\sim} \mathrm{Gal}(\overline{L}/\cup_{i=1}^d L(X_i^{1/p^\infty})),\ H_{\breve{L}} := \mathrm{Gal}(\overline{\breve{L}}/\breve{L}_\infty) = \mathrm{Gal}(\overline{L}/L_\infty),$$
$$\Gamma_{\breve{L}} := G_{\breve{L}}/H_{\breve{L}} = \mathrm{Gal}(\breve{L}_\infty/\breve{L}) \xrightarrow{\sim} \mathrm{Gal}(L_\infty/\cup_{i=1}^d L(X_i^{1/p^\infty})) \xrightarrow{\sim} \mathrm{Gal}(L(\mu_{p^\infty})/L) \xrightarrow{\sim} \mathbb{Z}_p^\times.$$

From the description above note that $G_{\breve{L}}$ can be identified with a subgroup of G_L, $H_{\breve{L}} \xrightarrow{\sim} H_L$ and $\Gamma_{\breve{L}}$ can be identified with a quotient of Γ_L.

2.1. Period rings.

In this subsection we will quickly recall and fix notations for the period rings to be used in the rest of this section. For details refer to [And06], [Bri06] and [Ohk13].

2.1.1. Crystalline period rings.

Let $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) := W(O_{L_\infty}^\flat)$ and $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) := W(O_{\overline{L}}^\flat)$ admitting the Frobenius on Witt vectors and continuous G_L-action (for the weak topology). We fix $\overline{\mu} := \varepsilon - 1$, where $\varepsilon := (1, \zeta_p, \zeta_{p^2}, \ldots) \in O_{F_\infty}^\flat$ and let $\mu := [\varepsilon] - 1, \xi := \mu/\varphi^{-1}(\mu) \in \mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$. For $g \in G_L$, we have $g(1+\mu) = (1+\mu)^{\chi(g)}$ where χ is the p-adic cyclotomic character. Moreover, we have a G_L-equivariant surjection $\theta : \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \to O_{\mathbb{C}_L}$ where $\mathbb{C}_L := \widehat{\overline{L}}$ and $\mathrm{Ker}\ \theta = \xi \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})$. The map θ further induces a Γ_L-equivariant surjection $\theta : \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) \to O_{\widehat{L}_\infty}$.

Recall that for $1 \leq i \leq d$, we fixed $X_i^\flat = (X_i, X_i^{1/p}, X_i^{1/p^2}, \ldots) \in O_{L_\infty}$ and we take $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ to be topological generators of Γ_L such that $\{\gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ'_L and γ_0 is a topological generator of Γ_L/Γ'_L and $\gamma_j(X_i^\flat) = \varepsilon X_i^\flat$ if $i = j$ and $X_i^\flat$ otherwise. Let us also fix Teichmüller lifts $[X_i^\flat] \in \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$. We set $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})\langle \xi^k/k!, k \in \mathbb{N}\rangle$. Let $t := \log(1+\mu) \in \mathbf{A}_{\mathrm{cris}}(O_{F_\infty})$ and set $\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) := \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})[1/p]$ and $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty})[1/t]$. For $g \in G_L$, we have $g(t) = \chi(g)t$. Furthermore, one can define period rings $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$, $\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty})$ and $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$. These rings are equipped with a Frobenius endomorphism φ, continuous Γ_L-action and an appropriate extension of the map θ. Rings with a subscript "cris" are equipped with a decreasing filtration and rings with a prefix "$\mathcal{O}$" are further equipped with an integrable connection satisfying Griffiths transversality with respect to the filtration (see [Abh21, §2.2] for definitions over R with similar notations). One can define variations of these rings over $\overline{L}$ which are further equipped with G_L-action. Moreover, from [MT20, Lemma 4.32] note that $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) = \mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})^{H_L}$ and $\mathbf{B}_{\mathrm{cris}}^+(O_{L_\infty}) = \mathbf{B}_{\mathrm{cris}}^+(O_{\overline{L}})^{H_L}$.

We have two O_L-algebra structures on $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$: a canonical structure coming from the definition of $\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$; a non-canonical $(\varphi, \Gamma_{\breve{L}})$-equivariant structure $O_L \to \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ given by the map $x \mapsto \sum_{\mathbf{k} \in \mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(x) \prod_{i=1}^d ([X_i^\flat] - X_i)^{[k_i]}$, in particular, $X_i \mapsto [X_i^\flat]$.

2.1.2. Rings of (φ, Γ)-modules.

For detailed explanations of objects defined in this subsubsection, see [And06]. Recall that $O_{L^\square}$ is a complete discrete valuation ring with uniformiser p imperfect residue field and O_L is a finite étale $O_{L^\square}$-algebra. Let $\mathbf{A}_{L^\square}^+$ denote the (p, μ)-adic completion of the localization $O_F[\![\mu]\!][[X_1^\flat]^{\pm 1}, \ldots, [X_d^\flat]^{\pm 1}]_{(p,\mu)}$. We have a natural embedding $\mathbf{A}_{L^\square}^+ \subset \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ and $\mathbf{A}_{L^\square}^+$ is stable under the Witt vector Frobenius and Γ_L-action on $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$; we equip $\mathbf{A}_{L^\square}^+$ with induced structures. Moreover, we have an embedding $\iota : O_{L^\square} \to \mathbf{A}_{L^\square}^+$ via the map $X_i \mapsto [X_i^\flat]$ and it extends to an isomorphism of rings $O_{L^\square}[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}_{L^\square}^+$. Equip $O_{L^\square}[\![\mu]\!]$ with finite and faithfully flat of degree p^{d+1} Frobenius endomorphism using the Frobenius on $O_{L^\square}$ and setting $\varphi(\mu) = (1+\mu)^p - 1$. Then the embedding ι and the isomorphism $O_{L^\square}[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}^+_{L^\square}$ are Frobenius-equivariant.

Let $\mathbf{A}^+_L$ denote the (p,μ)-adic completion of the unique extension of the embedding $\mathbf{A}^+_{L^\square} \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ along the finite étale map $O_{L^\square} \to O_L$ (see [CN17, Proposition 2.1]). We have a natural embedding $\mathbf{A}^+_L \subset \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ and $\mathbf{A}^+_L$ is stable under the induced Frobenius and Γ_L-action. Moreover, the embedding $\iota : O_{L^\square} \to \mathbf{A}^+_{L^\square} \subset \mathbf{A}^+_L$ and the isomorphism $O_{L^\square}[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}^+_{L^\square} \subset \mathbf{A}^+_L$ extend to a unique embedding $\iota : O_L \to \mathbf{A}^+_L$ and an isomorphism of rings $O_L[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}^+_L$. Equip $O_L[\![\mu]\!]$ with finite and faithfully flat of degree p^{d+1} Frobenius endomorphism using the Frobenius on O_L and setting $\varphi(\mu) = (1+\mu)^p - 1$. Then the embedding ι and the isomorphism $O_L[\![\mu]\!] \xrightarrow{\sim} \mathbf{A}^+_L$ are Frobenius-equivariant. In particular, the Frobenius $\varphi : \mathbf{A}^+_L \to \mathbf{A}^+_L$ is finite and faithfully flat of degree p^{d+1}. Let $u_\alpha := (1+\mu)^{\alpha_0}[X_1^\flat]^{\alpha_1} \cdots [X_d^\flat]^{\alpha_d}$ where $\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_d) \in \{0, 1, \ldots, p-1\}^{[0,d]}$ then we have $\varphi^*(\mathbf{A}^+_L) := \mathbf{A}^+_L \otimes_{\varphi, \mathbf{A}^+_L} \mathbf{A}^+_L \xrightarrow{\sim} \oplus_\alpha \varphi(\mathbf{A}^+_L) u_\alpha$.

Let $\mathbb{C}_L := \widehat{\overline{L}}$, $\tilde{\mathbf{A}} := W(\mathbb{C}_L^\flat)$ and $\tilde{\mathbf{B}} := \tilde{\mathbf{A}}[1/p]$ admitting the Frobenius on Witt vectors and continuous G_L-action (for the weak topology). Set $\mathbf{A}_L := \mathbf{A}^+_L[1/\mu]^\wedge$ equipped with induced Frobenius endomorphism and continuous Γ_L-action. Note that $\mathbf{A}_L$ is a complete discrete valuation ring with maximal ideal $p\mathbf{A}_L$, residue field $(O_L/p)((\mu))$ and fraction field $\mathbf{B}_L := \mathbf{A}_L[1/p]$. Similar to above, $\varphi : \mathbf{A}_L \to \mathbf{A}_L$ is finite and faithfully flat of degree p^{d+1} and we have $\varphi^*(\mathbf{A}_L) := \mathbf{A}_L \otimes_{\varphi, \mathbf{A}_L} \mathbf{A}_L \xrightarrow{\sim} \oplus_\alpha \varphi(\mathbf{A}_L) u_\alpha = (\oplus_\alpha \varphi(\mathbf{A}^+_L) u_\alpha) \otimes_{\varphi(\mathbf{A}^+_L)} \varphi(\mathbf{A}_L) \xleftarrow{\sim} \mathbf{A}^+_L \otimes_{\varphi, \mathbf{A}^+_L} \mathbf{A}_L$. Furthermore, we have a natural Frobenius and Γ_L-equivariant embedding $\mathbf{A}_L \subset \tilde{\mathbf{A}}^{H_L}$. Let $\mathbf{A}$ denote the p-adic completion of the maximal unramified extension of $\mathbf{A}_L$ inside $\tilde{\mathbf{A}}$ and set $\mathbf{B} := \mathbf{A}[1/p] \subset \tilde{\mathbf{B}}$, i.e. $\mathbf{A}$ is the ring of integers of $\mathbf{B}$. The rings $\mathbf{A}$ and $\mathbf{B}$ are stable under induced Frobenius and G_L-action and we have $\mathbf{A}_L = \mathbf{A}^{H_L}$ and $\mathbf{B}_L = \mathbf{B}^{H_L}$ stable under induced Frobenius and residual Γ_L-action.

2.1.3. Overconvergent rings. We begin by definining the ring of overconvergent coefficients stable under Frobenius and G_L-action (see [CC98] and [AB08]) Denote the natural valuation on $O^\flat_{\overline{L}}$ by $\upsilon^\flat$ extending the valuation on $O^\flat_{\overline{F}}$. Let $r > 0$ and let $\alpha \in O^\flat_{\overline{F}}$ such that $\upsilon^\flat(\alpha) = pr/(p-1)$. Set

$$\tilde{\mathbf{A}}^{\dagger,r} := \Big\{ \sum_{k \in \mathbb{N}} p^k[x_k] \in \tilde{\mathbf{A}} \text{ such that } \upsilon^\flat(x_k) + \tfrac{pr}{p-1}k \to +\infty \text{ as } k \to +\infty \Big\}.$$

The G_L-action and Frobenius φ on $\tilde{\mathbf{A}}$ induce commuting actions of G_L and φ on $\tilde{\mathbf{A}}^{\dagger,r}$ such that $\varphi(\tilde{\mathbf{A}}^{\dagger,r}) = \tilde{\mathbf{A}}^{\dagger,pr}$. Define the ring of *overconvergent coefficients* as $\tilde{\mathbf{A}}^\dagger := \cup_{r \in \mathbb{Q}_{>0}} \tilde{\mathbf{A}}^{\dagger,r} \subset \tilde{\mathbf{A}}$ equipped with induced Frobenius and G_L-action. Moreover, inside $\tilde{\mathbf{A}}$ we take $\mathbf{A}^{\dagger,r}_L := \mathbf{A}_L \cap \tilde{\mathbf{A}}^{\dagger,r}$ and $\mathbf{A}^{\dagger,r} := \mathbf{A} \cap \tilde{\mathbf{A}}^{\dagger,r}$. Define $\mathbf{A}^\dagger_L := \mathbf{A}_L \cap \mathbf{A}^\dagger_L = \cup_{r \in \mathbb{Q}_{>0}} \mathbf{A}^{\dagger,r}_L$ and $\mathbf{A}^\dagger := \mathbf{A} \cap \tilde{\mathbf{A}}^\dagger = \cup_{r \in \mathbb{Q}_{>0}} \mathbf{A}^{\dagger,r}$ equipped with induced Frobenius endomorphism and G_L-action from respective actions on $\tilde{\mathbf{A}}$; we have $\mathbf{A}^\dagger_L = (\mathbf{A}^\dagger)^{H_L}$. Upon inverting p in the definitions above one obtains $\mathbb{Q}_p$-algebras inside $\tilde{\mathbf{B}}$, i.e. set $\tilde{\mathbf{B}}^{\dagger,r} := \tilde{\mathbf{A}}^{\dagger,r}[1/p]$, $\tilde{\mathbf{B}}^\dagger := \tilde{\mathbf{A}}^\dagger[1/p]$, $\mathbf{B}^{\dagger,r} := \mathbf{A}^{\dagger,r}[1/p]$, $\mathbf{B}^\dagger := \mathbf{A}^\dagger[1/p]$, equipped with induced Frobenius and G_L-action. Moreover, set $\tilde{\mathbf{B}}^{\dagger,r}_L := (\tilde{\mathbf{B}}^{\dagger,r})^{H_L}$, $\tilde{\mathbf{B}}^\dagger_L := (\tilde{\mathbf{B}}^\dagger)^{H_L}$, $\mathbf{B}^{\dagger,r}_L := (\mathbf{B}^{\dagger,r})^{H_L} = \mathbf{A}^{\dagger,r}_L[1/p]$ and $\mathbf{B}^\dagger_L := (\mathbf{B}^\dagger)^{H_L} = \mathbf{A}^\dagger_L[1/p]$ equipped with induced Frobenius and residual Γ_L-action.

2.1.4. Analytic rings. In this subsection, we will define the Robba ring over L following [Ked05, §2] and [Ohk15, §1]. However, we will use the notations of [Ber02, §2] in the perfect residue field case (see [Ohk15, §1.10] for compatibility between different notations). Define

$$\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} := \cup_{r \geq 0} \cap_{s \geq r} \big(\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})\langle \tfrac{p}{[\overline{\mu}]^r}, \tfrac{[\overline{\mu}]^s}{p} \rangle [\tfrac{1}{p}]\big).$$

The ring $\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ can also be defined as $\cup_{r \in \mathbb{Q}_{>0}} \tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ where $\tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ denotes the Fréchet completion of $\tilde{\mathbf{B}}^{\dagger,r} = \tilde{\mathbf{A}}^{\dagger,r}[1/p]$ for a certain family of valuations (see [Ked05, §2] and [Ohk15, §1.6]). The Frobenius and G_L-action on $\tilde{\mathbf{B}}^{\dagger,r}$ respectively induce Frobenius and G_L-action on $\tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}}$ which extend to respective actions on $\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$. In particular, we have a Frobenius and G_L-equivariant inclusion $\tilde{\mathbf{B}}^\dagger \subset \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ (see [Ohk15, §1.6 & §1.10]). Set

$$\tilde{\mathbf{B}}^+_{\mathrm{rig}} := \cap_{n \in \mathbb{N}} \varphi^n(\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}))$$

equipped with an induced Frobenius endomorphism and G_L-action from the respective actions on $\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})$. Description of rings in [Ber02, Lemme 2.5, Exemple 2.8 & §2.3] directly extend to our situation as the aforementioned results do not depend on structure of the residue field of base ring O_L. Therefore, from loc. cit. it follows that $\tilde{\mathbf{B}}^+_{\mathrm{rig}} \subset \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ compatible with Frobenius and G_L-action. Moreover, we set $\tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},L} := (\tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}})^{H_L}$, $\tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L} := (\tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig}})^{H_L}$ and $\tilde{\mathbf{B}}^{+}_{\mathrm{rig},L} := (\tilde{\mathbf{B}}^{+}_{\mathrm{rig}})^{H_L} \subset \tilde{\mathbf{B}}^{\dagger}_{\mathrm{rig},L}$ equipped with induced Frobenius endomorphism and residual Γ_L-action.

Remark 2.1. Note that the definition $\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}$ and $\tilde{\mathbf{B}}^+_{\mathrm{rig}}$ as rings does not depend on L, in particular, one may define these rings using $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})$ and equip them with a Frobenius endomorphism compatible with the Frobenius endomorphism defined above.

Lemma 2.2. *We have* $(\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\varphi=1} = (\tilde{\mathbf{B}}^+_{\mathrm{rig}})^{\varphi=1} = \mathbb{Q}_p$.

Proof. Using Remark 2.1, note that the Frobenius invariant elements can be computed using results in the perfect residue field case. In particular, $(\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}})^{\varphi=1} = (\tilde{\mathbf{B}}^+_{\mathrm{rig}})^{\varphi=1} = \mathbb{Q}_p$, where the first equality follows from [Ber04, Proposition I.4.1] and the second equality from [Col02, Proposition 8.15]. ∎

Recall that from §2.1.2 we have a Frobenius-equivariant embedding $\iota : O_L \to \mathbf{A}^+_L$. From [Ohk15, §1.6] the ring $\mathbf{A}^{\dagger,r}_L$ has the following description:

$$\mathbf{A}^{\dagger,r}_L \xrightarrow{\sim} \Big\{ \sum_{k\in\mathbb{Z}} \iota(a_k)\mu^k \text{ such that } a_k \in O_L \text{ and for any } p^{-1/r} \leq \rho < 1, \lim_{k\to-\infty} |a_k|\rho^k = 0 \Big\}.$$

We have $\mathbf{B}^{\dagger,r}_L = \mathbf{A}^{\dagger,r}_L[1/p]$ and we set

$$\mathbf{B}^{\dagger,r}_{\mathrm{rig},L} := \Big\{ \sum_{k\in\mathbb{Z}} \iota(a_k)\mu^k \text{ such that } a_k \in L \text{ and for any } p^{-1/r} \leq \rho < 1, \lim_{k\to\pm\infty} |a_k|\rho^k = 0 \Big\}.$$

The ring $\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$ can also be defined as Fréchet completion of $\mathbf{B}^{\dagger,r}_L$ for a family of valuations induced by the inclusion $\mathbf{B}^{\dagger,r}_L \subset \tilde{\mathbf{B}}^{\dagger,r}$ (see [Ked05, §2] and [Ohk15, §1.6]). Define the *Robba ring* over L as $\mathbf{B}^{\dagger}_{\mathrm{rig},L} := \cup_{r\geq 0}\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$. The Frobenius and G_L-action on $\mathbf{B}^{\dagger,r}_L$ induce respective Frobenius and G_L-action on $\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$ which extend to respective actions on $\mathbf{B}^{\dagger}_{\mathrm{rig},L}$ (also see [Ohk15, §4.3] where Ohkubo constructs the differential action of Lie Γ_L; one may also obtain the action of Γ_L by exponentiating the action of Lie Γ_L). From the preceding discussion, we have a Frobenius and Γ_L-equivariant injection $\mathbf{B}^\dagger_L \subset \mathbf{B}^\dagger_{\mathrm{rig},L}$ and the former ring $\mathbf{B}^\dagger_L$ is also known as the *bounded Robba ring*. Furthermore, note that $\mathbf{B}^{\dagger,r}_L \subset \tilde{\mathbf{B}}^{\dagger,r}_L = (\tilde{\mathbf{B}}^{\dagger,r})^{H_L} \subset \tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},L}$, where the last term can also be described as the Fréchet completion of the middle term for a family of valuations induced by the inclusion $\tilde{\mathbf{B}}^{\dagger,r}_L \subset \tilde{\mathbf{B}}^{\dagger,r}$ (see [Ked05, §2] and [Ohk15, §1.6]).

To summarize, for $r \in \mathbb{Q}_{>0}$ we have the following commutative diagram with injective arrows:

$$\begin{tikzcd}
\mathbf{B}^{\dagger,r} \arrow[r] & \tilde{\mathbf{B}}^{\dagger,r} \arrow[dr] & \\
\mathbf{B}^{\dagger,r}_L \arrow[u] \arrow[r] \arrow[d] & \tilde{\mathbf{B}}^{\dagger,r}_L \arrow[u] \arrow[r] \arrow[d] & \tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \\
\mathbf{B}^{\dagger,r}_{\mathrm{rig},L} \arrow[r] & \tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig},L}, \arrow[ur] &
\end{tikzcd}$$

where in the second row, two rings on the left are obtained from the rings in first row by taking H_L-invariants and the rightmost ring in second row is obtained as Fréchet completion of the rightmost ring in first row. The bottom row is obtained as Fréchet completion of two rings on the left in the second row. These inclusions are compatible with Frobenius and Γ_L-action and these compatibilities are preserved after passing to respective Fréchet completions. In particular, we have a Frobenius and Γ_L-equivariant embedding $\mathbf{B}^\dagger_{\mathrm{rig},L} \subset \tilde{\mathbf{B}}^\dagger_{\mathrm{rig},L}$.

Definition 2.3. Define $\mathbf{B}^+_{\mathrm{rig},L} := \mathbf{B}^\dagger_{\mathrm{rig},L} \cap \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \subset \tilde{\mathbf{B}}^\dagger_{\mathrm{rig},L}$, equipped with induced Frobenius endomorphism and Γ_L-action.

Lemma 2.4. *The ring $\mathbf{B}^+_{\mathrm{rig},L}$ can be identified with the ring of convergent power series over the open unit disk in one variable over L, i.e.*

$$\mathbf{B}^+_{\mathrm{rig},L} \xrightarrow{\sim} \Big\{ \sum_{k\in\mathbb{N}} \iota(a_k)\mu^k \text{ such that } a_k \in L \text{ and for any } 0 \leq \rho < 1, \lim_{k\to+\infty} |a_k|\rho^k = 0\Big\},$$

Proof. Let $x \in \mathbf{B}^+_{\mathrm{rig},L} \subset \mathbf{B}^\dagger_{\mathrm{rig},L}$. Using the explicit description of $\mathbf{B}^{\dagger,r}_{\mathrm{rig},L}$ and $\mathbf{B}^{\dagger,r}_L$ for $r \in \mathbb{Q}_{>0}$, we can write $x = x^+ + x^-$ with x^+ convergent on the open unit disk over L and $x^- \in \mathbf{B}^\dagger_L$, in particular, $x^+ \in \tilde{\mathbf{B}}^+_{\mathrm{rig}}$. Moreover, using Remark 2.1 and [Ber02, Lemma 2.18, Corollaire 2.28], we have an exact sequence $0 \to \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) \to \tilde{\mathbf{B}}^{\dagger,r} \oplus \tilde{\mathbf{B}}^+_{\mathrm{rig}} \to \tilde{\mathbf{B}}^{\dagger,r}_{\mathrm{rig}} \to 0$, where $\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) = \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/p]$. So $x \in \mathbf{B}^+_{\mathrm{rig},L} \subset \tilde{\mathbf{B}}^+_{\mathrm{rig}}$ if and only if $x^- \in \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) \cap \mathbf{B}^\dagger_L = \mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \cap \mathbf{B}^\dagger_L = \mathbf{B}^+_L$, where we have used $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})^{H_L} = \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ (see [And06, Proposition 7.2]). Hence, x converges on the open unit disk over L. The other inclusion is obvious. ■

Remark 2.5. The topology on $\mathbf{B}^+_{\mathrm{rig},L}$ can be described as follows: Let $D(L,\rho)$ denote the closed disk of radius $0 < \rho < 1$ over L and let $\mathcal{O}(D(L,\rho))$ denote the ring of analytic functions, i.e. power series converging on the closed disk $D(L,\rho)$. Then $\mathcal{O}(D(L,\rho))$ is equipped with a topology induced by the supremum norm $\|x\|_\rho := \sup_{x\in D(L,\rho)} |f(x)| < +\infty$. We have $\mathbf{B}^+_{\mathrm{rig},L} = \lim_\rho \mathcal{O}(D(L,\rho)) \subset L[\![\mu]\!]$ and we equip it with the topology induced by the Fréchet limit of the topology on $\mathcal{O}(D(L,\rho))$ induced by the supremum norm, i.e. the topology on $\mathbf{B}^+_{\mathrm{rig},L}$ can be described by uniform convergence on $D(L,\rho)$ for $\rho \to 1^-$.

Lemma 2.6. *The natural map $\mathbf{B}^+_L \to \mathbf{B}^+_{\mathrm{rig},L}$ is faithfully flat.*

Proof. Note that $\mathbf{B}^+_L$ is a principal ideal domain and $\mathbf{B}^+_{\mathrm{rig},L}$ is a domain, so the map in claim is flat. To show that it is faithfully flat, it is enough to show that for any maximal ideal $\mathfrak{m} \subset \mathbf{B}^+_L$ we have $\mathfrak{m}\mathbf{B}^+_{\mathrm{rig},L} \neq \mathbf{B}^+_{\mathrm{rig},L}$. Note that if $\mathfrak{m} \subset \mathbf{B}^+_L$ is a maximal ideal, then $\mathfrak{m} = (f)$ where f is an irreducible distinguished polynomial in the sense of [Lan90, Chapter 5, §2]. Since any f as above admits a zero over the open unit disk, it follows f is not a unit in $\mathbf{B}^+_{\mathrm{rig},L}$. Hence, $\mathfrak{m}\mathbf{B}^+_{\mathrm{rig},L} \neq \mathbf{B}^+_{\mathrm{rig},L}$. ■

Remark 2.7. From §1.3 recall that $\varphi : L \to L$ is finite of degree p^d and we also have $\varphi(\mu) = (1+\mu)^p - 1$. Therefore, from the explicit description of $\mathbf{B}^+_{\mathrm{rig},L}$ in Lemma 2.4 it follows that the Frobenius endomorphism $\varphi : \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ is finite and faithfully flat of degree p^{d+1}.

2.1.5. Period rings for $\breve{L}$. Definitions above may be adopted almost verbatim to define corresponding period rings for $\breve{L}$, in particular, one recovers definitions of period rings in [Fon90], [CC98] and [Ber02], in particular, one obtain period rings $\mathbf{A}^+_{\breve{L}}$, $\mathbf{A}_{\breve{L}}$, $\mathbf{A}^\dagger_{\breve{L}}$, $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$ and $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}}$ equipped with a Frobenius endomorphism φ and $\Gamma_{\breve{L}}$-action. Note that we have a natural identification $\mathbf{A}^+_{\breve{L}} \xrightarrow{\sim} O_{\breve{L}}[\![\mu]\!]$ where the right hand side is equipped with a finite and faithfully flat of degree p Frobenius endomorphism using the natural Frobenius on $O_{\breve{L}}$ and setting $\varphi(\mu) = (1+\mu)^p - 1$ and a $\Gamma_{\breve{L}}$-action given as $g(\mu) = (1+\mu)^{\chi(g)} - 1$ for $g \in \Gamma_{\breve{L}}$. Moreover, the preceding isomorphism naturally extends to a Frobenius and $\Gamma_{\breve{L}}$-equivariant isomorphism $\mathbf{A}_{\breve{L}} \xrightarrow{\sim} O_{\breve{L}}[\![\mu]\!][1/\mu]^\wedge$, where $^\wedge$ denotes the p-adic completion.

Recall that the Frobenius-equivariant embedding $O_L \to O_{\breve{L}}$ is faithfully flat and it naturally extends to a Frobenius and $\Gamma_{\breve{L}}$-equivariant faithfully flat embedding $O_L[\![\mu]\!] \to O_{\breve{L}}[\![\mu]\!]$. Using Frobenius and $\Gamma_{\breve{L}}$-equivariant isomorphisms $\mathbf{A}^+_L \xrightarrow{\sim} O_L[\![\mu]\!]$ and $\mathbf{A}_{\breve{L}} \xrightarrow{\sim} O_{\breve{L}}[\![\mu]\!]$ we get a Frobenius and $\Gamma_{\breve{L}}$-equivariant faithfully flat embedding $\mathbf{A}^+_L \to \mathbf{A}^+_{\breve{L}}$ sending $[X_i^\flat] \mapsto X_i$. This further extends to a Frobenius and $\Gamma_{\breve{L}}$-equivariant faithfully flat embedding $\mathbf{A}_L \to \mathbf{A}_{\breve{L}}$.

We can equip $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ with a non-canonical O_L-algebra structure by first defining an injection $O_{L^\square} \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ via the map $X_i \mapsto [X_i^\flat]$ and extending it uniquely along the finite étale map

$O_{L^\square} \to O_L$, to an injection $O_L \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ (see [CN17, Proposition 2.1]). Note that the preceding maps are Frobenius-equivariant but not Γ_L-equivariant. The O_L-algebra structure on $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ naturally extends to a Frobenius-equivariant $O_{\breve{L}}$-algebra structure by sending $X_i^{1/p^n} \mapsto [(X_i^{1/p^n})^\flat]$ for all $1 \leq i \leq d$ and $n \in \mathbb{N}$. We can further extend this to a Frobenius and $\Gamma_{\breve{L}}$-equivariant embedding $\mathbf{A}_{\breve{L}}^+ = O_{\breve{L}}[\![\mu]\!] \to \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$.

Using the embeddings described above and following the definitions of various period rings discussed so far, we obtain a commutative diagram with injective arrows where the top horizontal arrows are Frobenius and Γ_L-equivariant and the rest are Frobenius and $\Gamma_{\breve{L}}$-equivariant:

$$\begin{tikzcd}
\tilde{\mathbf{B}}^+_{\mathrm{rig},L} & \mathbf{B}^+_{\mathrm{rig},L} \arrow[l] \arrow[r] \arrow[d] & \mathbf{B}^\dagger_{\mathrm{rig},L} \arrow[r] \arrow[d] & \tilde{\mathbf{B}}^\dagger_{\mathrm{rig},L} \\
& \mathbf{B}^+_{\mathrm{rig},\breve{L}} \arrow[ul] \arrow[r] & \mathbf{B}^\dagger_{\mathrm{rig},\breve{L}}. \arrow[ur] &
\end{tikzcd}$$

Remark 2.8. Similar to Lemma 2.4 we have

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \xrightarrow{\sim} \Big\{ \sum_{k \in \mathbb{N}} a_k \mu^k \text{ such that } a_k \in \breve{L} \text{ and for any } 0 \leq \rho < 1, \lim_{k \to +\infty} |a_k| \rho^k = 0 \Big\}.$$

The ring $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is equipped with a Fréchet topology similar to Remark 2.5. Moreover, since $\varphi : \breve{L} \xrightarrow{\sim} \breve{L}$ and $\varphi(\mu) = (1+\mu)^p - 1$, the Frobenius endomorphism on $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is finite and faithfully flat of degree p.

Lemma 2.9. *The rings $\mathbf{B}^+_{\mathrm{rig},L}$ and $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$ are Bézout domains and $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is flat.*

Proof. The first claim follows from [Ber02, Proposition 4.12]. Note that loc. cit. assumes the residue field of discrete valuation base field (L and $\breve{L}$ in our case) to be perfect, however the proof of loc. cit. only depends on [Laz62] and [Hel43] which are independent of this assumption. For the second claim, note that we can write $\mathbf{B}^+_{\mathrm{rig},\breve{L}} = \mathrm{colim}_{i \in I} M_i$, where I is the directed index set of finitely generated $\mathbf{B}^+_{\mathrm{rig},L}$-submodules of $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$. Since $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is a domain, M_i is torsion-free for each $i \in I$. Now recall that finitely generated torsion-free modules over a Bézout domain are finite projective (see [CE99, Chapter VII, Proposition 4.1] noting that Bézout domains are a special case of Prüfer domains), and therefore finite free by [Ked04, Proposition 2.5]. Moreover, directed colimit of finite free modules over a ring is flat (see [Sta23, Tag 058G]). Hence, it follows that $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is flat. ∎

Lemma 2.10. *The element $t/\mu = (\log(1+\mu))/\mu = \prod_{n \in \mathbb{N}}(\varphi^n([p]_q)/p)$ converges in $\mathbf{B}^+_{\mathrm{rig},L} \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}$. Moreover, $(t/\mu)\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mathbf{B}^+_{\mathrm{rig},L} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},L}$.*

Proof. The first claim follows from [Ber04, Exemple I.3.3] and [Laz62, Remarque 4.12]. For the second claim let $x = \sum_{k \in \mathbb{N}} x_k \mu^k \in \mathbf{B}^+_{\mathrm{rig},L}$ with $x_k \in L$ and let $y = \sum_{k \in \mathbb{N}} y_k \mu^k \in \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ with $y_k \in \breve{L}$ such that $ty/\mu = x$. Write $t/\mu = \sum_{k \in \mathbb{N}} a_k \mu^k$ with $a_k \in \mathbb{Q}_p$. Then we have $(\sum_{k \in \mathbb{N}} a_k \mu^k)(\sum_{k \in \mathbb{N}} y_k \mu^k) = \sum_{k \in \mathbb{N}} x_k \mu^k$. We will show that $y_k \in L$ for all $k \in \mathbb{N}$ using induction. Note that $a_0 y_0 = x_0 \in L$ so $y_0 = x_0/a_0 \in L$. Let $n \in \mathbb{N}$ and assume $y_k \in L$ for every $k \leq n$. Then we have $\sum_{k=0}^{n+1} a_k y_{n+1-k} = x_{n+1} \in L$ and by inductive assumption we get $y_{n+1} = (x_{n+1} \sum_{k=0}^{n} a_k y_{n+1-k})/a_0 \in L$. Hence, $y \in \mathbf{B}^+_{\mathrm{rig},L}$ implying $(t/\mu)\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mathbf{B}^+_{\mathrm{rig},L} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},L}$. ∎

Lemma 2.11. *We have $(t/\mu)\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \cap \mathbf{B}^+_{\mathrm{rig},\breve{L}} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},\breve{L}}$, therefore $(t/\mu)\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \cap \mathbf{B}^+_{\mathrm{rig},L} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},L}$ from Lemma 2.10.*

Proof. Let us first note that for each $n \in \mathbb{N}_{\geq 1}$ we have the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathbf{B}^+_{\mathrm{rig},\breve{L}} \arrow[r, "\varphi^{n-1}([p]_q)"] \arrow[d] & \mathbf{B}^+_{\mathrm{rig},\breve{L}} \arrow[r, "\mu \mapsto \zeta_{p^n} - 1"] \arrow[d] & \breve{L}(\zeta_{p^n}) \arrow[r] \arrow[d, "\phi_{\breve{L}}"] & 0 \\
0 \arrow[r] & \tilde{\mathbf{B}}^+_{\mathrm{rig}} \arrow[r, "\varphi^{n-1}([p]_q)"] & \tilde{\mathbf{B}}^+_{\mathrm{rig}} \arrow[r, "\theta \circ \varphi^{-n}"] & \mathbb{C}_L \arrow[r] & 0,
\end{tikzcd}$$

where left and middle vertical arrows are natural inclusions, the right vertical arrow is $\phi_{\breve{L}} : \breve{L}(\zeta_{p^n}) \xrightarrow{\sim} \breve{L}(\zeta_{p^n}) \subset \mathbb{C}_L$ given as $\sum_{k=0}^{e-1} a_k\zeta_{p^n}^k \mapsto \sum_{k=0}^{e-1} \varphi_{\breve{L}}^{-n}(a_k)\zeta_{p^n}^k$ with $e = [\breve{L}(\zeta_{p^n}) : \breve{L}]$ and $\varphi_{\breve{L}} : \breve{L} \xrightarrow{\sim} \breve{L}$ and $\theta : \tilde{\mathbf{B}}_{\mathrm{rig}}^+ \subset \mathbf{B}_{\mathrm{cris}}^+(O_{\overline{L}}) \to \mathbb{C}_L$ from §2.1.1. The top row is obviously exact and the bottom row is exact by [Ber02, Proposition 2.11, Proposition 2.12 & Remarque 2.14]. All vertical maps are injective and hence we obtain that $\varphi^n([p]_q)\tilde{\mathbf{B}}_{\mathrm{rig}}^+ \cap \mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \varphi^n([p]_q)\mathbf{B}_{\mathrm{rig},\breve{L}}^+$ for all $n \in \mathbb{N}$, in particular, $\varphi^n([p]_q)\tilde{\mathbf{B}}_{\mathrm{rig},L}^+ \cap \mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \varphi^n([p]_q)\mathbf{B}_{\mathrm{rig},\breve{L}}^+$.

Let $x \in (t/\mu)\tilde{\mathbf{B}}_{\mathrm{rig},L}^+ \cap \mathbf{B}_{\mathrm{rig},\breve{L}}^+$ and write $x = ty/\mu$ for some $y \in \tilde{\mathbf{B}}_{\mathrm{rig},L}^+$. We will show that $y \in \mathbf{B}_{\mathrm{rig},\breve{L}}^+$ by showing that it converges over each closed disk $D(\breve{L}, \rho)$ for $0 < \rho < 1$. Fix some $0 < \rho < 1$ and from Lemma 2.10 we write $t/\mu = \prod_{n\in\mathbb{N}}(\varphi^n([p]_q)/p) = \upsilon \prod_{n=1}^m(\varphi^n([p]_q)/p)$ for a unit $\upsilon \in \mathcal{O}(D(\breve{L}, \rho))^\times$ and $m \in \mathbb{N}$ depending on ρ. Then we have $y_0 = x = ([p]_q/p)y_1$ for some $y_1 \in \tilde{\mathbf{B}}_{\mathrm{rig},L}^+ \cap (p/[p]_q)\mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \mathbf{B}_{\mathrm{rig},\breve{L}}^+$. Repeating this for $1 \leq n \leq m$ we obtain $y_n \in \mathbf{B}_{\mathrm{rig},\breve{L}}^+$ such that $y_n = \varphi^n([p]_q/p)y_{n+1}$ for some $y_{n+1} \in \tilde{\mathbf{B}}_{\mathrm{rig},L}^+ \cap \varphi^n(p/[p]_q)\mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \mathbf{B}_{\mathrm{rig},\breve{L}}^+$. In particular, we have $y = \upsilon^{-1}y_{m+1} \in \mathcal{O}(D(\breve{L}, \rho))$. Since $\mathbf{B}_{\mathrm{rig},\breve{L}}^+ = \lim_\rho \mathcal{O}(D(\breve{L}, \rho))$ we get that $y \in \mathbf{B}_{\mathrm{rig},\breve{L}}^+$. ∎

2.1.6. φ-modules over certain period rings. Let $\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^\dagger}$ denote the category of finite free $\mathbf{B}_{\mathrm{rig},L}^\dagger$-modules equipped with an isomorphism $1 \otimes \varphi : \varphi^*M \xrightarrow{\sim} M$ and morphisms between objects are $\mathbf{B}_{\mathrm{rig},L}^\dagger$-linear maps compatible with $1 \otimes \varphi$ on both sides; denote by $\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^\dagger}^0$ the full subcategory of objects that are pure of slope 0 in the sense of [Ked04, §6.3]. Similarly, one can define the category $\varphi\text{-Mod}_{\mathbf{B}_L^\dagger}$ and denote by $\varphi\text{-Mod}_{\mathbf{B}_L^\dagger}^0$ the full subcategory of objects that are pure of slope 0 (as φ-modules over a discretely valued field).

Let $\text{Eff-}\varphi\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q}$ denote the category of effective and finite $[p]_q$-height $\mathbf{A}_L^+$-modules, i.e. finite free $\mathbf{A}_L^+$-module N equipped with a Frobenius-semilinear endomorphism $\varphi : N \to N$ such that the map $1 \otimes \varphi : \varphi^*(N) \to N$ is injective and its cokernel is killed by a finite power of $[p]_q$; denote by $\text{Eff-}\varphi\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} \otimes \mathbb{Q}_p$ the associated isogeny category. Similarly, define $\text{Eff-}\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^+}^{[p]_q}$ as the category of effective and finite $[p]_q$-height $\mathbf{B}_{\mathrm{rig},L}^+$-modules and $\text{Eff-}\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^+}^{[p]_q,0}$ as the full subcategory of objects that are pure of slope 0, i.e. M such $\mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},L}^+} M$ is pure of slope 0.

Lemma 2.12. (1) *There is a natural equivalence of categories* $\varphi\text{-Mod}_{\mathbf{B}_L^\dagger}^0 \xrightarrow{\sim} \varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^\dagger}^0$ *induced by the functor* $M \mapsto M \otimes_{\mathbf{B}_L^\dagger} \mathbf{B}_{\mathrm{rig},L}^\dagger$.

(2) *There is an exact equivalence of $\otimes$-categories* $\text{Eff-}\varphi\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} \otimes \mathbb{Q}_p \xrightarrow{\sim} \text{Eff-}\varphi\text{-Mod}_{\mathbf{B}_{\mathrm{rig},L}^+}^{[p]_q,0}$ *induced by the functor* $N \mapsto N \otimes_{\mathbf{A}_L^+} \mathbf{B}_{\mathrm{rig},L}^+$.

Proof. The claim in (1) follows from [Ked05, Theorem 6.3.3]. The equivalence of $\otimes$-categories in (2) follows from (1), [Kis06, Lemma 1.3.13] and [Ked04, Proposition 6.5], and the exactness follows since $\mathbf{B}_L^+ \to \mathbf{B}_{\mathrm{rig},L}^+$ is faithfully flat by Lemma 2.6. Note that in [Kis06] Kisin assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of [Kis06, Lemma 1.3.13] depends only on [Ked04, Proposition 6.5] and [Ked05, Theorem 6.3.3] which are independent of the structure of the residue field. In particular, the proof of [Kis06, Lemma 1.3.13] applies almost verbatim to our case. We recall the quasi-inverse functor from loc. cit. to be used in the sequel (see §4.5).

Let M_{rig}^+ be a finite height effective $\mathbf{B}_{\mathrm{rig},L}^+$-module pure of slope 0, then $M_{\mathrm{rig}}^\dagger := \mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},L}^+} M_{\mathrm{rig}}^+$ is pure of slope 0 and (1) implies that there exists a finite free $\mathbf{B}_L^\dagger$-module $M^\dagger$ pure of slope 0 such that $\mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_L^\dagger} M^\dagger \xrightarrow{\sim} M_{\mathrm{rig}}^\dagger \xleftarrow{\sim} \mathbf{B}_{\mathrm{rig},L}^\dagger \otimes_{\mathbf{B}_{\mathrm{rig},L}^+} M_{\mathrm{rig}}^+$. Choose a $\mathbf{B}_L^\dagger$-basis of $M^\dagger$ and a $\mathbf{B}_{\mathrm{rig},L}^+$-basis of M_{rig}^+. The composite of the isomorphisms above is given by a matrix with values in $\mathbf{B}_{\mathrm{rig},L}^\dagger$. By

[Ked04, Proposition 6.5], after modifying the chosen bases, we may assume the matrix to be identity, in particular, $M^\dagger$ and M^+_{rig} are spanned by a common basis. Let M denote the $\mathbf{B}^+_L$-span of this basis. Since $\mathbf{B}^+_L = \mathbf{B}^+_{\mathrm{rig},L} \cap \mathbf{B}^\dagger_L \subset \mathbf{B}^\dagger_{\mathrm{rig},L}$, we obtain that $M := M^+_{\mathrm{rig}} \cap M^\dagger \subset M^\dagger_{\mathrm{rig}}$ and $\mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_L} M \xrightarrow{\sim} M^+_{\mathrm{rig}}$ and $\mathbf{B}^\dagger_L \otimes_{\mathbf{B}^+_L} M \xrightarrow{\sim} M^\dagger$. Moreover, $M^\dagger$ is pure of slope 0, so there exists an $\mathbf{A}^\dagger_L$-lattice $M^\dagger_0 \subset M^\dagger$. Let $M'_0 := M \cap M^\dagger_0 \subset M^\dagger$ and set $M_0 := (\mathbf{A}^\dagger_L \otimes_{\mathbf{A}^+_L} M'_0) \cap M'_0[1/p] \subset M^\dagger$. Using [Kis06, Lemma 1.3.13] and the discussion above, $M_0 \subset M$ is a finite free φ-stable $\mathbf{A}^+_L$-submodule such that cokernel of the injective map $1 \otimes \varphi : \varphi^*(M_0) \to M_0$ is killed by some finite power of $[p]_q$. ∎

Remark 2.13. Let M be a finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module and $N \subset M$ a $\mathbf{B}^+_{\mathrm{rig},L}$-submodule. Then N is finite free if and only if it is finitely generated if and only if it is a closed submodule of M. Equivalences in the preceding statement essentially follow from [Kis06, Lemma 1.1.4]. Note that Kisin assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of loc. cit. depends on results of [Laz62, §7-§8], [Ked04, Lemma 2.4] and [Ber02, Proposition 4.12 & Lemme 4.13], where the proof of latter depends on [Laz62] and [Hel43]. Relevant results of [Laz62], [Ked04] and [Hel43] are independent of the structure of the residue field of L. Hence, we get the claim by using the proof of [Kis06, Lemma 1.1.4] almost verbatim.

We note some useful facts about $\mathbf{A}^+_L$-modules.

Lemma 2.14. *Let $O_K := O_F$, O_L or $O_{\breve{L}}$ and let $A := O_K[\![\mu]\!]$ equipped with a Frobenius endorphism extending the Frobenius on O_K by $\varphi(\mu) = (1+\mu)^p - 1$. Let N be a finitely generated A-module equipped with a Frobenius-semilinear endomorphism such that $1 \otimes \varphi : \varphi^*(N)[1/[p]_q] \xrightarrow{\sim} N[1/[p]_q]$. Then $N[1/p]$ is finite free over $A[1/p]$.*

Proof. The proof is essentially the same as [BMS18, Proposition 4.3]. Let J denote the smallest non-zero Fitting ideal of N over A. Set $K := O_K[1/p]$ and $\overline{A} = A/J$. From loc. cit. the claim can be reduced to checking that $\overline{A}[1/p] = 0$. Note that the Frobenius endomorphism on A and finite height condition on N are different from loc. cit. Therefore, we need some modifications in the arguments of loc. cit.; we us point out the differences in terms of their notations. Fix an algebraic closure $\overline{K}$ of K and consider the finite set $Z := \mathrm{Spec}(\overline{A}[1/p])(\overline{K})$ of $\overline{K}$-valued points of $\overline{A}[1/p]$. Let $Z' := \{x \in \mathfrak{m}$ such that $(1+x)^p - 1 \in Z\}$, where $\mathfrak{m} \subset O_{\overline{K}}$ is the maximal ideal. Then from the equality $(A/J)[1/[p]_q] = (A/\varphi(J))[1/[p]_q]$ we get that $Z \cap U = Z' \cap U$ where $U := \mathfrak{m} - \{\zeta_p - 1, \ldots, \zeta_p^{p-1} - 1\}$. All the arguments from loc. cit. then easily adapt to give an isomorphism $K[\mu]/(\mu^r) \xrightarrow{\sim} K[\mu]/(\varphi(\mu)^r)$ where $K = O_K[1/p]$. But then we get that $(\varphi(\mu)/\mu)^r$ is a unit in $K[\mu]$, whereas $\varphi(\mu)/\mu \in K[\mu]$ is an irreducible polynomial. Hence, we must have $r = 0$ and thus $(A/J)[1/p] = 0$, proving the claim. ∎

Remark 2.15. Let N be a finitely generated torsion-free $\mathbf{A}^+_L$-module. Then $D = \mathbf{A}_L \otimes_{\mathbf{A}^+_L} N$ is a finite free $\mathbf{A}_L$-module and $N \subset D$ an $\mathbf{A}^+_L$-submodule. Moreover, the $\mathbf{A}^+_L$-module $N' = N[1/p] \cap D$ is finite free. The claim essentially follows from [Fon90, Proposition B.1.2.4]. Note that Fontaine assumes the residue field of the discrete valuation base field (L in our case) to be perfect. However, the proof of [Fon90, Proposition B.1.2.4] only depends on [Lan90, Chapter 5, Theorem 3.1] which is independent of the structure of residue field of L. Therefore, one can adpat Fontaine's proof verbatim to show that N' is finite free.

Let N be a finite free $\mathbf{A}^+_{\breve{L}}$-module. Say that N is *effective* and of *finite* $[p]_q$*-height* if N is equipped with a Frobenius-semilinear endomorphism φ such that the natural map $1 \otimes \varphi : \varphi^*(N) \to N$ is injective and its cokernel is killed by some finite power of $[p]_q$.

Let $D_{\breve{L}}$ be a finite free étale φ-module over $\mathbf{A}_{\breve{L}}$. Let $\mathcal{S}(D_{\breve{L}})$ denote the set of finitely generated $\mathbf{A}^+_{\breve{L}}$-submodules $M \subset D_{\breve{L}}$ such that M is stable under induced φ from $D_{\breve{L}}$ and cokernel of the injective map $1 \otimes \varphi : \varphi^*(M) \to M$ is killed by some finite power of $[p]_q$. In [Fon90, §B.1.5.5], Fontaine functorially attached to $D_{\breve{L}}$ an $\mathbf{A}^+_{\breve{L}}$-submodule $j^+_*(D_{\breve{L}}) := \cup_{M \in \mathcal{S}(D_{\breve{L}})} M \subset D_{\breve{L}}$ (Fontaine uses the notation j^q_* to denote the functor j^+_*; we change notations to avoid obvious confusion).

Lemma 2.16. *The $\mathbf{A}^+_{\breve{L}}$-module $j^+_*(D_{\breve{L}})$ is free of rank $\leq \mathrm{rk}_{\mathbf{A}_{\breve{L}}} D_{\breve{L}}$. Moreover, if N is an effective $\mathbf{A}^+_{\breve{L}}$-module of finite $[p]_q$-height, then cokernel of the injective map $N \to j^+_*(\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}^+_{\breve{L}}} N)$ is killed by some finite power of μ.*

Proof. The first claim is shown in [Fon90, §B.1.5.5]. For the second claim note that N is finite free over $\mathbf{A}_{\breve{L}}^{+}$ and of finite $[p]_q$-height, therefore it is p-étale in the sense of [Fon90, §B.1.3.1] by the equivalence shown in [Fon90, Proposition B.1.3.3]. In particular, we get that $D_{\breve{L}} := \mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_{\breve{L}}^{+}} N$ is an étale φ-module and $N \in \mathcal{S}(D_{\breve{L}})$. Now from [Fon90, Proposition B.1.5.6] it follows that cokernel of the injective map $N \to j_*^{+}(D_{\breve{L}})$ is killed by some finite power of μ. ∎

2.2. p-adic representations and (φ, Γ)-modules. Let T be a finite free $\mathbb{Z}_p$-representation of G_L. From theory of (φ, Γ_L)-modules (see [Fon90] and [And06]) one can functorially associate to T a finite free étale (φ, Γ_L)-module $\mathbf{D}_L(T) := (\mathbf{A} \otimes_{\mathbb{Z}_p} T)^{H_L}$ over $\mathbf{A}_L$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$, i.e. $\mathbf{D}_L(T)$ is equipped with a Forbenius-semilinear endomorphism φ and a semilinear and continuous action of Γ_L commuting with φ and such that the natural map $1 \otimes \varphi : \varphi^*(\mathbf{D}_L(T)) \to \mathbf{D}_L(T)$ is an isomorphism. Moreover, we have $\tilde{\mathbf{D}}_L(T) := (\tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^{H_L} \xrightarrow{\sim} \tilde{\mathbf{A}}^{H_L} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T)$. Furthermore, by the theory overconvergence of p-adic and $\mathbb{Z}_p$-representations (see [CC98] and [AB08]) one can functorially associate to T a finite free étale (φ, Γ_L)-module $\mathbf{D}_L^{\dagger}(T) := (\mathbf{A}^{\dagger} \otimes_{\mathbb{Z}_p} T)^{H_L}$ over $\mathbf{A}_L^{\dagger}$ of rank $= \mathrm{rk}_{\mathbb{Z}_p} T$ and such that $\mathbf{A}_L \otimes_{\mathbf{A}_L^{\dagger}} \mathbf{D}_L^{\dagger}(T) \xrightarrow{\sim} \mathbf{D}_L(T)$. We have natural isomorphisms

$$\mathbf{A} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T, \quad \mathbf{A}^{\dagger} \otimes_{\mathbf{A}_L^{\dagger}} \mathbf{D}_L^{\dagger}(T) \xrightarrow{\sim} \mathbf{A}^{\dagger} \otimes_{\mathbb{Z}_p} T, \tag{2.1}$$

comaptible with (φ, Γ_L)-actions. More generally, the constructions described above are functorial and induce equivalence of categories

$$\mathrm{Rep}_{\mathbb{Z}_p}(G_L) \xrightarrow{\sim} (\varphi, \Gamma_L)\text{-}\mathrm{Mod}_{\mathbf{A}_L}^{\text{ét}} \xleftarrow{\sim} (\varphi, \Gamma_L)\text{-}\mathrm{Mod}_{\mathbf{A}_L^{\dagger}}^{\text{ét}}. \tag{2.2}$$

Similar statements are also true for p-adic representations of G_L. For a p-adic representation V of G_L, set $\mathbf{D}_{\mathrm{rig},L}^{\dagger}(V) := \mathbf{B}_{\mathrm{rig},L}^{\dagger} \otimes_{\mathbf{B}_L^{\dagger}} \mathbf{D}_L^{\dagger}(V)$ which is the unique finite free (φ, Γ_L)-module over $\mathbf{B}_{\mathrm{rig},L}^{\dagger}$ of rank $= \dim_{\mathbb{Q}_p} V$ and pure of slope 0 functorially attached to V (see [Ber02], [Ked05] and [Ohk15]). Moreover, the preceding functor induces an equivalence of categories between p-adic representations of G_L and finite free (φ, Γ_L)-modules over $\mathbf{B}_{\mathrm{rig},L}^{\dagger}$ which are pure of slope 0 (see [Ohk15, Lemma 4.5.7]) and we have a natural (φ, G_L)-equivariant isomorphism

$$\tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger} \otimes_{\mathbf{B}_{\mathrm{rig},L}^{\dagger}} \mathbf{D}_{\mathrm{rig},L}^{\dagger}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}_{\mathrm{rig}}^{\dagger} \otimes_{\mathbb{Q}_p} V. \tag{2.3}$$

Remark 2.17. We have variations of the results mentioned above for p-adic (resp. $\mathbb{Z}_p$-representations) of $G_{\breve{L}}$ as well (see [Fon90], [CC98] and [Ber02] for details).

Finally, let V be a p-adic representation of G_L and $T \subset V$ a G_L-stable $\mathbb{Z}_p$-lattice. Since $G_{\breve{L}}$ is a subgroup of G_L, therefore by restriction V is a p-adic representation of $G_{\breve{L}}$ and $T \subset V$ a $G_{\breve{L}}$-stable $\mathbb{Z}_p$-lattice. Furthermore, we have a $\Gamma_{\breve{L}}$-equivariant embedding $\mathbf{A}_L \subset \mathbf{A}_{\breve{L}}$ (via the map $[X_i^{\flat}] \mapsto X_i$) and thus we have isomorphisms of étale $(\varphi, \Gamma_{\breve{L}})$-modules $\mathbf{D}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T)$ and $\tilde{\mathbf{D}}_{\breve{L}}(T) := (\tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T)^{H_{\breve{L}}} \xrightarrow{\sim} \tilde{\mathbf{A}}^{H_{\breve{L}}} \otimes_{\mathbf{A}_{\breve{L}}} \mathbf{D}_{\breve{L}}(T)$. Similar statements are also true for V.

2.3. Crystalline representations. Let $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L)$ denote the category of p-adic crystalline representations of G_L (see [Bri06, §3.3]) and let $\mathrm{MF}_L^{\mathrm{wa}}(\varphi, \partial)$ denote the category of weakly admissible filtered (φ, ∂)-modules over L (see [Bri06, Définition 4.21]). Then the following functor induces an exact equivalence of $\otimes$-categories:

$$\begin{aligned} \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L) &\xrightarrow{\sim} \mathrm{MF}_L^{\mathrm{wa}}(\varphi, \partial) \\ V &\longmapsto \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) := (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V)^{G_L}, \end{aligned} \tag{2.4}$$

with an exact quasi-inverse given as $D \mapsto \mathcal{O}\mathbf{V}_{\mathrm{cris},L}(D) := (\mathrm{Fil}^0(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L D))^{\partial=0, \varphi=1}$ (see [Bri06, Corollaire 4.37]). In particular, if V is a p-adic crystalline representation of G_L then $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is a rank $= \dim_{\mathbb{Q}_p} V$ weakly admissible filtered (φ, ∂)-module over L. Moreover, as a representation of $G_{\breve{L}}$ one can attach to V a rank $= \dim_{\mathbb{Q}_p} V$ filtered φ-module over $\breve{L}$, denoted as $\mathbf{D}_{\mathrm{cris},\breve{L}}(V)$. Then from [BT08, Proposition 4.14] and the map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ sending $X_i \mapsto [X_i^\flat]$ we obtain an isomorphism of filtered φ-modules over $\breve{L}$

$$\breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris},\breve{L}}(V). \tag{2.5}$$

The representation V is said to be positive if all its Hodge-Tate weights are ≤ 0 and in this case we have $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = (\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V)^{G_L}$. We equip $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ with a $(\varphi, G_{\breve{L}})$-equivariant L-algebra structure via the composition $L \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ where the first map is the non-canonical L-algebra structure on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ (see §2.1.1).

Lemma 2.18. *There exists a natural $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear and Frobenius-equivariant isomorphism*

$$\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0} \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V),$$

induced by the surjective map $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ given by $X_i \mapsto [X_i^\flat]$ for $1 \leq i \leq d$.

Proof. Let $J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) :=$ p-adic closure of the ideal $([X_1^\flat] - X_1, \ldots, [X_d^\flat] - X_d) \subset \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$. Then we have a projection,

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V) \longrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V), \tag{2.6}$$

via the map $X_i \mapsto [X_i^\flat]$ with kernel given as $J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$. Moreover, using the non-canonical L-algebra structure on $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, we have an L-linear map $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ given as $x \mapsto \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(x) \prod_{i=1}^d([X_i^\flat] - X_i)^{[k_i]}$, where we write $\prod_{i=1}^d \partial_i^{k_i}(x) = \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x)$ for notational convenience. The map above extends $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linearly to a map

$$\begin{aligned} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) &\longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \\ a \otimes x &\longmapsto a \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(x) \prod_{i=1}^d([X_i^\flat] - X_i)^{[k_i]}, \end{aligned} \tag{2.7}$$

and it provides a section to the projection given above. In particular, we obtain a $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear direct sum decomposition

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) = (J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \oplus (\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)).$$

Note that the image of the section (2.7) lies in $(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0}$. Moreover, since V is crystalline we have $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L V$ and then one can easily show that $(J\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0} = 0$. Therefore, from the direct sum decomposition we conclude that $(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0} \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Note that (2.6) and (2.7) are evidently compatible with Frobenius on either side, therefore the isomorphism in the claim is compatible with Frobenius. Hence, we get the claim. ∎

Remark 2.19. Using the $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear map in (2.7) we equip $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ with a G_L-action by transport of structure. In particular, the action of $g \in G_L$ on $a \otimes x \in \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is given by the formula $g(a \otimes x) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(x) \prod_{i=1}^d(g([X_i^\flat]) - [X_i^\flat])^{[k_i]}$.

Remark 2.20. Using the description in Remark 2.19 we have that $\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of G_L as well. Moreover, we note that the H_L-action on $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ in the tensor product $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is trivial and $\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})^{H_L} = \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ by [MT20, Lemma 4.32]. Therefore, we get that

$$(\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{H_L} = \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V). \tag{2.8}$$

We equip $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ with the residual Γ_L-action.

Lemma 2.21. *For $x \in \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and $g \in \Gamma_L$, the series $\sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^d \partial_i^{k_i}(x) \prod_{i=1}^d(g([X_i^\flat]) - [X_i^\flat])^{[k_i]}$ converges in $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. In particular, $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under Γ_L-action.*

Proof. Let $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ be topological generators of Γ_L as in §2.1.1, in particular, $\gamma_j([X_i]^\flat) = (1+\mu)[X_i^\flat]$ if $i = j$ and 0 otherwise. Simplifying the sum in the claim for γ_j we get it as $\sum_{\mathbf{k}\in\mathbb{N}^d} \mu^{[k_j]}[X_j^\flat] \prod_{i=1}^d \partial_i^{k_i}(x)$. Recall that the connection ∂ on $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is p-adically quasi-nilpotent, i.e. there exists an O_L-lattice $M \subset \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ stable under ∂ such that ∂ is nilpotent modulo p. In particular, we have $\partial : M \to M \otimes \Omega^1_{O_L}$. Let $\{e_1, \ldots, e_h\}$ denote an O_L-basis of M. Then we may check on the chosen basis that $\varphi(M) \subset p^{-r}M$ for some fixed $r \in \mathbb{N}$. Moreover, recall that we have $L \otimes_{\varphi,L} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, so we may write $x = \sum_{j=1}^h a_j\varphi(e_j)$ for $a_j \in L$. Since $\partial_i(\varphi(e_j)) = p\varphi(\partial_i(e_j))$ for all $1 \leq i \leq d$ and $1 \leq j \leq n$, we get that

$$\sum_{\mathbf{k}\in\mathbb{N}^d} \mu^{[k_j]}[X_j^\flat] \prod_{i=1}^d \partial_i^{k_i}(\varphi(e_i)) = p^{-dr} \sum_{\mathbf{k}\in\mathbb{N}^d} \mu^{[k_j]}[X_i^\flat] \prod_{i=1}^d p^{k_i}p^r\varphi(\partial_i^{k_i}(e_i)),$$

converges in $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Therefore, using Leibniz rule we are reduced to showing that the sum $\sum_{\mathbf{k}\in\mathbb{N}^d} \mu^{[k_j]}[X_j^\flat] \prod_{i=1}^d \partial_i^{k_i}(a)$ converges in $\mathbf{B}^+_{\mathrm{rig},L}$ for any $a \in L$. This follows easily since we have $\partial_i^k(X_i^n)/k! = 0$ for $n < k$, $\partial_i^k(X_i^n)/k! = \binom{n}{k}X_i^{n-k}$ for $n \geq k$ and $\partial_i^k(X_i^{-n})/k! = (-1)^k\binom{n+k-1}{k}X_i^{-(n+k)}$ for $n \in \mathbb{N}$. Hence, the lemma is proved. ■

Lemma 2.22. *The action of Γ_L on $\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is trvial modulo μ.*

Proof. Note that $g(\mu) = (1+\mu)^{\chi(g)} - 1$ for $g \in \Gamma_L$ and χ the p-adic cyclotomic character. Using Lemma 2.21 and for $g \in \Gamma_L$ and $a \otimes x \in \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, this action is given by the formula $g(a \otimes x) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^x \partial_i^{k_i}(x) \prod_{i=1}^d (g([X_i^\flat]) - [X_i^\flat])^{[k_i]}$. Note that

$$(g-1)(a \otimes x) = ((g-1)a) \otimes x + g(a) \otimes ((g-1)x), \tag{2.9}$$

where $g(x)$ is given by the series in Lemma 2.21. So, $(g-1)x = \sum_{\mathbf{k}\in\mathbb{N}^d_+} \prod_{i=1}^x \partial_i^{k_i}(x) \prod_{i=1}^d ((g-1)[X_i^\flat])^{[k_i]}$, where $\mathbb{N}^d_+ = \mathbb{N}^d \setminus \{(0,0,\ldots,0)\}$. Using the explicit description of $\mathbf{B}^+_{\mathrm{rig},L}$ in Lemma 2.4 note that $(g-1)\mathbf{B}^+_{\mathrm{rig},L} \subset \mu\mathbf{B}^+_{\mathrm{rig},L}$ and from the proof of Lemma 2.21 note that $(g-1)[X_i^\flat] \in \mu\mathbf{B}^+_L$. Therefore, an argument similar to the proof of Lemma 2.21 shows that $(g-1)x$ converges in $\mu\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. So from (2.9) it follows that $(g-1)(a \otimes x) \in \mu\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. ■

3. Wach modules

In this section we will describe Wach modules in the imperfect residue field case and finite $[p]_q$-height representations of G_L and relate them to crystalline representations. Our definition is a direct and natural generalization of Wach modules in the perfect residue field case (see [Ber04, Définition III.4.1]).

3.1. Wach modules over $\mathbf{A}^+_L$.

Inside $\mathbf{A}_{\mathrm{inf}}(O_{F_\infty})$ fix $q := [\varepsilon]$, $\mu := [\varepsilon] - 1 = q - 1$ and $[p]_q := \tilde{\xi} := \varphi(\mu)/\mu$.

Definition 3.1. Let $a, b \in \mathbb{Z}$ with $b \geq a$. A *Wach module* over $\mathbf{A}^+_L$ with weights in the interval $[a, b]$ is a finite free $\mathbf{A}^+_L$-module N equipped with a continuous and semilinear action of Γ_L satisfying the following assumptions:

(1) The action of Γ_L on $N/\mu N$ is trivial.

(2) There is a Frobenius-semilinear operator $\varphi : N[1/\mu] \to N[1/\varphi(\mu)]$ commuting with the action of Γ_L such that $\varphi(\mu^b N) \subset \mu^b N$ and cokernel of the induced injective map $(1 \otimes \varphi) : \varphi^*(\mu^b N) \to \mu^b N$ is killed by $[p]_q^{b-a}$.

Define the $[p]_q$-*height* of N to be the largest value of $-a$ for $a \in \mathbb{Z}$ as above. Say that N is *effective* if one can take $b = 0$ and $a \leq 0$. A Wach module over $\mathbf{B}^+_L$ is a finitely generated module M equipped with a Frobenius-semilinear operator $\varphi : M[1/\mu] \to M[1/\varphi(\mu)]$ commuting with the action of Γ_L such that there exists a φ-stable (after inverting μ) and Γ_L-stable $\mathbf{A}^+_L$-submodule $N \subset M$ with N a Wach module over $\mathbf{A}^+_L$ (equipped with induced (φ, Γ_L)-action) and $N[1/p] = M$.

Denote the category of Wach modules over $\mathbf{A}_L^+$ as $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q}$ with morphisms between objects being $\mathbf{A}_L^+$-linear Γ_L-equivariant and φ-equivariant morphisms (after inverting μ).

Definition 3.2. Let N be a Wach module over $\mathbf{A}_L^+$. Define a decreasing filtration on N called the *Nygaard filtration*, for $k \in \mathbb{Z}$, as

$$\mathrm{Fil}^k N := \{x \in N \text{ such that } \varphi(x) \in [p]_q^k N\}.$$

From the definition it is clear that N is effective if and only if $\mathrm{Fil}^0 N = N$. Similarly, we can define a Nygaard filtration on $M := N[1/p]$ and it satisfies $\mathrm{Fil}^k M = (\mathrm{Fil}^k N)[1/p]$.

Extending scalars along $\mathbf{A}_L^+ \to \mathbf{A}_L$ induces a functor $(\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} \to (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L}^{\text{ét}}$ and we make the following claim:

Proposition 3.3. *The following natural functor is fully faithful:*

$$\begin{aligned} (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} &\longrightarrow (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L}^{\text{ét}} \\ N &\longmapsto \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N. \end{aligned}$$

Proof. We need to show that for Wach modules N and N', we have a bijection

$$\mathrm{Hom}_{(\varphi,\Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q}}(N, N') \xrightarrow{\sim} \mathrm{Hom}_{(\varphi,\Gamma)\text{-Mod}_{\mathbf{A}_L}^{\text{ét}}}(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} N, \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N') \tag{3.1}$$

Note that $\mathbf{A}_L^+ \to \mathbf{A}_L = \mathbf{A}_L^+[1/\mu]^\wedge$ is injective, in particular, the map in (3.1) is injective. To check that (3.1) is surjective let $D_L = \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N$, $D_L' = \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N'$ and take an $\mathbf{A}_L$-linear and (φ, Γ_L)-equivariant map $f : D_L \to D_L'$. Base changing f along the embedding $\mathbf{A}_L \to \mathbf{A}_{\breve{L}}$ (see §2.1.5) we obtain an $\mathbf{A}_{\breve{L}}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant map $f_{\breve{L}} : D_{\breve{L}} \to D_{\breve{L}}'$. Using the definition and notation preceding Lemma 2.16 we further obtain an $\mathbf{A}_{\breve{L}}^+$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant map $f_{\breve{L}} : j_*^+(D_{\breve{L}}) \to j_*^+(D_{\breve{L}}')$ where we abuse notations by writing $f_{\breve{L}}$ instead of $j_*^+(f_{\breve{L}})$. From Lemma 2.16 note that for some $s \in \mathbb{N}$ and $N_{\breve{L}} := \mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} N$, we have $\mu^s N_{\breve{L}} \subset j_*^+(D_{\breve{L}})$ and its cokernel is killed by some finite power of μ. Hence, $N_{\breve{L}}[1/\mu] \xrightarrow{\sim} j_*^+(D_{\breve{L}})[1/\mu]$. Similarly, one can also show that $N_{\breve{L}}'[1/\mu] \xrightarrow{\sim} j_*^+(D_{\breve{L}}')[1/\mu]$.

Now from the map $f_{\breve{L}} : j_*^+(D_{\breve{L}}) \to j_*^+(D_{\breve{L}}')$ we obtain an induced $\Gamma_{\breve{L}}$-equivariant map $f_{\breve{L}} : N_{\breve{L}}[1/\mu] = j_*^+(D_{\breve{L}}')[1/\mu] \to j_*^+(D_{\breve{L}}')[1/\mu] = N_{\breve{L}}'[1/\mu]$ and from Lemma 3.4 we get that $f_{\breve{L}}(N_{\breve{L}}) \subset N_{\breve{L}}'$. It is easy to see that $N := N_{\breve{L}} \cap D_L \subset D_{\breve{L}}$ and $N' := N_{\breve{L}}' \cap D_L' \subset D_{\breve{L}}'$, so we conclude that $f(N) = f_{\breve{L}}(N_{\breve{L}}) \cap f(D_L) \subset N_{\breve{L}}' \cap D_L' = N'$. This proves the surjectivity of (3.1). ∎

Lemma 3.4. *Let N and N' be Wach modules over $\mathbf{A}_{\breve{L}}^+$ and let $f : N[1/\mu] \to N'[1/\mu]$ be an $\mathbf{A}_{\breve{L}}^+$-linear and $\Gamma_{\breve{L}}$-equivariant map. Then $f(N) \subset N'$.*

Proof. The proof is similar to the proof of [Abh21, Lemma 5.31]. Assume $f(N) \subset \mu^{-k}N'$ for some $k \in \mathbb{N}$ and consider the reduction of f modulo μ, which is again $\Gamma_{\breve{L}}$-equivariant. By definition we have that $\Gamma_{\breve{L}}$ acts trivially over $N/\mu N$, whereas $\mu^{-k}N'/\mu^{-k+1}N' \xrightarrow{\sim} N'/\mu N'(-k)$, i.e. the action of $\Gamma_{\breve{L}}$ on $\mu^{-k}N'/\mu^{-k+1}N'$ is given by χ^{-k} where χ is the p-adic cyclotomic character, in particular, $(\mu^{-k}N'/\mu^{-k+1}N')^{\Gamma_{\breve{L}}} = 0$. Since f is $\Gamma_{\breve{L}}$-equivariant, we must have $k = 0$, i.e. $f(N) \subset N'$. ∎

Analogous to above, one can define categories $(\varphi, \Gamma)\text{-Mod}_{\mathbf{B}_L^+}^{[p]_q}$ and $(\varphi, \Gamma)\text{-Mod}_{\mathbf{B}_L}^{\text{ét}}$ and a functor from the former to latter by extending scalars along $\mathbf{B}_L^+ \to \mathbf{B}_L$. Then passing to associated isogeny catgeories in Proposition 3.3 we get the following:

Corollary 3.5. *The natural functor $(\varphi, \Gamma)\text{-Mod}_{\mathbf{B}_L^+}^{[p]_q} \to (\varphi, \Gamma)\text{-Mod}_{\mathbf{B}_L}^{\text{ét}}$ is fully faithful.*

Composing the functor in Proposition 3.3 with the equivalence in (2.2), we obtain a fully faithful functor

$$\begin{aligned} \mathbf{T}_L : (\varphi, \Gamma)\text{-Mod}_{\mathbf{A}_L^+}^{[p]_q} &\longrightarrow \text{Rep}_{\mathbb{Z}_p}(G_L) \\ N &\longmapsto (\mathbf{A} \otimes_{\mathbf{A}_L^+} N)^{\varphi=1} \xrightarrow{\sim} (W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} N)^{\varphi=1}. \end{aligned} \tag{3.2}$$

Lemma 3.6. *Let N be Wach module of $[p]_q$-height s and let $T := \mathbf{T}_L(N)$. Then we have a G_L-equivariant isomorphism*

$$\mathbf{A}^+[1/\mu] \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \mathbf{A}^+[1/\mu] \otimes_{\mathbb{Z}_p} T. \tag{3.3}$$

Moreover, if N is effective, then we have G_L-equivariant inclusions $\mu^s(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) \subset \mathbf{A}^+ \otimes_{\mathbf{A}_L^+} N \subset \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T$.

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have that $\mathbf{T}_L(\mu^r N(-r)) = \mathbf{T}_L(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which Γ_L acts via χ^{-r} where χ is the p-adic cyclotomic character). Therefore, it is enough to show both the claims for effective Wach modules. So assume N is effective. Since N is finite free over $\mathbf{A}_L^+$, using Definition 3.1 (2) and tensor product Frobenius we obtain an isomorphism $\varphi : \mathbf{A}_{\text{inf}}(O_{\overline{L}})[1/\xi] \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \mathbf{A}_{\text{inf}}(O_{\overline{L}})[1/\tilde{\xi}] \otimes_{\mathbf{A}_L^+} N$. So from [MT20, Proposition 6.15] we get G_L-equivariant inclusions

$$\mu^s(\mathbf{A}_{\text{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T) \subset \mathbf{A}_{\text{inf}}(O_{\overline{L}}) \otimes_{\mathbf{A}_L^+} N \subset \mathbf{A}_{\text{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T \subset \tilde{\mathbf{A}} \otimes_{\mathbf{A}_L^+} N.$$

Moreover, from (2.1) we have $\mathbf{A} \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \mathbf{A} \otimes_{\mathbb{Z}_p} T$. Therefore, taking the following intersection inside $\tilde{\mathbf{A}} \otimes_{\mathbf{A}_L^+} N \xrightarrow{\sim} \tilde{\mathbf{A}} \otimes_{\mathbb{Z}_p} T$ we obtain G_L-equivariant inclusions:

$$\mu^s(\mathbf{A}_{\text{inf}}(O_{\overline{L}}) \cap \mathbf{A}) \otimes_{\mathbb{Z}_p} T \subset (\mathbf{A}_{\text{inf}}(O_{\overline{L}}) \cap \mathbf{A}) \otimes_{\mathbf{A}_L^+} N \subset (\mathbf{A}_{\text{inf}}(O_{\overline{L}}) \cap \mathbf{A}) \otimes_{\mathbb{Z}_p} T.$$

Since $\mathbf{A}^+ = \mathbf{A}_{\text{inf}}(O_{\overline{L}}) \cap \mathbf{A}$ we get that the natural map in (3.3) is bijective and $\mu^s(\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T) \subset \mathbf{A}^+ \otimes_{\mathbf{A}_L^+} N \subset \mathbf{A}^+ \otimes_{\mathbb{Z}_p} T$ (for N effective), as desired. ∎

3.2. Finite $[p]_q$-height representations. In this subsection we generalize the definition of finite $[p]_q$-height representations from [Abh21, Definition 4.9] in the imperfect residue field case. Let T be a finite free $\mathbb{Z}_p$-representation of G_L, $V := T[1/p]$ and set $\mathbf{D}_L^+(T) := (\mathbf{A}^+ \otimes_{\mathbb{Z}_p} T)^{H_L}$ be the (φ, Γ_L)-module over $\mathbf{A}_L^+$ associated to T and let $\mathbf{D}_L^+(V) := \mathbf{D}_L^+(T)[1/p]$ be the (φ, Γ_L)-module over $\mathbf{B}_L^+$ associated to V.

Definition 3.7. A *finite $[p]_q$-height* $\mathbb{Z}_p$-representation of G_L is a finite free $\mathbb{Z}_p$-module T admitting a linear and continuous action of G_L such that there exists a finite free $\mathbf{A}_L^+$-submodule $\mathbf{N}_L(T) \subset \mathbf{D}_L(T)$ satisfying the following:

(1) $\mathbf{N}_L(T)$ is a Wach module in the sense of Definition 3.1.

(2) We have $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$.

Set the $[p]_q$-height of T to be the $[p]_q$-height of $\mathbf{N}_L(T)$. Say T is *positive* if $\mathbf{N}_L(T)$ is effective.

A finite $[p]_q$-height p-adic representation of G_L is a finite dimensional $\mathbb{Q}_p$-vector space admitting a linear and continuous action of G_L such that there exists a G_L-stable $\mathbb{Z}_p$-lattice $T \subset V$ with T of finite $[p]_q$-height. We set $\mathbf{N}_L(V) = \mathbf{N}_L(T)[1/p]$ satisfying analogous properties. Set $[p]_q$-height of V to be the $[p]_q$-height of T. Say V is positive if $\mathbf{N}_L(V)$ is effective.

Remark 3.8. For T a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_L and $r \in \mathbb{N}$. We set $\mathbf{N}_L(T(r)) := \mu^{-r}\mathbf{N}_L(T)(r)$, in particular, property of being finite $[p]_q$-height is invariant under Tate twists.

Lemma 3.9. *Let T be a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_L. Then,*

(1) *If T is positive then $\mu^s \mathbf{D}_L^+(T) \subset \mathbf{N}_L(T) \subset \mathbf{D}_L^+(T)$.*

(2) *The $\mathbf{A}_L^+$-module $\mathbf{N}_L(T)$ is unique.*

Proof. Since $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$ and this scalar extension is fully faithful by Proposition 3.3, we obtain that $\mathbf{T}_L(\mathbf{N}_L(T)) \xrightarrow{\sim} T$ as representations of G_L (here $\mathbf{T}_L$ is the functor defined in (3.2)). This also implies that Lemma 3.6 holds for $\mathbf{N}_L(T)$, so taking H_L-invariants there we obtain $\mu^s \mathbf{D}_L^+(T) \subset \mathbf{N}_L(T) \subset \mathbf{D}_L^+(T)$ which shows (1). The claim in (2) follows from Proposition 3.3, or using an argument similar to [Abh21, Proposition 4.13]. ∎

Remark 3.10. Let V be a finite $[p]_q$-height p-adic representation of G_L and $T \subset V$ a finite $[p]_q$-height G_L-stable $\mathbb{Z}_p$-lattice. Then we have $\mathbf{N}_L(V) = \mathbf{N}_L(T)[1/p]$ and from Lemma 3.9 we get that if V is positive then $\mu^s \mathbf{D}_L^+(V) \subset \mathbf{N}_L(V) \subset \mathbf{D}_L^+(V)$. Moreover, from Corollary 3.5 (or [Abh21, Proposition 4.13]) it follows that $\mathbf{N}_L(V)$ is unique, in particular, it is independent of choice of the lattice T by . Alternatively, note that since we have $\mathbf{N}_L(V(r)) = \mu^{-r}\mathbf{N}_L(V)(r)$, without loss of generality we may assume that V is positive and $T' \subset V$ another finite $[p]_q$-height G_L-stable $\mathbb{Z}_p$-lattice. Then $\mu^s \mathbf{D}_L^+(V) \subset \mathbf{N}_L(T')[1/p] \subset \mathbf{D}_L^+(V)$ and using the argument in the proof of [Abh21, Proposition 4.13] almost verbatim gives $\mathbf{N}_L(V) = \mathbf{N}_L(T)[1/p] \xrightarrow{\sim} \mathbf{N}_L(T')[1/p]$ compatible with (φ, Γ_L)-action.

Remark 3.11. From the definition of finite $[p]_q$-height representations, Lemma 3.9 and the fully faithful functor in (3.2) it follows that the data of a finite $[p]_q$-height representation is equivalent to the data of a Wach module.

3.3. Wach modules are crystalline. The goal of this subsection is to prove Theorem 3.12 and Corollary 3.16. To prove our results we need certain period rings similar to [Abh21, §4.3.1] which we denote as $\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ below. We define these as follows: let $\varpi = \zeta_p - 1$ and set $\mathbf{A}_{L,\varpi}^+ := \mathbf{A}_L^+[\varphi^{-1}(\mu)] \subset \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$. Restricting the map θ on $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ (see §2.1.1) to $\mathbf{A}_{L,\varpi}^+$ we get a surjection $\theta : \mathbf{A}_{L,\varpi}^+ \twoheadrightarrow O_L[\varpi]$. Define $\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ to be the p-adic completion of the divided power envelope of the map θ with respect to Ker θ. Moreover, consider the surjective map $\theta_L : O_L \otimes_{\mathbb{Z}} \mathbf{A}_{L,\varpi}^+ \twoheadrightarrow O_L[\varpi]$ given as $x \otimes y \mapsto x\theta(y)$. Define $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ to be the p-adic completion of the divided power envelope of the map θ_L with respect to Ker θ_L. Similar to [Abh21, §4.3.1] one can show that $\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \subset \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ and $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ stable under Frobenius and Γ_L-action on latter. We equip $\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ and $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ with induced structures, in particular, a filtration (same as filtration by divided powers of Ker θ and Ker θ_L respectively, see [Abh21, Remark 4.23]) and a connection ∂_A on $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ satisfying Griffiths transversality and such that $(\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}})^{\partial_A = 0} = \mathbf{A}_{L,\varpi}^{\mathrm{PD}}$.

Theorem 3.12. *Let N be a Wach module over $\mathbf{A}_L^+$ then $V := \mathbf{T}_L(N)[1/p]$ is a p-adic crystalline representation of G_L.*

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have that $\mathbf{T}_L(\mu^r N(-r)) = \mathbf{T}_L(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which Γ_L acts via χ^{-r} where χ is the p-adic cyclotomic character). Therefore, it is enough to show the claim for effective Wach modules. So assume N is effective. Note that N is free over $\mathbf{A}_L^+$ and $\mathbf{T}_L(N)$ is a finite $[p]_q$-height $\mathbb{Z}_p$-representation of G_L in the sense of Definition 3.7 (see Remark 3.11). Then the results of [Abh21, §4.3-§4.5] can be adapted to the case of base ring O_L almost verbatim since all objects appearing in loc. cit. admit a natural variation for O_L. In particular, proofs of [Abh21, Theorem 4.25, Proposition 4.28] can be adapted to our case to get that $V = \mathbf{T}_L(N)[1/p]$ is a crystalline representation of G_L.

Set $D_L := (\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p])^{\Gamma_L} \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, then from Proposition 3.14 it follows that D_L is a finite L-vector space of dimension $= \mathrm{rk}_{\mathbf{A}_L^+} N$ equipped with tensor product Frobenius and a connection induced from the connection on $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ satisfying Griffiths transversality with respect to the filtration defined as $\mathrm{Fil}^k D_L := \big(\sum_{i+j=k} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} \mathrm{Fil}^j N[1/p]\big)^{\Gamma_L}$ where $N[1/p]$ is equipped with Nygaard filtration of Definition 3.2. Then from Proposition 3.14 we have a natural isomorphism

$\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L} D_L \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]$. Now consider the following diagram:

$$\begin{tikzcd}
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L D_L \arrow[r, "(3.6)", "\sim"'] \arrow[d, "(3.7)"'] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{A}_L^+} N[1/p] \arrow[d, "(3.3)"', "\wr"] \\
\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[r] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V,
\end{tikzcd} \tag{3.4}$$

where the left vertical arrow is extension of the inclusion $D_L \subset \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ from (3.7) along $L \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, the top horizontal arrow is extension of the isomorphism in Proposition 3.14 along $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, the right vertical arrow is extension of the isomorphism (3.3) in Lemma 3.6 along $\mathbf{A}^+[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ and the bottom horizontal arrow is the natural injective map (see [Bri06, Proposition 3.22]). Commutativity and compatibility of the diagram with (φ, G_L)-action and connection follows from (3.7). Bijectivity of the top horizontal arrow and the right vertical arrow imply that the left vertical arrow and the bottom horizontal arrow are bijective as well. Hence, V is a crystalline representation of G_L. ■

Remark 3.13. In diagram (3.4), taking the G_L-fixed part of the left vertical arrow we get that

$$D_L \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \tag{3.5}$$

compatible with Frobenius and connection. Moreover, since the bottom horizontal arrow of the diagram (3.4) is compatible with filtrations (see [Bri06, Proposition 3.35]), an argument similar to the proof of [Abh21, Proposition 4.49] shows that the isomorphism in (3.5) is compatible with filtrations, where we consider the Hodge filtration on $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$.

Following result was used in the proof of Theorem 3.12:

Proposition 3.14. *Let N be an effective Wach module over $\mathbf{A}_L^+$, then $D_L := (\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p])^{\Gamma_L}$ is a finite L-vector space of dimension $= \mathrm{rk}_{\mathbf{A}_L^+} N$ equipped with Frobenius, filtration and a connection satisfying Griffiths transversality with respect to the filtration. Moreover, we have a natural comparison isomorphism*

$$\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L} D_L \xrightarrow{\sim} \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p], \tag{3.6}$$

compatible with Frobenius, filtration, connection and Γ_L-action.

Proof. We will adapt the proof of [Abh21, Proposition 4.28]. Following [Abh21, §4.4.1], for $n \in \mathbb{N}$ define a p-adically complete ring $S_n^{\mathrm{PD}} := \mathbf{A}_L^+\langle \frac{\mu}{p^n}, \frac{\mu^2}{2!p^{2n}}, \ldots, \frac{\mu^k}{k!p^{kn}}, \ldots \rangle$. The p-adically completed divided power ring S_n^{PD} is equipped with a continuous action of Γ_L and we have a Frobenius homomorphism $\varphi : S_n^{\mathrm{PD}} \to S_{n-1}^{\mathrm{PD}}$, in particular, $\varphi^n(S_n^{\mathrm{PD}}) \subset S_0^{\mathrm{PD}} \subset \mathbf{A}_{L,\varpi}^{\mathrm{PD}}$, where the latter inclusion is obvious. The reader should note that in [Abh21, §4.4.1] we consider a further completion of S_n^{PD} with respect to certain filtration by PD-ideals, denoted $\widehat{S}_n^{\mathrm{PD}}$ in loc. cit. However, such a completion is not strictly necessary and all proofs of loc. cit. can be carried out without it. In particular, many good properties of $\widehat{S}_n^{\mathrm{PD}}$ restrict to good properties on S_n^{PD} as well (for example, (φ, Γ_L)-action above).

Now consider the O_F-linear homomorphism of rings $\iota : O_L \to S_n^{\mathrm{PD}}$ sending $X_j \mapsto [X_j^\flat]$ for $1 \leq j \leq d$. Using ι define an O_F-linear morphism of rings $f : O_L \otimes_{O_F} S_n^{\mathrm{PD}} \to S_n^{\mathrm{PD}}$ via $a \otimes b \mapsto \iota(a)b$. Let $\mathcal{O}S_n^{\mathrm{PD}}$ denote the p-adic completion of the divided power envelope of $O_L \otimes_{O_F} S_n^{\mathrm{PD}}$ with respect to Ker f. The divided power ring $\mathcal{O}S_n^{\mathrm{PD}}$ is equipped with a continuous action of Γ_L, an integrable connection and we have a Frobenius $\varphi : \mathcal{O}S_n^{\mathrm{PD}} \to \mathcal{O}S_{n-1}^{\mathrm{PD}}$, in particular, $\varphi^n(\mathcal{O}S_n^{\mathrm{PD}}) \subset \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$. Moreover, we have $O_L = (\mathcal{O}S_n^{\mathrm{PD}})^{\Gamma_L}$ and with $V_j := \frac{X_j \otimes 1}{1 \otimes [X_j^\flat]}$ for $1 \leq j \leq d$, we have p-adically closed divided power ideals

$$J^{[i]}\mathcal{O}S_n^{\mathrm{PD}} := \Big\langle \tfrac{\mu^{[k_0]}}{p^{nk_0}} \prod_{j=1}^{d} (1 - V_j)^{[k_j]},\ \mathbf{k} = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1} \text{ such that } \sum_{j=0}^{d} k_j \geq i \Big\rangle.$$

Equip $\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N$ with tensor product filtration, tensor product Frobenius and an integrable connection induced from the connection on $\mathcal{O}S_n^{\mathrm{PD}}$. Then $D_n := (\mathcal{O}S_n^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p])^{\Gamma_L}$ is an L-vector space equipped with an integrable connection and we have a Frobenius morphism $\varphi : D_n \to D_{n-1}$. In particular, $\varphi^n(D_n) \subset D_L = (\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p])^{\Gamma_L} \subset (\mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{A}_L^+} N[1/p])^{G_L}$, where the last inclusion follows since $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) = \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})^{H_L}$ (see [MT20, Corollary 4.34]). Let $T := \mathbf{T}_L(N)$ be the associated finite free $\mathbb{Z}_p$-representation of G_L and $V := T[1/p]$, then we have

$$\begin{aligned} D_L \subset (\mathcal{O}\mathbf{B}_{\mathrm{cris}}^+(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p])^{G_L} \subset &(\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p])^{G_L} \\ &\xrightarrow{\sim} (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V)^{G_L} = \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \end{aligned} \tag{3.7}$$

where the isomorphism follows by taking G_L-fixed elements of extension along $\mathbf{A}^+[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ of the isomorphism in Lemma 3.6. Recall that $\varphi^n(D_n) \subset D_L$, or equivalently, the L-linear map $1 \otimes \varphi^n : L \otimes_{\varphi^n, L} D_n \to D_L$ is injective, we get that $L \otimes_{\varphi^n, L} D_n$ is a finite dimensional L-vector space. Moreover, φ is finite free over L, so it follows that D_n is a finite dimensional L-vector space equipped with an integrable connection. Furthermore, for $n \geq 1$ similar to the proof of [Abh21, Lemmas 4.32 & 4.36], one can show that $\log \gamma_i = \sum_{k \in \mathbb{N}} (-1)^k \frac{(\gamma_i - 1)^{k+1}}{k+1}$ converge as a series of operators on $\mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N$, where $\{\gamma_0, \gamma_1, \ldots, \gamma_d\}$ are topological generators of Γ_L (see §2.1).

Lemma 3.15. *Let $m \geq 1$ (let $m \geq 2$ if $p = 2$), then we have a Γ_L-equivariant isomorphism via the natural map $a \otimes b \otimes x \mapsto ab \otimes x$:*

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_{O_L} D_m \xrightarrow{\sim} \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]. \tag{3.8}$$

Proof. Compatibility of (3.8) with Γ_L-action is obvious from the definitions, so we only need to check that it is bijective. We will first show that (3.8) is injective. Note that we have an injective ring homomorphism $\mathcal{O}S_m^{\mathrm{PD}}[1/p] \xrightarrow{\varphi^m} \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}[1/p] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$. Since D_m is a finite dimensional L-vector space, we get that the following map is injective

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_{O_L} D_m = \mathcal{O}S_m^{\mathrm{PD}}[1/p] \otimes_L D_m \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, L} D_m. \tag{3.9}$$

Recall that $V = T[1/p]$ and consider the following composition

$$\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, L} D_m \xrightarrow{1 \otimes \varphi^m} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L D_L \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V), \tag{3.10}$$

where the first map is injective because $1 \otimes \varphi^m : L \otimes_{\varphi^m, L} D_m \to D_L$ is injective and injectivity of the second map follows from (3.7), in particular, (3.10) is injective. Furthermore, similiar to (3.9), note that $N[1/p]$ is a finite free $\mathbf{B}_L^+$-module, so it follows that the map $\mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p] = \mathcal{O}S_m^{\mathrm{PD}}[1/p] \otimes_{\mathbf{B}_L^+} N[1/p] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, \mathbf{B}_L^+} N[1/p]$ is injective. Also, recall that we have an isomorphism $1 \otimes \varphi : \mathbf{B}_L^+ \otimes_{\varphi, \mathbf{B}_L^+} N[1/p, 1/[p]_q] \xrightarrow{\sim} N[1/p, 1/[p]_q]$. So $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, \mathbf{B}_L^+} N[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p]$, since $[p]_q$ is invertible in $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$. Combining the preceding two observations, we get that the following composition is injective:

$$\mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p] \longrightarrow \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, \mathbf{B}_L^+} N[1/p] \xrightarrow[\sim]{1 \otimes \varphi^m} \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p]. \tag{3.11}$$

Now consider the following diagram

$$\begin{tikzcd} \mathcal{O}S_m^{\mathrm{PD}} \otimes_{O_L} D_m \arrow[r, "(3.9)"] \arrow[d, "(3.8)"'] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\varphi^m, L} D_m \arrow[r, "(3.10)"] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_R \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[d] \\ \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p] \arrow[r, "(3.11)"] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}_L^+} N[1/p] \arrow[r] & \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V, \end{tikzcd}$$

where the right vertical arrow is the natural injective map (see [Bri06, Proposition 3.22]) and the bottom right horizontal map is extension of the isomorohism in Lemma 3.6 along $\mathbf{A}^+[1/\mu] \to \mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$. The diagram commutes by definition and it follows that the left vertical arrow, i.e. (3.8) is injective.

Now let us check the surjectivity of the map (3.8). Define the following operators on $\mathcal{O}N_m^{\mathrm{PD}} := \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]$:

$$\partial_i := \begin{cases} -(\log \gamma_0)/t & \text{for } i = 0, \\ (\log \gamma_i)/(tV_i) & \text{for } 1 \leq i \leq d, \end{cases}$$

where $V_i = \frac{X_i \otimes 1}{1 \otimes [X_i^\flat]}$ for $1 \leq i \leq d$ (see [Abh21, §4.4.2]). Using the fact that for $g \in \Gamma_L$ and $x \in \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N$ we have $(g-1)(ax) = (g-1)a \cdot x + g(a)(g-1)x$ and from the equality $\log(\gamma_i) = \lim_{n \to +\infty}(\gamma_i^{p^n} - 1)/p^n$, it is easy to see that ∂_i satisfy the Leibniz rule for all $0 \leq i \leq d$. In particular, the operator $\partial : \mathcal{O}N_m^{\mathrm{PD}} \to \mathcal{O}N_m^{\mathrm{PD}} \otimes_{\mathcal{O}S_m^{\mathrm{PD}}} \Omega^1_{\mathcal{O}S_m^{\mathrm{PD}}/O_L}$ given as $x \mapsto \partial_0(x)dt + \sum_{i=1}^d \partial_i(x)d[X_i^\flat]$ defines a connection on $\mathcal{O}N_m^{\mathrm{PD}}$. The connection ∂ is integrable since the operators ∂_i commute with each other (see [Abh21, Lemma 4.38]) and using the finite $[p]_q$-height property of N it is easy to show that ∂ is p-adically quasi-nilpotent as well (see [Abh21, Lemma 4.39]).

For $x \in N[1/p]$, similar to the proof of [Abh21, Lemma 4.39 & Lemma 4.41], it follows that the following sum converges in $D_m = (\mathcal{O}N_m^{\mathrm{PD}})^{\Gamma_L} = (\mathcal{O}N_m^{\mathrm{PD}})^{\partial=0}$:

$$y = \sum_{\mathbf{k} \in \mathbb{N}^{d+1}} \partial_0^{k_0} \circ \partial_1^{k_1} \circ \cdots \circ \partial_d^{k_d}(x) \tfrac{t^{[k_0]}}{p^{mk_0}} (1 - V_1)^{[k_1]} \cdots (1 - V_d)^{[k_d]}. \tag{3.12}$$

By choosing a basis of N and using the formula in (3.12) on basis elements, we can define a linear transformation α on the finite free $\mathcal{O}S_m^{\mathrm{PD}}[1/p]$-module $\mathcal{O}N_m^{\mathrm{PD}}$. Now similar to the proof of [Abh21, Lemma 4.43] it can easily be deduced that for some large enough $N \in \mathbb{N}$ we can write $p^N \det \alpha \in 1 + J^{[1]}\mathcal{O}S_m^{\mathrm{PD}}$, i.e. $\det \alpha$ is a unit in $\mathcal{O}S_m^{\mathrm{PD}}[1/p]$ and α defines an automorphism of $\mathcal{O}N_m^{\mathrm{PD}}$. Finally, as the formula in (3.12) converges in D_m, it follows that the map $\mathcal{O}S_m^{\mathrm{PD}} \otimes_{O_L} D_m \to \mathcal{O}S_m^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p]$ is surjective. Hence, (3.8) is bijective, proving the lemma. ■

Note that D_L is an L-vector space equipped with tensor product Frobenius, a filtration given as $\mathrm{Fil}^k D_L := \big(\sum_{i+j=k} \mathrm{Fil}^i \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} \mathrm{Fil}^j N[1/p]\big)^{\Gamma_L}$, where $N[1/p]$ is equipped with the Nygaard filtration of Definition 3.2. Moreover, D_L is equipped with an integrable connection induced from the connection on $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ satisfying Griffiths transversality with respect to the filtration since the the same is true for the connection on $\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$. Now consider the following diagram:

$$\begin{tikzcd} \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L,\varphi^m} D_m \arrow[r, "1\otimes\varphi^m"] \arrow[d, "(3.8)"', "\wr"] & \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L} D_L \arrow[d, "(3.6)"] \\ \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+,\varphi^m} N[1/p] \arrow[r, "\sim"] & \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} N[1/p], \end{tikzcd} \tag{3.13}$$

where the left vertical arrow is extension of the isomorphism (3.8) in Lemma 3.15 along $\varphi^m : \mathcal{O}S_m^{\mathrm{PD}} \to \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}$ and the bottom horizontal isomorphism follows from an argument similar to [Abh21, Lemma 4.46]. By description of the arrows it follows that the diagram is commutative and (φ, Γ_L)-equivariant. Taking Γ_L-invariants for the composition of left vertical and bottom horizontal isomorphisms gives an L-linear isomorphism $O_L \otimes_{O_L,\varphi^m} D_m \xrightarrow{\sim} D_L$. So it follows that the top horizontal arrow in the diagram (3.13) is bijective as well. The preceding observation together with the bijectivity of left vertical and bottom horizontal arrows imply that the right vertical arrow is bijective as well, in particular, the comparison in (3.6) is an isomorphism compatible with Frobenius, connection and Γ_L-action. Compatibilty of (3.6) with filtrations follows from an argument similar to [Abh21, Corollary 4.54] (using the filtration compatible isomorphism (3.5) in Remark 3.5). This concludes our proof. ■

There exists another relation between the Wach module N and $\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Let us equip N with a Nygaard filtration as in Definition 3.2. Then we note that $(N/\mu N)[1/p]$ is a φ-module over L since

$[p]_q = p \bmod \mu N$ and $N/\mu N$ is equipped with a filtration $\mathrm{Fil}^k(N/\mu N)$ given as the image of $\mathrm{Fil}^k N$ under the surjection $N \twoheadrightarrow N/\mu N$. We equip $(N/\mu N)[1/p]$ with induced filtration, in particular, it is a filtered φ-module over L.

Corollary 3.16. *Let N be a Wach module over $\mathbf{A}_L^+$ and $V := \mathbf{T}_L(N)[1/p]$ the associated crystalline representation from Theorem 3.12. Then we have $(N/\mu N)[1/p] \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ as filtered φ-modules over L.*

Proof. For $r \in \mathbb{N}$ large enough, the Wach module $\mu^r N(-r)$ is always effective and we have that $\mathbf{T}_L(\mu^r N(-r)) = \mathbf{T}_L(N)(-r)$ (the twist $(-r)$ denotes a Tate twist on which Γ_L acts via χ^{-r} where χ is the p-adic cyclotomic character). Therefore, it is enough to show the claim for effective Wach modules. So assume N is effective and set $M := N[1/p]$ equipped with induced Frobenius, Γ_L-action and Nygaard filtration. Note that the L-vector space M/μ is equipped with a Frobenius-semilinear operator φ induced from M such that $1 \otimes \varphi : \varphi^*(M/\mu) \xrightarrow{\sim} M/\mu$ since $[p]_q = p \mod \mu$. The filtration $\mathrm{Fil}^k(M/\mu)$ is the image of $\mathrm{Fil}^k M$ under the surjective map $M \twoheadrightarrow M/\mu$. From the discussion before Theorem 3.12 recall that we have a period ring $\mathcal{O}\mathbf{A}_{R,\varpi}^{\mathrm{PD}} \subset \mathcal{O}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ equipped with a Frobenius, filtration, connection and Γ_L-action. Moreover, from Theorem 3.12 we have $D_L = (\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(V))^{\Gamma_L}$ equipped with a Frobenius, filtration and connection such that $D_L \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ compatible with supplementary structures (see (3.5)). Consider the following diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mu M \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M/\mu \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}) \otimes_{\mathbf{A}_L^+} M \arrow[r] & \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} M \arrow[r] & L(\zeta_p) \otimes_L M/\mu \arrow[r] & 0 \\
0 \arrow[r] & (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}}) \otimes_{O_L} D_L \arrow[r] \arrow[u, "\wr"] & \mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L} D_L \arrow[r] \arrow[u, "\wr", "(3.6)"'] & L(\zeta_p) \otimes_L D_L \arrow[r] \arrow[u, "\wr"] & 0.
\end{tikzcd}$$

Note that $(\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} M) \cap M = (\mathrm{Fil}^1\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \cap \mathbf{A}_L^+) \otimes_{\mathbf{A}_L^+} M = \mu M$, so the vertical maps from first to second row are natural inclusions and the second row is exact. Moreover, the middle vertical arrow from second to third row is the isomorphism (3.6) in Proposition 3.14, from which it can easily be shown that the left vertical arrow is also an isomorphism and therefore the right vertical arrow is also an isomorphism. Taking $\mathrm{Gal}(L(\zeta_p)/L)$-invariants of the right arrow gives $M/\mu \xleftarrow{\sim} D_L \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ where the last isomorphism is compatible with Frobenius, filtration and connection as mentioned in the proof of Theorem 3.12 (see (3.5)).

Note that the isomorphism $D_L \xrightarrow{\sim} M/\mu$ is compatible with Frobenius and we need to check the compatibility between respective filtrations. In the diagram above, the middle term of the second row is equipped with tensor product filtration so the image of $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{\mathbf{A}_L^+} M)$ under the surjective map from second to third term is given as $L(\zeta_p) \otimes_L \mathrm{Fil}^k(M/\mu)$. Similarly, the middle term of the third row is equipped with tensor product filtration so the image of $\mathrm{Fil}^k(\mathcal{O}\mathbf{A}_{L,\varpi}^{\mathrm{PD}} \otimes_{O_L} D_L)$ under the surjective map from second to third term is given as $L(\zeta_p) \otimes_L \mathrm{Fil}^k D_L$. Since the isomorphism (3.6) in Proposition 3.14 is compatible with filtrations, we get $L(\zeta_p) \otimes_L \mathrm{Fil}^k D_L \xrightarrow{\sim} L(\zeta_p) \otimes_L \mathrm{Fil}^k(M/\mu)$. Taking $\mathrm{Gal}(L(\zeta_p)/L)$-invariants in the preceding isomorphism gives $\mathrm{Fil}^k D_L \xrightarrow{\sim} \mathrm{Fil}^k(M/\mu)$. This concludes our proof. ∎

4. Crystalline implies finite height

The goal of this section is to prove the following claim:

Theorem 4.1. *Let T be a finite free $\mathbb{Z}_p$-representation of G_L such that $V := T[1/p]$ is a p-adic crystalline representation of G_L. Then there exists a unique Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}_L^+$ satisfying Definition 3.7. In other words, T is of finite $[p]_q$-height.*

Before carrying out the proof of Theorem 4.1, we note the following corollaries: let $\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L)$ denote the category of $\mathbb{Z}_p$-lattices inside p-adic crystalline representations of G_L. Then combining Theorem 3.12 and Theorem 4.1 and [Abh21, Proposition 4.14] (for compatibility with tensor products), we obtain the following:

Corollary 4.2. *The Wach module functor induces an equivalence of $\otimes$-catgeories*

$$\mathrm{Rep}_{\mathbb{Z}_p}^{\mathrm{cris}}(G_L) \xrightarrow{\sim} (\varphi,\Gamma)\text{-}\mathrm{Mod}_{\mathbf{A}_L^+}^{[p]_q}$$
$$T \longmapsto \mathbf{N}_L(T),$$

with a quasi-inverse $\otimes$-functor given as $N \mapsto \mathbf{T}_L(N) := (W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} N)^{\varphi=1}$.

Passing to associated isogeny categories, we obtain the following:

Corollary 4.3. *The Wach module functor induces an exact equivalence of $\otimes$-categories $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_L) \xrightarrow{\sim} (\varphi,\Gamma)\text{-}\mathrm{Mod}_{\mathbf{B}_L^+}^{[p]_q}$ via $V \mapsto \mathbf{N}_L(V)$, with an exact quasi-inverse $\otimes$-functor given as $M \mapsto \mathbf{V}_L(M) := (W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L^+} M)^{\varphi=1}$.*

In the rest of this section we will carry out the proof of Theorem 4.1 and Corollary 4.2 by constructing $\mathbf{N}_L(T)$ and show Corollary 4.3 as a consquence. In §4.1 we collect important properties of classical Wach modules, i.e. the perfect residue field case. In §4.2 we use ideas from [Kis06; KR09] to show that classical Wach modules are compatible with Kisin-Ren modules, and we further show that in our setting, a finite $[p]_q$-height module on the open unit disk over $\breve{L}$ descends to a finite $[p]_q$-height module on the open unit disk over L, similar to [BT08]. On the module thus obtained, we use results of §2.3 to construct an action of Γ_L and study its properties in §4.3. Then in §4.4 we check that our construction is compatible with the theory of étale (φ,Γ_L)-modules. Finally, in §4.5 we construct the promised Wach module $\mathbf{N}_L(T)$ and prove Theorem 4.1 and Corollary 4.3.

For a p-adic representation of G_L, note that the property of being crystalline and of finite $[p]_q$-height is invariant under twisting the representation by χ^r for $r \in \mathbb{N}$. So from now onwards we will assume that V is a p-adic positive crystalline representation of G_L, i.e. all its Hodge-Tate weights are ≤ 0. We have $T \subset V$ a G_L-stable $\mathbb{Z}_p$-lattice.

4.1. Classical Wach modules. Recall that $G_{\breve{L}}$ is a subgroup of G_L, so from [BT08, Proposition 4.14] it follows that V is a p-adic positive crystalline representation of $G_{\breve{L}}$ and $T \subset V$ a $G_{\breve{L}}$-stable $\mathbb{Z}_p$-lattice. Note that $\breve{L}$ is an unramified extension of $\mathbb{Q}_p$ with perfect residue field, therefore the $G_{\breve{L}}$-representation V is of finite $[p]_q$-height (see [Col99] and [Ber04]). Let the $[p]_q$-height of V be $s \in \mathbb{N}$. One associates to V a finite free $(\varphi,\Gamma_{\breve{L}})$-module over $\mathbf{B}_{\breve{L}}^+$ of rank $= \dim_{\mathbb{Q}_p} V$ called the Wach module $\mathbf{N}_{\breve{L}}(V)$ and to T a finite free $(\varphi,\Gamma_{\breve{L}})$-module over $\mathbf{A}_{\breve{L}}^+$ of rank $= \dim_{\mathbb{Q}_p} V$ called the Wach module $\mathbf{N}_{\breve{L}}(T)$ (see [Wac96; Wac97; Ber04] and [Abh21, §4.1] for a recollection). Let $\tilde{\mathbf{D}}_L^+(T) := (\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T)^{H_L}$ be the (φ,Γ_L)-module over $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})^{H_L}$ (see [And06, Proposition 7.2]) associated to T and let $\tilde{\mathbf{D}}_L^+(V) := \tilde{\mathbf{D}}_L^+(T)[1/p]$ over $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) = \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})^{H_L}$ associated to V.

Lemma 4.4 ([Ber04])**.** (1) *$\mathbf{N}_{\breve{L}}(T) = \mathbf{N}_{\breve{L}}(V) \cap \mathbf{D}_{\breve{L}}(T) \subset \mathbf{D}_{\breve{L}}(V)$.*

(2) *$\mu^s \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T \subset \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbf{A}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(T) \subset \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T$ and taking H_L-invariants gives $\mu^s \tilde{\mathbf{D}}_L^+(T) \subset \mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{A}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(T) \subset \tilde{\mathbf{D}}_L^+(T)$. Similar claims are also true for V.*

By properties of Wach modules, we have functorial isomorphisms of étale (φ,Γ_L)-modules, where the second isomorphism in first row follows from [Ber04, Théorème III.3.1]

$$\begin{aligned} \mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(T) &\xrightarrow{\sim} \mathbf{D}_{\breve{L}}(T) \text{ and } \mathbf{A}_{\breve{L}}^\dagger \otimes_{\mathbf{A}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}^\dagger(T), \\ \mathbf{B}_{\breve{L}} \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V) &\xrightarrow{\sim} \mathbf{D}_{\breve{L}}(V) \text{ and } \mathbf{B}_{\breve{L}}^\dagger \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}^\dagger(V), \\ \mathbf{B}_{\mathrm{rig},\breve{L}}^\dagger \otimes_{\mathbf{B}_{\breve{L}}^+} \mathbf{N}_{\breve{L}}(V) &\xrightarrow{\sim} \mathbf{D}_{\mathrm{rig},\breve{L}}^\dagger(V). \end{aligned} \tag{4.1}$$

Let $\mathbf{N}_{\mathrm{rig},\breve{L}}(V) := \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ equipped with (diagonally) induced Frobenius-semilinear operator φ and $\Gamma_{\breve{L}}$-action. From [Ber04, Proposition II.2.1], recall that we have a natural inclusion $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ which extends $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-linearly to a Frobenius and $\Gamma_{\breve{L}}$-equivariant inclusion $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ such that its cokernel is killed by $(t/\mu)^s \in \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ (see [Ber04, Propositions II.3.1 & III.2.1]). In particular, we obtain a $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V). \tag{4.2}$$

Moreover, note that from loc. cit. we have a natural isomorphism of filtered φ-modules $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)/\mu\mathbf{N}_{\mathrm{rig},\breve{L}}(V) = \mathbf{N}_{\breve{L}}(V)/\mu\mathbf{N}_{\breve{L}}(V)$ such that the largest Hodge-Tate weight of V equals s, i.e. the $[p]_q$-height of V. Since t/μ is a unit in $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ and $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \subset \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$, extending scalars in (4.2) gives a $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism

$$\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V). \tag{4.3}$$

Lemma 4.5. *The following diagram is commutative and $(\varphi, G_{\breve{L}})$-equivariant:*

$$\begin{tikzcd}
\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \arrow[r, "\sim"] \arrow[d, "\wr"] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \arrow[d, "\wr"] \\
\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \arrow[r, equal] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V.
\end{tikzcd}$$

Proof. The left vertical arrow is an isomorphism since V is a crystalline representation of $G_{\breve{L}}$. The top vertical arrow is scalar extension of (4.3) along $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \to \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$. Next, from Lemma 4.4 (2) we have a $(\varphi, G_{\breve{L}})$-equivariant isomorphism $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbb{Z}_p} T$ and extending this isomorphism along $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \to \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ gives the isomorphism in right vertical arrow. The commutativity of the diagram follows since the top horizontal arrow is also the $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear extension of the natural inclusion $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ (see [Ber04, §II.2]). ∎

4.2. Kisin's construction. Our goal is to construct a Wach module $\mathbf{N}_L(T)$ over $\mathbf{A}^+_L$. To this end, we will adapt some ideas from [BT08] and [KR09] generalizing the results of Kisin in [Kis06] to first construct a finite $[p]_q$-height module over $\mathbf{B}^+_{\mathrm{rig},L}$.

Let $E(X) := \frac{(1+X)^p-1}{X} \in \mathbb{Z}_p[\![X]\!]$ denote the cyclotomic polynomial. We equip $\mathbb{Z}_p[\![X]\!]$ with the cyclotomic Frobenius operator φ given by identity on $\mathbb{Z}_p$ and setting $\varphi(X) = (1+X)^p - 1$ and for $n \in \mathbb{N}$ we set $E_n(X) := \varphi^n(E(X))$. In particular, $\zeta_{p^{n+1}} - 1$ is a simple zero of $E_n(X)$. For $X = \mu$, we will write $E_n(X) = \tilde{\xi}_n$ for $n \in \mathbb{N}$ and $\varphi(\mu)/\mu = \tilde{\xi} = \tilde{\xi}_0 = E(\mu) = [p]_q$.

Remark 4.6. Define $\phi_L : \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ as the map given by Frobenius on L and $\phi_L(\mu) = \mu$, i.e. $\sum_{k\in\mathbb{N}} \iota(a_k)\mu^k \mapsto \sum_{k\in\mathbb{N}} \iota(\phi_L(a_k))\mu^k$. Then $\mathbf{B}^+_{\mathrm{rig},L}$ is finite free of rank p^d over $\mathbf{B}^+_{\mathrm{rig},L}$ via the map ϕ_L, in particular, flat. Similarly, let $\phi_{\breve{L}} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ denote the map given by Frobenius on $\breve{L}$ and $\phi_{\breve{L}}(\mu) = \mu$. From §2.1.4 recall that we have an injection $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ which is evidently compatible with ϕ_L on left and $\phi_{\breve{L}}$ on right.

Remark 4.7. We have $t/\mu \in \mathbf{B}^+_{\mathrm{rig},L}$ and we can write $t/\mu = \prod_{n\in\mathbb{N}}(\tilde{\xi}_n/p)$ (see [Ber04, Exemple I.3.3] and [Laz62, Remarque 4.12]). The zeros of t/μ are $\zeta_{p^{n+1}} - 1$ for all $n \in \mathbb{N}$. Moreover, we have $\phi_L^{-n}(t/\mu) = t/\mu$, therefore the zeros of $\phi_L^{-n}(t/\mu)$ are $\zeta_{p^{n+1}} - 1$ as well.

Now let $\widehat{\mathbf{B}}_{\breve{L},n}$ denote the completion of $\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\breve{L}}$ with respect to the maximal ideal generated by $\mu - (\zeta_{p^{n+1}} - 1)$. Moreover, since $\zeta_{p^{n+1}} - 1$ is a simple root of $\tilde{\xi}_n$ we obtain that $(\mu - (\zeta_{p^{n+1}} - 1)) = (\tilde{\xi}_n) \subset \widehat{\mathbf{B}}_{\breve{L},n}$. The local ring $\widehat{\mathbf{B}}_{\breve{L},n}$ naturally admits an action of $\Gamma_{\breve{L}}$ obtained by the diagonal action of $\Gamma_{\breve{L}}$ on the tensor product $\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\breve{L}}$. We put a filtration on $\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n]$ as $\mathrm{Fil}^r \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] = \tilde{\xi}_n^r \widehat{\mathbf{B}}_{\breve{L},n}$ for $r \in \mathbb{Z}$. We have inclusions $\mathbf{B}^+_{\breve{L}} \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}} \subset \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n]$.

Let $D_L := \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and $D_{\breve{L}} := \mathbf{D}_{\mathrm{cris},\breve{L}}(V)$ then using the φ-equivariant injection $L \to \breve{L}$, we have an isomorphism of filtered φ-modules $\breve{L} \otimes_L D_L \xrightarrow{\sim} D_{\breve{L}}$ from (2.5). Note that D_L (resp. $D_{\breve{L}}$) is an effective filtered φ-module over L (resp. over $\breve{L}$), i.e. $\mathrm{Fil}^0 D_L = D_L$ (resp. $\mathrm{Fil}^0 D_{\breve{L}} = D_{\breve{L}}$) and we have a φ-equivariant inclusion $D_L \subset D_{\breve{L}}$. Consider a map

$$i_n : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \xrightarrow{\phi_{\breve{L}}^{-n} \otimes \varphi_{D_{\breve{L}}}^{-n}} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}},$$

where $\phi_{\breve{L}} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is defined in Remark 4.6 and $\varphi_{D_{\breve{L}}}$ is the Frobenius-semilinear operator on $D_{\breve{L}}$. Since the residue field of $\breve{L}$ is perfect, the map i_n is well-defined and it extends to a map

$$i_n : \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}} \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}.$$

Define a $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-module

$$\mathcal{M}_{\breve{L}}(D_{\breve{L}}) := \{x \in \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}, \text{ such that } \forall n \in \mathbb{N}, i_n(x) \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}})\},$$

where $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}$ is equipped with tensor product Frobenius and $\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}}$ is equipped with tensor product filtration. By [Kis06, Lemma 1.2.2] and [KR09, Lemma 2.2.1], the $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-module $\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is finite free of rank $= \dim_{\breve{L}} D_{\breve{L}}$ stable under φ and $\Gamma_{\breve{L}}$ such that cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) \to \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is killed by $\tilde{\xi}^s$ and the action of $\Gamma_{\breve{L}}$ is trivial modulo μ. Moreover, from [KR09, Lemma 2.2.2] there exists a unique $\breve{L}$-linear section $\alpha : \mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \to \mathcal{M}_{\breve{L}}(D_{\breve{L}})[\mu/t]$ such that the image $\alpha(\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}}))$ is $\Gamma_{\breve{L}}$-invariant. Furthermore, the section α is φ-equivariant and it induces an isomorphism

$$1 \otimes \alpha : \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} (\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})[\mu/t]. \tag{4.4}$$

Finally, from [KR09, Proposition 2.2.6] we have a natural isomorphism $D_{\breve{L}} \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ compatible with Frobenius and filtration and under the isomorphism above the image of $D_{\breve{L}}$ coincides with $\alpha(\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu\mathcal{M}_{\breve{L}}(D_{\breve{L}}))$.

Next, we note that the $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}}$-module $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is pure of slope 0 using [Kis06, Theorem 1.3.8] and [KR09, Proposition 2.3.3]. Then from [KR09, Corollay 2.4.2] one obtains an $\mathbf{A}^+_{\breve{L}}$-module $N_{\breve{L}}$ finite free of rank $= \dim_{\breve{L}} D_{\breve{L}}$ equipped with a Frobenius-semilinear endomorphism φ and semilinear and continuous action of $\Gamma_{\breve{L}}$ such that cokernel of the injective map $1 \otimes \varphi : \varphi^*(N_{\breve{L}}) \to N_{\breve{L}}$ is killed by $\tilde{\xi}^s$, the action of $\Gamma_{\breve{L}}$ is trivial modulo μ and $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{A}^+_{\breve{L}}} N_{\breve{L}} \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ compatible with $(\varphi, \Gamma_{\breve{L}})$-action.

Lemma 4.8. *There is a natural* $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$*-linear and* $(\varphi, \Gamma_{\breve{L}})$*-equivariant isomorphism* $\beta : \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$*. Moreover, it restricts to a* $\mathbf{B}^+_{\breve{L}}$*-linear and* $(\varphi, \Gamma_{\breve{L}})$*-equivariant isomorphism* $\beta : N_{\breve{L}}[1/p] \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(V)$*.*

Proof. Recall that by definition $\mathbf{N}_{\mathrm{rig},\breve{L}}(V) = \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$, and consider the following diagram:

$$\begin{tikzcd} \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}} \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathbf{N}_{\mathrm{rig},\breve{L}}(V)[\mu/t] \\ \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} (\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu) \arrow[r, "\sim", "1\otimes\alpha"'] & \mathcal{M}_{\breve{L}}(D_{\breve{L}})[\mu/t], \arrow[u, "\beta", "\wr"'] \end{tikzcd} \tag{4.5}$$

where the top horizontal arrow is (4.2), the bottom horizontal arrow is (4.4) and the left vertical arrow follows from $D_{\breve{L}} \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu$. For the right vertical arrow β, we consider $\mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ and

$\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ as submodules of $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)$ and construct the map as follows: from [KR09, Lemma 2.1.2] the action of $\Gamma_{\breve{L}}$ on $\mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ is "$\mathbb{Z}_p$-analytic" in the sense of [KR09, §2.1.3] and we have $D_{\breve{L}} \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)/\mu$. So from the equivalence of categories in [KR09, Proposition 2.2.6] and its proof, it follows that we have an isomorphism $\beta : \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(\mathbf{N}_{\mathrm{rig},\breve{L}}(V)/\mu) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ as $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-submodules of $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}$ compatible with $(\varphi, \Gamma_{\breve{L}})$-action and whose reduction modulo μ induces isomorphisms $\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu \xrightarrow{\sim} D_{\breve{L}} \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)/\mu$. Commutativity of the diagram follows from the uniqueness of f and noting that the composition of left, top and right arrow provides a section $\mathcal{M}_{\breve{L}}(D_{\breve{L}})/\mu \to \mathcal{M}_{\breve{L}}(D_{\breve{L}})[\mu/t]$ with the same property as α. This shows the first claim. For the second claim, note that $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ is pure of slope 0, so from [KR09, Corollary 2.4.2] we conclude that the isomorphism β induces an isomorphism $\beta : N_{\breve{L}}[1/p] \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(V)$ compatible with $(\varphi, \Gamma_{\breve{L}})$-action. ■

From (2.5) we have an isomorphism of filtered φ-modules $\breve{L} \otimes_L D_L \xrightarrow{\sim} D_{\breve{L}}$.

Definition 4.9. Define

$$\begin{aligned}\mathcal{M}_L(D_L) &:= \{x \in \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L, \text{ such that } \forall n \in \mathbb{N}, i_n(x) \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}})\}\\ &= (\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L) \cap \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}.\end{aligned}$$

From §2.1.5 recall that we have a φ-equivariant injection $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$, therefore by definition $\mathcal{M}_L(D_L)$ is stable under the induced tensor product Frobenius semilinear-operator φ on $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}$. Using Lemma 4.8 and the discussion preceding (4.2) we have φ-equivariant inclusions $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}$. Moreover, from Lemma 2.9 recall that $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is flat and from Lemma 2.10 we have $\mathbf{B}^+_{\mathrm{rig},L} \cap (t/\mu)\mathbf{B}^+_{\mathrm{rig},\breve{L}} = (t/\mu)\mathbf{B}^+_{\mathrm{rig},\breve{L}}$, or equivalently, $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] = \mathbf{B}^+_{\mathrm{rig},L}$. So it follows that we have φ-equivariant inclusions

$$\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L \subset \mathcal{M}_L(D_L) \subset (\mu/t)^s \mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L. \tag{4.6}$$

Therefore, similar to (4.2), we obtain a φ-equivariant isomorphism

$$\mathcal{M}_L(D_L)[\mu/t] \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L, \tag{4.7}$$

Note that extending scalars of the isomorphism $L(\zeta_{p^n+1}) \xrightarrow{\sim} (L(\zeta_{p^{n+1}}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L})/(\mu - (\zeta_{p^{n+1}} - 1))$ along $L \to \breve{L}$, gives $\breve{L}(\zeta_{p^n+1}) \xrightarrow{\sim} (\breve{L}(\zeta_{p^{n+1}}) \otimes_{\breve{L}} \mathbf{B}^+_{\mathrm{rig},L})/(\mu - (\zeta_{p^{n+1}} - 1))$. Let $I \subset \breve{L}(\zeta_{p^{n+1}}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L}$ denote the maximal ideal generated by $\mu - (\zeta_{p^{n+1}} - 1)$, and let $(\breve{L}(\zeta_{p^{n+1}}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L})_I$ denote the localization at I. Then the natural map $(\breve{L}(\zeta_{p^{n+1}}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L})_I \to \widehat{\mathbf{B}}_{\breve{L},n}$ is obtained as completion of a discrete valuation ring and we get the following:

Lemma 4.10. *The composition of maps* $\mathbf{B}^+_{\mathrm{rig},L} \to \breve{L}(\zeta_{p^{n+1}}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L} \to (\breve{L}(\zeta_{p^{n+1}}) \otimes_L \mathbf{B}^+_{\mathrm{rig},L})_I \to \widehat{\mathbf{B}}_{\breve{L},n}$ *is flat for all* $n \in \mathbb{N}$.

Lemma 4.11. *Consider* $\widehat{\mathbf{B}}_{\breve{L},n}$ *as a* $\mathbf{B}^+_{\mathrm{rig},L}$*-algebra via the composition* $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}} \xrightarrow{\phi_{\breve{L}}^{-n}} \widehat{\mathbf{B}}_{\breve{L},n}$.

(1) *The homomorphism*

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \longrightarrow \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}} \xleftarrow{\sim} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_L D_L,$$

induced by i_n *is an isomorphism.*

(2) *The isomorphism in* (1) *induces an isomorphism*

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \xrightarrow{\sim} \sum_{i \in \mathbb{N}} \tilde{\xi}_n^{-i} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^i D_L$$

(3) *The φ-equivariant homomorphism $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \to \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ obtained by extending $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-linearly the φ-equivariant inclusion $\mathcal{M}_L(D_L) \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is an isomorphism. Moreover, $\mathcal{M}_L(D_L)$ is a finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module of rank $= \dim_L D_L$.*

Proof. The proof follows in a manner similar to [Kis06, Lemma 1.2.1]. For (1), note that using (2.5) we have $D_{\breve{L}} \xrightarrow{\sim} \breve{L} \otimes_L D_L$, so we can write

$$\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} (\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_L D_L \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}.$$

Moreover, we have $\phi_{\breve{L}}^{-n} \otimes \varphi_{D_{\breve{L}}}^{-n} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}} \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}$, so extending scalars of the isomorphism above along $\phi_{\breve{L}}^{-n} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \widehat{\mathbf{B}}_{\breve{L},n}$ gives $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{i_n, \mathbf{B}^+_{\mathrm{rig},\breve{L}}} (\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_L D_L) \xrightarrow{\sim} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\breve{L}} D_{\breve{L}}$.

To show (2), let us write for $k \in \mathbb{N}$

$$\mathcal{M}_{L,k}(D_L) := \{x \in \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L \text{ such that } i_k(x) \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},k}[1/\tilde{\xi}_k] \otimes_{\breve{L}} D_{\breve{L}})\}.$$

Then we have $\mathcal{M}_L(D_L) = \cap_{k\in\mathbb{N}} \mathcal{M}_{L,k}(D_L) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L$. By flatness of $\widehat{\mathbf{B}}_{\breve{L},n}$ over $\mathbf{B}^+_{\mathrm{rig},L}$ (see Lemma 4.10) and of $\phi_L : \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ (see Remark 4.6), we get that $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) = \cap_{k\in\mathbb{N}} (\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L)) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L$. To prove our claim, it suffices to show the following two equalities:

$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L) = \sum_{r\in\mathbb{N}} \tilde{\xi}_n^{-r} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L,$$
$$\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L) = \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_L D_L, \quad \text{for } k \neq n.$$

For the first equality note that we have $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L) \subset \sum_{r\in\mathbb{N}} \tilde{\xi}_n^{-r} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L$ by definition. For the converse, note that we have $\tilde{\xi}_n^{-1} = \frac{1}{p}\varphi^n(\mu/t)\varphi^{n+1}(t/\mu) \in \mathbf{B}_{\mathrm{rig},L}[\mu/t]$ and $\phi_L^{-n}(\tilde{\xi}_n^{-1}) = \tilde{\xi}_n^{-1}$. So for any $r \in \mathbb{N}$ and $\tilde{\xi}_n^{-r} a \otimes d \in \tilde{\xi}_n^{-r} \widehat{\mathbf{B}}_{\breve{L},n} \otimes_L \mathrm{Fil}^r D_L$, we have $\tilde{\xi}_n^{-r} \otimes \varphi^n(d) \in \mathcal{M}_{L,n}(D_L)$ since $i_n(\tilde{\xi}_n^{-r} \otimes \varphi^n(d)) = \tilde{\xi}_n^{-r} \otimes d \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_{\breve{L}} D_{\breve{L}})$. Therefore, $\tilde{\xi}_n^{-r} a \otimes d = a \otimes i_n(\tilde{\xi}_n^{-r} \otimes \varphi^n(d)) \in \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,n}(D_L)$. For the second equality again note that by definition we have $\widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L) \subset \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_L D_L$. For the converse, note that $\tilde{\xi}_k$ is a unit in $\widehat{\mathbf{B}}_{\breve{L},n}$ since $\zeta_{p^{k+1}} - 1$ is not a root of $\tilde{\xi}_n$ because $n \neq k$. So for any $j, r \in \mathbb{N}$ we take $\tilde{\xi}_n^{-r} \tilde{\xi}_k^{-j} a \otimes d \in \widehat{\mathbf{B}}_{\breve{L},n}[1/\tilde{\xi}_n] \otimes_L \mathrm{Fil}^j D_L$. Moreover, $\tilde{\xi}_n$ is a unit in $\widehat{\mathbf{B}}_{\breve{L},k}$ and $i_k(\tilde{\xi}_n^{-r} \tilde{\xi}_k^{-j} \otimes \varphi^k(d)) = \tilde{\xi}_n^{-r} \tilde{\xi}_k^{-j} \otimes d \in \mathrm{Fil}^0(\widehat{\mathbf{B}}_{\breve{L},k}[1/\tilde{\xi}_k] \otimes_{\breve{L}} D_{\breve{L}})$ so we have $\tilde{\xi}_n^{-r} \tilde{\xi}_k^{-j} \otimes \varphi^k(d) \in \mathcal{M}_{L,k}(D_L)$. Therefore, $\tilde{\xi}_n^{-r} \tilde{\xi}_k^{-j} a \otimes d = a \otimes i_k(\tilde{\xi}_n^{-r} \tilde{\xi}_k^{-j} \otimes \varphi^k(d)) \in \widehat{\mathbf{B}}_{\breve{L},n} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_{L,k}(D_L)$.

For (3), note that we have inclusions $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_L D_L \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s \mathbf{B}_{\mathrm{rig},\breve{L}} \otimes_L D_L$, where the first two inclusions follow since the map $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is flat (see Lemma 2.9) and $\mathcal{M}_L(D_L) \subset (\mu/t)^s \mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L$ from (4.6). So we get that $(t/\mu)^s$ kills the cokernel of $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \to \mathcal{M}_{\breve{L}}(D_{\breve{L}})$. Moreover, note that $\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \subset (\mu/t)^s \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}}$ is a closed submodule by [Kis06, Lemma 1.1.5, Lemma 1.2.2] and since $\mathbf{B}^+_{\mathrm{rig},L} \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is a closed subring, we get that $\mathcal{M}_L(D_L) \subset (\mu/t)^s \mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L$ is closed and hence finite free by Remark 2.13 and of rank $= \dim_L D_L$ by the isomorphism shown below.

Let us write $\mathbf{B}^+_{\mathrm{rig},L} = \lim_\rho \mathcal{O}(D(L,\rho))$ as the limit of ring of analytic functions on closed disks $D(L,\rho)$ of radius $0 < \rho < 1$ (see Remark 2.5); similarly write $\mathbf{B}^+_{\mathrm{rig},\breve{L}} = \lim_\rho \mathcal{O}(D(\breve{L},\rho))$. Since $\mathcal{M}_L(D_L)$ and $\mathcal{M}_{\breve{L}}(D_{\breve{L}})$ are free, we have $\mathcal{M}_L(D_L) \xrightarrow{\sim} \lim_\rho(\mathcal{O}(D(L,\rho)) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L))$ and $\mathcal{M}_{\breve{L}}(D_{\breve{L}}) \xrightarrow{\sim} \lim_\rho(\mathcal{O}(D(\breve{L},\rho)) \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}}))$. Then to show our claim, it is enough to show that the map

$$\mathcal{O}(D(\breve{L},\rho)) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \longrightarrow \mathcal{O}(D(\breve{L},\rho)) \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}}), \tag{4.8}$$

is a bijection. Note that $\mathcal{O}(D(\breve{L},\rho))$ is a domain, so injectivity of (4.8) can be checked after passing to the fraction field of $\mathcal{O}(D(\breve{L},\rho))$. To check that (4.8) is surjective, let Q denote the cokernel of (4.8) and we will show that $Q=0$. Note that Q is a finitely generated $S := \mathcal{O}(D(\breve{L},\rho))$-module killed by $(t/\mu)^s$ and S is a principal ideal domain (see [Bos14, Chapter 2, Corollary 10]). So by the structure theorem of finitely generated modules over S we write $Q = \oplus S/\mathfrak{a}_i$ where $\mathfrak{a}_i = (a_i)$ for some nonzero primary elements $a_i \in S$ and such that $a_i|(t/\mu)^s$ for each i. Note that $\sqrt{\mathfrak{a}_i}$ is a maximal ideal of S and $Q_{\sqrt{\mathfrak{a}_i}} = S/\mathfrak{a}_i$, so to get $Q=0$ it is enough to show that $Q_{\sqrt{\mathfrak{a}_i}}=0$. From [Bos14, Chapter 2, Corollary 13] note that each maximal ideal $\sqrt{\mathfrak{a}_i}$ corresponds to a zero of $(t/\mu)^s$, in particular, we are reduced to showing that Q vanishes at zeros of t/μ. This follows from (2). Hence, we get that (4.8) is an isomorphism and passing to the limit over ρ we obtain $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$. ∎

Lemma 4.12. *We have following properties for the $\mathbf{B}^+_{\mathrm{rig},L}$-module $\mathcal{M}_L(D_L)$:*

(1) *Cokernel of the injective map $1\otimes\varphi : \varphi^*(\mathcal{M}_L(D_L)) \to \mathcal{M}_L(D_L)$ is killed by $[p]_q^s$.*

(2) *$\mathcal{M}_L(D_L)$ is pure of slope 0, i.e. the $\mathbf{B}^\dagger_{\mathrm{rig},L}$-module $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L)$ is pure of slope 0 in the sense of [Ked04, §6.3].*

Proof. For (1), let us first note the following commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{M}_L(D_L) \arrow[r] \arrow[d] & \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L \arrow[r] \arrow[d] & Q \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{M}_{\breve{L}}(D_{\breve{L}}) \arrow[r] & \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}} \arrow[r] & \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} Q \arrow[r] & 0.
\end{tikzcd}$$

All maps are φ-equivariant and vertical maps are injective (see (4.6), Lemma 4.8, Defnition 4.9 and Lemma 4.11 (3)). From Remark 2.7, Remark 2.8, Lemma 2.10 and Lemma 2.9 recall that the maps $\varphi_L : \mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ and $\varphi_{\breve{L}} : \mathbf{B}^+_{\mathrm{rig},\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ are faithfully flat (we write φ with subscripts to avoid confusion), $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is flat and $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] = \mathbf{B}^+_{\mathrm{rig},L}$. Using Lemma 4.11 (3) and $D_{\breve{L}} \xrightarrow{\sim} \breve{L} \otimes_L D_L$ from (2.5), we get $\varphi^*_{\breve{L}}(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi^*_L(\mathcal{M}_L(D_L))$ and $\varphi^*_L(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi^*_L(\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]} \varphi^*_L(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L) \xrightarrow{\sim} \varphi^*_{\breve{L}}(\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}})$. So from preceding discussion and exactness of both rows in the diagram above, it follows that

$$\begin{aligned}
\varphi^*_L(\mathcal{M}_L(D_L)) &= (\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi^*_L(\mathcal{M}_L(D_L))) \cap (\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \varphi^*_L(\mathcal{M}_L(D_L))) \\
&\xrightarrow{\sim} \varphi^*_L(\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L D_L) \cap \varphi^*_{\breve{L}}(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) \subset \varphi^*_{\breve{L}}(\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} D_{\breve{L}}).
\end{aligned}$$

Now let $x \in \mathcal{M}_L(D_L) \subset \mathcal{M}_{\breve{L}}(D_{\breve{L}})$, then there exists $y \in \varphi^*(\mathcal{M}_{\breve{L}}(D_{\breve{L}}))$ such that $(1\otimes\varphi)y = \tilde{\xi}^s x$. Recall that $1\otimes\varphi : \varphi^*(D_L) \xrightarrow{\sim} D_L$ and $\varphi(\mu/t) = (\tilde{\xi}\mu)/(pt)$, therefore the cokernel of $1\otimes\varphi : \varphi^*((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \to (\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L$ is killed by $\tilde{\xi}^s$, in particular, $\tilde{\xi}^s x \in (1\otimes\varphi)\varphi^*((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L)$. Since $1\otimes\varphi$ is injective on $\varphi^*((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\breve{L}} D_{\breve{L}})$, therefore we get that $y \in \varphi^*((\mu/t)^s\mathbf{B}^+_{\mathrm{rig},L} \otimes_L D_L) \cap \varphi^*(\mathcal{M}_{\breve{L}}(D_{\breve{L}})) = \varphi^*(\mathcal{M}_L(D_L))$. In particular, the cokernel of $1\otimes\varphi : \varphi^*(\mathcal{M}_L(D_L)) \to \mathcal{M}_L(D_L)$ is killed by $\tilde{\xi}^s$.

For (2), note that from Lemma 4.11 (3) $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L) \xrightarrow{\sim} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$. Moreover, from [Ked04, Theorem 6.10] we obtain a slope filtration on $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathcal{M}_L(D_L)$ such that base changing this slope filtration along $\mathbf{B}^\dagger_{\mathrm{rig},L} \to \mathbf{B}^\dagger_{\mathrm{rig},\breve{L}}$ gives a slope filtration on $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$. However, from [Kis06, Theorem 1.3.8] and [KR09, Proposition 2.3.3] $\mathbf{B}^\dagger_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathcal{M}_{\breve{L}}(D_{\breve{L}})$ is pure of slope 0. Therefore, we must have that $\mathcal{M}_L(D_L)$ is pure of slope 0. ∎

4.3. Stability under Galois action. In this subsection we will define and study a finite free slope 0 (φ, Γ_L)-module $\mathbf{N}_{\mathrm{rig},L}(V)$ over $\mathbf{B}^+_{\mathrm{rig},L}$ from the $\mathbf{B}^+_{\mathrm{rig},L}$-module in Definition 4.9. From §2.1.4 recall that we have identifications $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} = (\tilde{\mathbf{B}}^+_{\mathrm{rig}})^{H_L} = \cap_{n\in\mathbb{N}}\varphi^n(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}))$ where the last equality follows since $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) = \mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})^{H_L}$ (see §2.1.1). Moreover, using the isomorphism in Lemma 2.18 and Remark 2.20 we see that $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is equipped with an action of Γ_L. We have $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ and we make the following claim:

Lemma 4.13. *The $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}$-module $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of Γ_L. For $a \otimes x \in \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ this action can be explicitly described by the formula*

$$g(a \otimes x) = g(a) \otimes \sum_{\mathbf{k}\in\mathbb{N}^d} \prod_{i=1}^{d} \partial_i^{k_i}(x) \prod_{i=1}^{d} (g([X_i^\flat]) - [X_i^\flat])^{[k_i]}, \ \ \textit{for } g \in \Gamma_L.$$

Proof. The non-canonical $(\varphi, G_{\breve{L}})$-equivariant L-algebra structure on $\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ from §2.1.1 extends to a $(\varphi, G_{\breve{L}})$-equivariant $\breve{L}$-algebra structure and it provides $(\varphi, G_{\breve{L}})$-equivariant L-algebra and $\breve{L}$-algebra structures on $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$ via the composition $L \to \breve{L} \to \mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \twoheadrightarrow \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})$, where the last map is the projection map described before Lemma 2.18. Moreover, recall that we have $L \otimes_{\varphi^n, L} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ for all $n \in \mathbb{N}$. So we can write

$$\begin{aligned}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) &\xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)\\ &\xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\varphi_{\breve{L}}^{-n},\breve{L}} (\breve{L} \otimes_{\varphi_{\breve{L}}^{n},\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)).\end{aligned}$$

Applying φ^n to the isomorphism above and simplifying gives $\varphi^n(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \xrightarrow{\sim} \varphi^n(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Note that the Frobenius endomorphism φ on $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris}}(V)$ commutes with the action of Γ_L. Therefore, the following is stable under Γ_L-action:

$$\begin{aligned}\cap_{n\in\mathbb{N}}\varphi^n(\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) &= (\cap_{n\in\mathbb{N}}\varphi^n(\mathcal{O}\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}))) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)\\ &= \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V).\end{aligned}$$

The second claim follows from Lemma 2.21. ∎

Extending the isomorphism in (4.2) along the map $\mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \to \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t]$ (see §2.1.5), we obtain an isomorphism $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$. Recall that for $g \in \Gamma_L$ we have $g(t) = \chi(g)t$ and $g(\mu) = (1+\mu)^{\chi(g)} - 1$, where χ is the p-adic cyclotomic character. Now using $\mathbf{D}_{\mathrm{cris},\breve{L}}(V) = \breve{L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ we get isomorphisms $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ and we equip the last term with a Γ_L-action by transport of structure via this isomorphism. In particular, the preceding discussion induces an action of Γ_L over $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V) = \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$. Our first objective is to show that $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V) \subset \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ is stable under the action of Γ_L. We will do this by embedding everything into $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$.

Let us fix some elements in $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$. For $n \in \mathbb{N}$, let $n = (p-1)f(n) + r(n)$ with $r(n), f(n) \in \mathbb{N}$ and $0 \leq r(n) < p-1$. Set $t^{\{n\}} := \frac{t^n}{f(n)!p^{f(n)}}$ and $\Lambda := \{\sum_{n\in\mathbb{N}} a_n t^{\{n\}}$ with $a_n \in O_F$ such that $a_n \to 0$ as $n \to +\infty\} = O_F[t, (t^{p-1}/p)^{[k]}, k \in \mathbb{N}]^\wedge$ where $^\wedge$ denotes the p-adic completion. Then we have an isomorphism of rings

$$O_F[\mu, (\mu^{p-1}/p)^{[k]}, k \in \mathbb{N}]^\wedge \xrightarrow{\sim} \Lambda,$$

via the map $\mu \mapsto \exp(t) - 1$ with the inverse map given as $t \mapsto \log(1+\mu)$ (see [Bri08, Lemme 6.2.13]). Furthermore, for $r \in \mathbb{N}$ and $A := \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$, $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})$, $\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ or $\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$ set

$$I^{(r)}A := \{a \in A \text{ such that } \varphi^n(a) \in \mathrm{Fil}^r A \text{ for all } n \in \mathbb{N}\}. \tag{4.9}$$

Lemma 4.14. *We note the following facts:*

(1) $t^{p-1} \in p\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$, $t^{\{n\}} \in \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ *and* t/μ *is a unit in* $\Lambda \subset \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$.

(2) *For* $r \in \mathbb{N}$ *we have* $I^{(r)}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = \mu^r \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$ *and* $I^{(p-1)}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty}) = \mu^{p-1}\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$.

(3) *Let* $S = O_F[\![\mu]\!]$, *then the natural map* $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})\widehat{\otimes}_S \Lambda \rightarrow \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ *defined via* $\sum_{k\in\mathbb{N}} a_k \otimes (\mu^{p-1}/p)^{[k]} \mapsto \sum_{k\in\mathbb{N}} a_k(\mu^{p-1}/p)^{[k]}$ *is continuous for the* p*-adic topology and an isomorphism of* $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})$*-algebras.*

(4) *The ideal* $I^{(r)}\mathbf{A}_{\mathrm{cris}}(O_{L_\infty})$ *is topologically generated by* $t^{\{s\}}$ *for* $s \geq r$.

(5) *The natural map* $\mathbf{A}_{\mathrm{inf}}(O_{L_\infty})/I^{(r)} \rightarrow \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})/I^{(r)}$ *is injective and the cokernel is killed by* $m!p^m$ *where* $m = \lfloor \frac{r}{p-1} \rfloor$.

Similar statements are true for $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})$ *and* $\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}})$.

Proof. All claims except (3) follow from [Fon94, §5.2] and [Tsu99, §A3]. The proof of claim in (3) follows in a manner similar to the proof of [Bri08, Proposition 6.2.14]. ∎

Remark 4.15. Inside $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$ we have subrings $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{inf}}(O_{L_\infty})[1/p]$, $\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) := \mathbf{A}_{\mathrm{cris}}(O_{L_\infty})[1/p]$ and $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}$ and we equip these with a filtration induced from the natural filtration on $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty})$ (see §2.1.1). Then one can define ideals similar to (4.9) for these rings and from Lemma 4.14 (7) we obtain isomorphisms $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})/I^{(r)} \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty})/I^{(r)}$ and $\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/I^{(r)} \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})/I^{(r)}$.

Proposition 4.16. *The* $\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})$*-module*

$$\mathbf{N}_{\breve{L},\infty}(V) := \mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \subset (\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V)^{H_L} = \tilde{\mathbf{D}}^+_L(V),$$

is stable under the residual action of Γ_L *on* $\tilde{\mathbf{D}}^+_L(V)$ *and we equip* $\mathbf{N}_{\breve{L},\infty}(V)$ *with this action. Then we have a natural* Γ_L*-equivariant embedding* $\mathbf{N}_{\breve{L},\infty}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ *(see Remark 2.19 and (2.8) for* Γ_L*-action on the latter).*

Proof. From Lemma 4.4 (2) consider the exact sequence

$$0 \longrightarrow \mu^s\tilde{\mathbf{D}}^+_L(V) \longrightarrow \mathbf{N}_{\breve{L},\infty}(V) \longrightarrow \mathbf{N}_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_L(V) \longrightarrow 0, \tag{4.10}$$

where we know that $\mu^s\tilde{\mathbf{D}}^+_L(V) \subset \tilde{\mathbf{D}}^+_L(V)$ is stable under the action of Γ_L. Therefore, to show that the middle term above is stable under the action of Γ_L, it is enough to show that for the inclusion $\mathbf{N}_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_L(V) \subset \tilde{\mathbf{D}}^+_L(V)/\mu^s\tilde{\mathbf{D}}^+_L(V) \subset (\mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \otimes_{\mathbb{Q}_p} V)^{H_L} \subset \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \otimes_{\mathbb{Q}_p} V$, the image of the first term in the last term is stable under the action of G_L.

From Lemma 4.5, we have a $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$-linear and $(\varphi, G_{\breve{L}})$-equivariant isomorphism $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$. In view of Remark 4.15, let us set

$$M := (I^{(s)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V) \cap (\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)) \subset \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V.$$

Then we obtain a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mu^s\tilde{\mathbf{D}}^+_L(V) \arrow[r] \arrow[d] & \mathbf{N}_{\breve{L},\infty}(V) \arrow[r] \arrow[d] & \mathbf{N}_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_L(V) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M \arrow[r] & \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \arrow[r] & (\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V))/M \arrow[r] & 0.
\end{tikzcd}$$

The left vertical arrow is injective by Lemma 4.4 (2) and the middle arrow is obviously injective.

Lemma 4.17. *The inclusion* $\mathbf{N}_{\breve{L},\infty}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ *induces a* $\Gamma_{\breve{L}}$*-equivariant isomorphism* $\mathbf{N}_{\breve{L},\infty}(V)/\mu^s\tilde{\mathbf{D}}^+_L(V) \xrightarrow{\sim} (\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V))/M$.

Proof. First, we observe that by Lemma 4.4 (2) we have

$$\begin{aligned} M \cap \mathbf{N}_{\check{L},\infty}(V) &= (I^{(s)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V) \cap \mathbf{N}_{\check{L},\infty}(V) \\ &\subset (I^{(s)}\mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V) \cap \tilde{\mathbf{D}}^+_{L}(V) \subset \mu^s \tilde{\mathbf{D}}^+_{L}(V). \end{aligned}$$

Therefore, we get that the rightmost vertical map in the diagram is injective. Next, we need to show that $\mathbf{N}_{\check{L},\infty}(V) + M = \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V)$. The left expression is clearly contained in the right. To show the other direction, let $x \in \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V)$. Then for $m \in \mathbb{N}$ large enough $p^m x \in \mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{A}^+_{\check{L}}} \mathbf{N}_{\check{L}}(T)$. By the isomorphism in Lemma 4.14 (4), for $r = \lceil \frac{s}{p-1} \rceil$, $k \in \mathbb{N}$ and $x_k \in \mathbf{N}_{\check{L}}(T)$ such that $x_k \to 0$ as $k \to +\infty$, we can write

$$p^m x = \sum_{k \in \mathbb{N}} x_k (\mu^{p-1}/p)^{[k]} = \sum_{k \leq r-1} x_k (\mu^{p-1}/p)^{[k]} + \sum_{k \geq r} x_k (\mu/p)^{[k]}.$$

Clearly, the first sum in the rightmost expression is in $\mathbf{N}_{\check{L},\infty}(V)$. Moreover, from Lemma 4.14 (1) there exists $v \in \Lambda^\times$ such that $\mu^{p-1}/p = v t^{p-1}/p$. Therefore, we obtain that the second sum is in $(I^{(s)}\mathbf{A}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Z}_p} T) \cap (\mathbf{A}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{A}^+_{\check{L}}} \mathbf{N}_{\check{L}}(T)) \subset M$. Hence, $x \in \mathbf{N}_{\check{L},\infty}(V) + M$. ∎

Let us now consider the diagram (4.11) below with the following description: in (4.11) the bottom horizontal arrow is a (φ, G_L)-equivariant isomorphism since V is a crystalline representation of G_L. The left vertical arrow from fourth to third row is induced by the projection $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ via $X_i \mapsto [X_i^\flat]$, it admits a section as in (2.7), it is evidently φ-equivariant and it is G_L-equivariant since the codomain is equipped with a G_L-action by transport of structure from the domain (see Remark 2.19). The right vertical arrow from fourth to third row is also induced by the projection $\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \twoheadrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$, it admits a natural section $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \to (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes V)^{\partial=0}$ and it is naturally (φ, G_L)-equivariant. The horizontal arrow in third row is the isomorphism in Lemma 2.18 and it is (φ, G_L)-equivariant by the preceding discussion and Remark 2.19. Commutativity of lower square follows from this. The left vertical arrow from third to second row is an isomorphism since $\check{L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{D}_{\mathrm{cris},\check{L}}(V)$ by (2.5) and its $(\varphi, G_{\check{L}})$-equivariance can either be checked by the explicit formula in Remark 2.19 or by observing that the non-canonical map $L \to \check{L} \to \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}})$ is $(\varphi, G_{\check{L}})$-equivariant (see proof of Lemma 4.13). The horizontal arrow in second row is a $(\varphi, G_{\check{L}})$-equivariant isomorphism since V is a crystalline representation of $G_{\check{L}}$. Commutativity of the middle square follows since the outer square between second and fourth row as well as the lower square are commutative. Commutativity and $(\varphi, G_{\check{L}})$-equivariance of the top square follows from Lemma 4.5.

$$\begin{tikzcd} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\check{L}} \mathbf{D}_{\mathrm{cris},\check{L}}(V) \arrow[r, "\sim"] \arrow[d, hook] & \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V) \arrow[d, hook] \\ \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\check{L}} \mathbf{D}_{\mathrm{cris},\check{L}}(V) \arrow[r, "\sim"] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \arrow[d, equal] \\ \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[r, "\sim"] \arrow[u, "\wr"] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V \\ (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))^{\partial=0} \arrow[r, "\sim"] \arrow[u, "\wr"] & (\mathcal{O}\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V)^{\partial=0}. \arrow[u, "\wr"'] \end{tikzcd} \tag{4.11}$$

Furthermore, in the diagram (4.11), the image of composition of top two left vertical maps inside $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of G_L by Remark 2.19. So the image of composition of top two right vertical maps inside $\mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V$ is stable under the action of G_L and it follows that its image $(\mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\check{L}}} \mathbf{N}_{\check{L}}(V))/M \subset \mathbf{B}^+_{\mathrm{cris}}(O_{\overline{L}})/I^{(s)} \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \otimes_{\mathbb{Q}_p} V$ is stable under the action of G_L. Therefore, from the preceding lemma we obtain that the image of $\mathbf{N}_{\check{L},\infty}(V)/\mu^s \tilde{\mathbf{D}}^+_{L}(V) \subset \mathbf{B}_{\mathrm{inf}}(O_{\overline{L}})/\mu^s \otimes_{\mathbb{Q}_p} V$ is stable under the action of G_L. Hence, from (4.10) we conclude that $\mathbf{N}_{\breve{L},\infty}(V)$ is stable under the action of Γ_L and the following natural composition is Γ_L-equivariant:

$$\mathbf{B}_{\mathrm{inf}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V). \tag{4.12}$$

∎

Recall that $\mathbf{N}_{\mathrm{rig},\breve{L}}(V) = \mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ and we note the following:

Corollary 4.18. *Extending $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}$-linearly the embedding $\mathbf{N}_{\breve{L},\infty}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ from Proposition 4.16 gives an identification of $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}$-submodules of $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$*

$$\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}_{\mathrm{inf}}(O_{L_\infty})} \mathbf{N}_{\breve{L},\infty}(V) = \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) = \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V),$$

stable under the Γ_L-action on $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$.

Proof. The equality in the claim follows from definitions and the compatibility of Γ_L-actions follows from (4.12). Using (4.2) we have $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) = \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Therefore, we can identify $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ as a $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}$-submodule of $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Then the stability of $\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V)$ under Γ_L-action follows from Proposition 4.16. ∎

Recall that from Definition 4.9 we have a $\mathbf{B}^+_{\mathrm{rig},L}$-submodule $\mathcal{M}_L(\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \subset \mathcal{M}_{\breve{L}}(\mathbf{D}_{\mathrm{cris},\breve{L}}(V))$ stable under $(\varphi, \Gamma_{\breve{L}})$-action and from Lemma 4.8 a $\mathbf{B}^+_{\mathrm{rig},\breve{L}}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism $\beta : \mathcal{M}_{\breve{L}}(\mathbf{D}_{\mathrm{cris},\breve{L}}(V)) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$. Define a $\mathbf{B}^+_{\mathrm{rig},L}$-submodule of $\mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ as

$$\mathbf{N}_{\mathrm{rig},L}(V) := \beta(\mathcal{M}_L(\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))) \subset \mathbf{N}_{\mathrm{rig},\breve{L}}(V). \tag{4.13}$$

Since the map $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ constructed in §2.1.5 is $(\varphi, \Gamma_{\breve{L}})$-equivariant, from (4.13) we obtain a $\mathbf{B}^+_{\mathrm{rig},L}$-linear and $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism $\beta : \mathcal{M}_L(\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},L}(V)$. In particular, from Lemma 4.11 (3) we get that $\mathbf{N}_{\mathrm{rig},L}(V)$ is a finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module of rank $= \dim_{\mathbb{Q}_p} V$ and the natural map $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \to \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ is a $(\varphi, \Gamma_{\breve{L}})$-equivariant isomorphism, since β is $(\varphi, \Gamma_{\breve{L}})$-equivariant. Moreover, from Lemma 4.12 it follows that $\mathbf{N}_{\mathrm{rig},L}(V)$ is of finite $[p]_q$-height and pure of slope 0. Now consider the following diagram:

$$\begin{tikzcd}
\mathcal{M}_L(\mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V))[\mu/t] \arrow[r, "\sim"] \arrow[d] \arrow[rr, bend left, "\beta", "\sim"'] & \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[d] & \mathbf{N}_{\mathrm{rig},L}(V)[\mu/t] \arrow[l, "\sim"'] \arrow[d] \\
\mathcal{M}_{\breve{L}}(\mathbf{D}_{\mathrm{cris},\breve{L}}(V))[\mu/t] \arrow[r, "\sim"] \arrow[rr, bend right, "\sim", "\beta"'] & \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) & \mathbf{N}_{\mathrm{rig},\breve{L}}(V)[\mu/t]. \arrow[l, "\sim"']
\end{tikzcd} \tag{4.14}$$

In the diagram, all vertical arrows are natural inclusions. In the bottom row, the left to right horizontal arrow is the inverse of the composition of lower horizontal and left vertical arrow of diagram (4.5), the right to left horizontal arrow is inverse of (4.2), the curved arrow is the map β in Lemma 4.8 and the resulting triangle commutes by diagram (4.5). In the top row, the left to right horizontal arrow is the isomorphism in (4.7), the curved arrow is from (4.13), the right to left horizontal arrow is the inverse of β composed with natural inclusion and the resulting triangle commutes by definition. Therefore, two inner squares commute by definition and all maps are $(\varphi, \Gamma_{\breve{L}})$-equivariant.

Using the diagram (4.14) and Defintion 4.9, we may also write $\mathbf{N}_{\mathrm{rig},L}(V) = (\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \cap \mathbf{N}_{\mathrm{rig},\breve{L}}(V) \subset \mathbf{B}^+_{\mathrm{rig},\breve{L}}[\mu/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V)$. In particular, below we will consider $\mathbf{N}_{\mathrm{rig},L}(V)$ as a $\mathbf{B}^+_{\mathrm{rig},L}$-submodule of $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Furthermore, from Lemma 2.21 recall that

$\mathbf{B}^+_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}^+_{\mathrm{cris}}(O_{L_\infty}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of Γ_L and we equip the former with induced Γ_L-action. Since $g(t) = \chi(g)t$ and $g(\mu) = (1+\mu)^{\chi(g)} - 1$ for $g \in \Gamma_L$ and χ the p-adic cyclotomic character, the preceding Γ_L-action extends to $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$.

Proposition 4.19. *The $\mathbf{B}^+_{\mathrm{rig},L}$-submodule $\mathbf{N}_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ is stable under the action of Γ_L. Moreover, the preceding inclusion extends to a $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]$-*linear *and (φ, Γ_L)-compatible isomorphism*

$$\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V). \tag{4.15}$$

Proof. From Corollary 4.18 and the discussion after (4.13), we have

$$\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V),$$

stable under the action of Γ_L. Moreover, using Lemma 2.21 and the discussion after (4.14), we have a Γ_L-equivariant embedding $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \subset \mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Therefore, inside $\mathbf{B}_{\mathrm{cris}}(O_{L_\infty}) \otimes_{\mathbb{Q}_p} \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$, the following intersection is stable under Γ_L-action:

$$\begin{aligned}
(\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)) &\cap (\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)) \\
&= (\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)) \cap (\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)) \\
&= (\tilde{\mathbf{B}}^+_{\mathrm{rig},L} \cap \mathbf{B}^+_{\mathrm{rig},L}[\mu/t]) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) = \mathbf{N}_{\mathrm{rig},L}(V).
\end{aligned}$$

The first equality follows from (4.14) and the second equality follows from Lemma 2.11 and the fact that $\mathbf{N}_{\mathrm{rig},L}(V)$ is finite free over $\mathbf{B}^+_{\mathrm{rig},L}$. This proves the first claim. For the second claim, note that by definition, extending the (φ, Γ_L)-equivariant inclusion $\mathbf{N}_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ along the map $\mathbf{B}^+_{\mathrm{rig},L} \to \mathbf{B}^+_{\mathrm{rig},L}$ coincides with the top right horizontal arrow of the diagram (4.14). Hence, the isomorphism in (4.15) follows. ∎

Corollary 4.20. *The action of Γ_L on $\mathbf{N}_{\mathrm{rig},L}(V)$ is trvial modulo μ.*

Proof. Note that $g(\mu) = (1+\mu)^{\chi(g)} - 1$ and $g(t) = \chi(g)t$ for $g \in \Gamma_L$ and χ the p-adic cyclotomic character, in particular, $(g-1)(\mu/t) = \mu u_g(\mu/t)$ for some $u_g \in \mathbf{B}^+_L$. Therefore, using Lemma 2.22 it follows that the action of Γ_L is trivial modulo μ on $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xleftarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)$ (see (4.15)).

From Proposition 4.19 we have (φ, Γ_L)-equivariant inclusion $\mathbf{N}_{\mathrm{rig},L}(V) \subset \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$. Let $x \in \mathbf{N}_{\mathrm{rig},L}(V)$, then for $g \in \Gamma_L$ we have $(g-1)x \in \mathbf{N}_{\mathrm{rig},L}(V) \subset \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ and $(g-1)x \in \mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)$. So inside $\mathbf{N}_{\mathrm{rig},\breve{L}}(V)[\mu/t]$,

$$\begin{aligned}
\mathbf{N}_{\mathrm{rig},\breve{L}}(V) &\cap (\mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)) \\
&= (\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)) \cap (\mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)) \\
&= (\mathbf{B}^+_{\mathrm{rig},\breve{L}} \cap \mu\mathbf{B}^+_{\mathrm{rig},L}[\mu/t]) \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) = \mu\mathbf{N}_{\mathrm{rig},L}(V),
\end{aligned}$$

where the first equality follows from the isomorphism $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ (see the discussion after (4.13)), the second equality follows since $\mathbf{N}_{\mathrm{rig},L}(V)$ is free over $\mathbf{B}^+_{\mathrm{rig},L}$ and the last equality follows from Lemma 2.10. Hence, $(g-1)\mathbf{N}_{\mathrm{rig},L}(V) \subset \mu\mathbf{N}_{\mathrm{rig},L}(V)$ for $g \in \Gamma_L$. ∎

4.4. Compatibility with (φ, Γ_L)-modules. From §2.2 recall that $\mathbf{D}^\dagger_{\mathrm{rig},L}(V)$ is a pure of slope 0 finite free (φ, Γ_L)-module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$ functorially attached to V. The following result is a generalisation of [Ber02, Proposition 3.5 & Théorème 3.6] from the perfect residue field case to L:

Proposition 4.21. *There are natural (φ, G_L)-equivariant isomorphisms*

(1) $\tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V$.

(2) $\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},L}} \mathbf{D}^\dagger_{\mathrm{rig},L}(V)$.

Proof. For (1), recall that from Lemma 4.13, there is a $\tilde{\mathbf{B}}^+_{\mathrm{rig}}$-linear and (φ, G_L)-equivariant map

$$\tilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \longrightarrow \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V,$$

where the isomorphism is from Lemma 2.18. Extending the isomorphism in (4.2) along $\tilde{\mathbf{B}}^+_{\mathrm{rig},L}[\mu/t] \to \tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t]$ and using (2.5) we obtain a φ-equivariant isomorphism

$$\tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\breve{L}} \mathbf{D}_{\mathrm{cris},\breve{L}}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V).$$

The preceding isomorphism fits into a commutative diagram compatibly with (4.11)

$$\begin{tikzcd}
\tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[r] \arrow[d, "\wr"] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[dr, "\sim"] & \\
\tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(V) \arrow[r, "\sim"] & \tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V \arrow[r] & \mathbf{B}_{\mathrm{cris}}(O_{\overline{L}}) \otimes_{\mathbb{Q}_p} V,
\end{tikzcd} \tag{4.16}$$

where the left horizontal arrow in bottom row is induced from the isomorphism $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbb{Z}_p} T$ (see Lemma 4.4 (2)), the slanted isomorphism is the isomorphism in the third row of (4.11) and the rest are natural injective maps. Since the slanted isomorphism is (φ, G_L)-equivariant we obtain that the isomorphism $\tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V$ is (φ, G_L)-equivariant, showing (1). For (2), extending the isomorphism in (1) along $\tilde{\mathbf{B}}^+_{\mathrm{rig}}[1/t] \to \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t]$ and using (2.3) we obtain (φ, G_L)-equivariant isomorphisms

$$\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V \xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^\dagger_{\mathrm{rig},L}} \mathbf{D}^\dagger_{\mathrm{rig},L}(V). \qquad \blacksquare$$

From the discussion after (4.13) and Proposition 4.19 we have that $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V)$ is a pure of slope 0 finite free (φ, Γ_L)-module over $\mathbf{B}^\dagger_{\mathrm{rig},L}$ of rank $= \dim_{\mathbb{Q}_p} V$. Therefore, by the equivalence of categories in [Ohk15, Lemma 4.5.7] there exists a unique finite free étale (φ, Γ_L)-module $D^\dagger_L$ over $\mathbf{B}^\dagger_L$ of rank $= \dim_{\mathbb{Q}_p} V$ such that $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} D^\dagger_L$ compatible with (φ, Γ_L)-action.

Corollary 4.22. *There exists a natural (φ, G_L)-equivariant isomorphism $\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_L} V$ inducing natural (φ, Γ_L)-equivariant isomorphisms $D^\dagger_L \xrightarrow{\sim} \mathbf{D}^\dagger_L(V)$ and $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} \mathbf{D}^\dagger_L(V)$.*

Proof. Consider the following diagram:

$$\begin{tikzcd}
\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \arrow[r, "\sim"] \arrow[d] & \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},\breve{L}}} \mathbf{N}_{\mathrm{rig},\breve{L}}(V) \arrow[r, "\sim"] \arrow[d] & \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V \arrow[d] \\
\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \arrow[r, "\sim"] & \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \arrow[r, "\sim"] & \tilde{\mathbf{B}}^\dagger_{\mathrm{rig}}[1/t] \otimes_{\mathbb{Q}_p} V.
\end{tikzcd}$$

In the top row, the left horizontal arrow is induced by the isomorphism $\mathbf{B}^+_{\mathrm{rig},\breve{L}} \otimes \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},\breve{L}}(V)$ (see the discussion after (4.13)) and the right horizontal arrow is induced by the isomorphism $\mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbf{A}^+_{\breve{L}}} \mathbf{N}_{\breve{L}}(T) \xrightarrow{\sim} \mathbf{A}_{\mathrm{inf}}(O_{\overline{L}})[1/\mu] \otimes_{\mathbb{Z}_p} T$ (see Lemma 4.4 (2)). In the bottom row, the left horizontal arrow is induced by the (φ, Γ_L)-equivariant isomorphism $\mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^+_{\mathrm{rig},L}[\mu/t] \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V)$ (see (4.15) in Proposition 4.19) and the right horizontal arrow is induced from Propositon 4.21 (1). The left and right vertical arrow are natural maps and the middle vertical arrow is induced from (4.2) and (2.5). Commutativity of the left square follows from (4.14) and commutativity of the right square follows from (4.16). This shows the first claim.

For the second claim, set $V' := (\tilde{\mathbf{B}}^\dagger \otimes_{\mathbf{B}^\dagger_L} D^\dagger_L)^{\varphi=1}$, it is a p-adic representation of G_L with $\dim_{\mathbb{Q}_p} V' = \dim_{\mathbb{Q}_p} V$ (see [AB08, Théorème 4.35]). Now we note that $V' \subset (\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^\dagger_L} D^\dagger_L)^{\varphi=1} \xrightarrow{\sim} (\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V))^{\varphi=1} \xrightarrow{\sim} (\tilde{\mathbf{B}}^\dagger_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{\varphi=1} = V$, where the first isomorphism follows from the discussion before the claim, the second isomorphism follows from (1) and the last equality follows from Lemma 2.2. Therefore, $V' \xrightarrow{\sim} V$ as G_L-representations and it implies that $D^\dagger_L = \mathbf{D}^\dagger_L(V') \xrightarrow{\sim} \mathbf{D}^\dagger_L(V)$ as étale (φ, Γ_L)-modules over $\mathbf{B}^\dagger_L$. It is straightforward to verify that this isomorphism is compatible with the commutative diagram above. This concludes our proof. ∎

Remark 4.23. As indicated before the Corollary, for a p-adic crystalline representation of V, combining the (φ, Γ_L)-equivariant isomorphism $\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_{\mathrm{rig},L}} \mathbf{N}_{\mathrm{rig},L}(V) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} \mathbf{D}^\dagger_L(V)$ together with the inverse of the isomorphism (4.15), gives a $\mathbf{B}^\dagger_{\mathrm{rig},L}$-linear (φ, Γ_L)-equivariant isomorphism

$$\mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_L \mathcal{O}\mathbf{D}_{\mathrm{cris},L}(V) \xrightarrow{\sim} \mathbf{B}^\dagger_{\mathrm{rig},L} \otimes_{\mathbf{B}^\dagger_L} \mathbf{D}^\dagger_L(V). \tag{4.17}$$

The isomorphism (4.17) generalises [Ber02, Proposition 3.7] from the perfect residue field case to L.

4.5. Obtaining Wach module. The finite free $\mathbf{B}^+_{\mathrm{rig},L}$-module $\mathbf{N}_{\mathrm{rig},L}(V)$ is of finite $[p]_q$-height s and pure of slope 0 (see Lemma (4.12)), therefore from Lemma 2.12 (2) there exists a unique finite free $\mathbf{B}^+_L$-module of rank $= \dim_{\mathbb{Q}_p} V$ and finite $[p]_q$-height s whose extension of scalars along $\mathbf{B}^+_L \to \mathbf{B}^+_{\mathrm{rig},L}$ gives $\mathbf{N}_{\mathrm{rig},L}(V)$. In particular, from the proof of Lemma 2.12 we note the following:

Definition 4.24. Define $\mathbf{N}_L(V) := \mathbf{N}_{\mathrm{rig},L}(V) \cap \mathbf{D}^\dagger_L(V) \subset \mathbf{D}^\dagger_{\mathrm{rig},L}(V)$.

The $\mathbf{B}^+_L$-module $\mathbf{N}_L(V)$ is finite free of rank $= \dim_{\mathbb{Q}_p} V$ and it is equipped with an induced Frobenius-semilinear endomorphism φ such that cokernel of the injective map $(1 \otimes \varphi) : \varphi^*(\mathbf{N}_L(V)) \to \mathbf{N}_L(V)$ is killed by $[p]_q^s$ since $\mathbf{N}_{\mathrm{rig},L}(V)$ is of finite $[p]_q$-height s and $1 \otimes \varphi : \varphi^*(\mathbf{D}^\dagger_L(V)) \xrightarrow{\sim} \mathbf{D}^\dagger_L(V)$. Moreover, we have $\mathbf{N}_L(V) \subset \mathbf{D}^+_L(V)$ because inside $\mathbf{D}^\dagger_{\mathrm{rig},L}(V)$ we have

$$\begin{aligned}\mathbf{N}_L(V) = \mathbf{N}_{\mathrm{rig},L}(V) \cap \mathbf{D}^\dagger_L(V) &\subset (\tilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V)^{H_L} \cap (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V)^{H_L}\\ &\subset ((\tilde{\mathbf{B}}^+_{\mathrm{rig}} \otimes_{\mathbb{Q}_p} V) \cap (\mathbf{B}^\dagger \otimes_{\mathbb{Q}_p} V))^{H_L}\\ &\subset ((\tilde{\mathbf{B}}^+_{\mathrm{rig}} \cap \mathbf{B}^\dagger) \otimes_{\mathbb{Q}_p} V)^{H_L} = (\mathbf{B}^+ \otimes_{\mathbb{Q}_p} V)^{H_L} = \mathbf{D}^+_L(V).\end{aligned}$$

Furthermore, since $\mathbf{N}_{\mathrm{rig},L}(V)$ and $\mathbf{D}^\dagger_L(V)$ are stable under compatible action of Γ_L (see Proposition 4.19 and Corollary 4.22), we conclude that $\mathbf{N}_L(V)$ is stable under Γ_L-action. In particular, from the preceding discussion and Lemma 2.12 we get (φ, Γ_L)-equivariant isomorphisms

$$\mathbf{B}^+_{\mathrm{rig},L} \otimes_{\mathbf{B}^+_L} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{N}_{\mathrm{rig},L}(V) \text{ and } \mathbf{B}^\dagger_L \otimes_{\mathbf{B}^+_L} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{D}^\dagger_L(V). \tag{4.18}$$

Lemma 4.25. *The action of Γ_L on $\mathbf{N}_L(V)$ is trvial modulo μ.*

Proof. Let $g \in \Gamma_L$ and $x \in \mathbf{N}_L(V)$. Then, $(g-1)x \in \mathbf{N}_L(V) \subset \mathbf{D}^\dagger_L(V)$. Moreover, from Corollary 4.20 we have $(g-1)x \in \mu\mathbf{N}_{\mathrm{rig},L}(V)$. Therefore, inside $\mathbf{D}^\dagger_{\mathrm{rig},L}(V)$, from (4.18) we get

$$(g-1)x \in \mathbf{D}^\dagger_L(V) \cap \mu\mathbf{N}_{\mathrm{rig},L}(V) = (\mathbf{B}^\dagger_L \cap \mu\mathbf{B}^+_{\mathrm{rig},L}) \otimes_{\mathbf{B}^+_L} \mathbf{N}_L(V) = \mu\mathbf{N}_L(V).$$ ∎

Definition 4.26. Define the Wach module over $\mathbf{A}^+_L = \mathbf{B}^+_L \cap \mathbf{A}_L \subset \mathbf{B}_L$ as

$$\mathbf{N}_L(T) := \mathbf{N}_L(V) \cap \mathbf{D}_L(T) \subset \mathbf{D}_L(V).$$

Proof of Theorem 4.1. We will show that $\mathbf{N}_L(T)$ from Definition 4.26 satisfies all axioms of Definition 3.7. From the definition, note that $\mathbf{N}_L(T)$ is a finitely generated torsion-free $\mathbf{A}_L^+$-module and an elementary computation shows that $\mathbf{N}_L(T) \cap \mu^n \mathbf{N}_L(V) = \mu^n \mathbf{N}_L(T)$ for all $n \in \mathbb{N}$, in particular, $\mathbf{N}_L(T)/\mu$ is p-torsion free. Moreover, we have $\mathbf{N}_L(T)[1/p] = \mathbf{N}_L(V)$ and a simple diagram chase shows that $(\mathbf{N}_L(T)/p)[\mu] = (\mathbf{N}_L(T)/\mu)[p] = 0$ and $(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T))/p = (\mathbf{N}_L(T)/p)[1/\mu]$. So we have $\mathbf{N}_L(T)/p^n \subset (\mathbf{N}_L(T)/p^n)[1/\mu] = \mathbf{A}_L \otimes_{\mathbf{A}_L^+} N/p^n$ for all $n \in \mathbb{N}$ and therefore $\mathbf{N}_L(T) \cap p^n(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)) = p^n \mathbf{N}_L(T)$, in particular, $\mathbf{N}_L(V) \cap (\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)) = \mathbf{N}_L(T)$. Now using Remark 2.15 it follows that $\mathbf{N}_L(T)$ is finite free $\mathbf{A}_L^+$-module of rank $= \mathrm{rk}_{\mathbf{B}_L^+} \mathbf{N}_L(V) = \dim_{\mathbb{Q}_p} V$. Alternatively, to get the preceding statement, one can also use [Ber04, Lemme II.1.3] (the proof of loc. cit. does not require the residue field of discrete valuation base field, L in our case, to be perfect).

From the definition it also follows that $\mathbf{N}_L(T) \cap p^n \mathbf{D}_L(T) = p^n \mathbf{N}_L(T)$, in particular, we have $\mathbf{N}_L(T)/p^n \subset \mathbf{D}_L(T)/p^n$ and therefore $(\mathbf{N}_L(T)/p^n)[1/\mu] \subset \mathbf{D}_L(T)/p^n$. So we get that $(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T))/p^n \subset \mathbf{D}_L(T)/p^n$, or equivalently, $(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T)) \cap p^n \mathbf{D}_L(T) = p^n(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T))$. Note that we have $(\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T))[1/p] = \mathbf{B}_L \otimes_{\mathbf{B}_L^+} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{D}_L(V)$, where the last isomorphism follows from (4.18). Therefore, we get that $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) = \mathbf{D}_L(T) \cap (\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T))[1/p] \xrightarrow{\sim} \mathbf{D}_L(T) \cap \mathbf{D}_L(V) = \mathbf{D}_L(T)$. Next, $\mathbf{N}_L(T)$ is equipped with an induced Frobenius-semilinear endomorphism φ. We have $\varphi : \mathbf{A}_L^+ \to \mathbf{A}_L^+$ is finite and faithfully flat of degree p^{d+1} and $\varphi^*(\mathbf{A}_L) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{A}_L$ and similarly $\varphi^*(\mathbf{B}_L^+) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{B}_L^+$ (see §2.1.2). Therefore, we get that $\varphi^*(\mathbf{N}_L(V)) = \mathbf{B}_L^+ \otimes_{\varphi, \mathbf{B}_L^+} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{N}_L(V)$ and $\varphi^*(\mathbf{D}_L(T)) = \mathbf{A}_L \otimes_{\varphi, \mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{A}_L^+ \otimes_{\varphi, \mathbf{A}_L^+} \mathbf{D}_L(T)$. Then it easily follows that $\varphi^*(\mathbf{N}_L(T)) = \varphi^*(\mathbf{N}_L(V)) \cap \varphi^*(\mathbf{D}_L(T)) \subset \varphi^*(\mathbf{D}_L(V))$. Now since $1 \otimes \varphi$ is injective on $\varphi^*(\mathbf{D}_L(V))$, $1 \otimes \varphi : \varphi^*(\mathbf{D}_L(T)) \xrightarrow{\sim} \mathbf{D}_L(T)$ and cokernel of $1 \otimes \varphi : \varphi^*(\mathbf{N}_L(V)) \to \mathbf{N}_L(V)$ is killed by $[p]_q^s$, we get that cokernel of the injective map $1 \otimes \varphi : \varphi^*(\mathbf{N}_L(T)) \to \mathbf{N}_L(T)$ is killed by $[p]_q^s$. Finally, note that $\mathbf{N}_L(T)$ is equipped with an induced Γ_L-action such that Γ_L acts trivially on $\mathbf{N}_L(T)/\mu\mathbf{N}_L(T)$ (follows easily from Lemma 4.25) and we have $\mathbf{A}_L \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{D}_L(T)$. Hence, we conclude that T is of finite $[p]_q$-height. ∎

Corollary 4.27. *There exists a natural isomorphism of étale $(\varphi, \Gamma_{\breve{L}})$-modules $\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}(T)$ and a natural isomorphism of Wach modules $\mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(T)$.*

Proof. Note that we have an injection of étale $(\varphi, \Gamma_{\breve{L}})$-modules $\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \subset \mathbf{D}_{\breve{L}}(T)$ and $(W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_L} \mathbf{D}_L(T))^{\varphi=1} \xrightarrow{\sim} T \xleftarrow{\sim} (W(\mathbb{C}_L^\flat) \otimes_{\mathbf{A}_{\breve{L}}} \mathbf{D}_{\breve{L}}(T))^{\varphi=1}$ as $G_{\breve{L}}$-representations. So we get that $\mathbf{A}_{\breve{L}} \otimes_{\mathbf{A}_L} \mathbf{D}_L(T) \xrightarrow{\sim} \mathbf{D}_{\breve{L}}(T)$. Furthermore, we have a $(\varphi, \Gamma_{\breve{L}})$-equivariant injection of Wach modules $\mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \subset \mathbf{N}_{\breve{L}}(T)$. So by the unniquess of a Wach module attached to T (see Lemma 3.9), it follows that $\mathbf{A}_{\breve{L}}^+ \otimes_{\mathbf{A}_L^+} \mathbf{N}_L(T) \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(T)$. ∎

Proof of Corollary 4.3. The equivalence of $\otimes$-categories follows from Theorem 4.1 and we are left to show exactness of the functor $\mathbf{N}_L$ since exactness of the quasi-inverse functor follows from Proposition 3.3 and the exact equivalence in (2.2). From §2.1.5 recall that $\mathbf{A}_L^+ \to \mathbf{A}_{\breve{L}}^+$ is faithfully flat, therefore $\mathbf{B}_L^+ \to \mathbf{B}_{\breve{L}}^+$ is faithfully flat. Moreover, for a p-adic crystalline representation V of G_L, from Corollary 4.27 we have $\mathbf{B}_{\breve{L}}^+ \otimes_{\mathbf{B}_L^+} \mathbf{N}_L(V) \xrightarrow{\sim} \mathbf{N}_{\breve{L}}(V)$. So given an exact sequence

$$0 \to V_1 \to V_2 \to V_3 \to 0, \tag{4.19}$$

of p-adic crystalline representations of G_L it is enough to show that the sequence

$$0 \to \mathbf{N}_{\breve{L}}(V_1) \to \mathbf{N}_{\breve{L}}(V_2) \to \mathbf{N}_{\breve{L}}(V_3) \to 0 \tag{4.20}$$

is exact. Furthermore, note that (4.19) is exact if and only if it is exact after tensoring with $\mathbb{Q}_p(r)$ for $r \in \mathbb{Z}$. Similarly, (4.20) is exact if and only if it is exact after tensoring with $\mu^{-r}\mathbf{B}_{\breve{L}}^+(r)$. So we may assume that (4.19) is an exact sequence of positive crystalline representations, i.e. the Wach modules in (4.20) are effective. Moreover, the map $\mathbf{B}^+_{\breve{L}} \to \mathbf{B}^+_{\mathrm{rig},\breve{L}}$ is faithfully flat (by an argument similar to Lemma 2.6), so it is enough to show that the following sequence is exact:

$$0 \to \mathbf{N}_{\mathrm{rig},\breve{L}}(V_1) \to \mathbf{N}_{\mathrm{rig},\breve{L}}(V_2) \to \mathbf{N}_{\mathrm{rig},\breve{L}}(V_3) \to 0.$$

Exactness of the preceding sequence follows from Lemma 4.8, [Kis06, Theorem 1.2.15], [KR09, Proposition 2.2.6] and exactness of the functor $\mathbf{D}_{\mathrm{cris},\breve{L}}$. ■

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